CS 174: Combinatorics and Discrete Probability Fall 2012

Homework 6

Due: Thursday, October 11, 2012 by **9:30am**

Instructions: *You should upload your homework solutions on bspace. You are strongly encouraged to type out your solutions using LaTeX. You may also want to consider using mathematical mode typing in some office suite if you are not familiar with LaTeX. If you must handwrite your homeworks, please write clearly and legibly. We will not grade homeworks that are unreadable. You are encouraged to work in groups of 2-4, but you* **must** *write solutions on your own. Please review the homework policy carefully on the class homepage.*

Note: You *must* justify all your answers. In particular, you will get no credit if you simply write the final answer without any explanation.

Problem 1. *(Exercise 5.16 from MU – 6 points)* Let G be a random graph generated using the $G_{n,p}$ model. Write the answers for these questions in the limit as $n \to \infty$, so you should ignore constant and lower order terms.

(a) A *clique* of k vertices in a graph is a subset of k vertices such that all $\binom{k}{2}$ edges between these vertices lie in the graph. For what value of p, as a function of n, is the expected number of cliques of five vertices in G equal to 1?

(b) A $K_{3,3}$ is a complete bipartite graph with three vertices on each side. In other words, it is a graph with six vertices and nine edges; the six distinct vertices are arranged in two groups of three, and the nine edges connect each of the nine pairs over vertices with one vertex in each group. For what value of p, as a function of n, is the expected number of $K_{3,3}$ subgraphs of G equal to 1?

(c) For what value of p, as a function of n, is the expected number of Hamiltonian cycles in the graph equal to 1?

Problem 2. *(Exercise 5.21 from MU – 8 points)* In hashing with *open addressing*, the hash table is implemented as an array and there are no linked lists or chaining. Each entry in the array either contains one hashed item or is empty. The hash function defines, for each key k, a *probe sequence* $h(k,0), h(k,1), \ldots$ of table locations. To insert the key k, we first examine the sequence of table locations in the order defined by the key's probe sequence until we find an empty location; then we insert the item at that position. When searching for an item in the hash table, we examine the sequence of table locations in the order defined by the key's probe sequence until either the item is found or we have found an empty location in the sequence. If an empty location is found, this means the item is not present in the table.

An open-address hash table with $2n$ entries is used to store n items. Assume that the table location $h(k,j)$ is uniform over the $2n$ possible table locations and that all $h(k,j)$ are independent.

(a) Show that, under these conditions, the probability of an insertion requiring more than m probes is at most 2^{-m}.

(b) Show that, for $i = 1, 2, \ldots, n$, the probability that the i^{th} insertion requires more than $2\log(n)$ probes is at most $1/n^2$.

(c) Now, let X_i denote the number of probes required by the i^{th} insertion. You showed above that $\Pr(X_i \geq 2\log(n)) \leq 1/n^2$. Let the random variable $X = \max_{1 \leq i \leq n} X_i$ denote the maximum number of probes required by any of the n insertions. Show that $\Pr(X > 2\log(n)) \leq 1/n$.

(d) Use the above to conclude that the expected length of the longest probe sequence, $\mathbb{E}[X] = O(\log(n))$.

Problem 3 *(Exercise 6.2 from MU – 8 points)*

(a) Prove that, for every integer n, there exists a colouring of the edges of the complete graph, K_n, using two colours – say red and black – so that the total number of monochromatic K_4 is at most $\binom{n}{4}/2^5$.

(b) Give a Las Vegas algorithm for finding such a colouring (one with at most $\binom{n}{4}/2^5$ monochromatic K_4) that runs in expected polynomial time in n. Recall that a Las Vegas algorithm always returns a correct output, but its worst case running time may be unbounded.

(c) Show how to construct such a colouring deterministically in polynomial time using the method of conditional expectations.

Problem 4 *(Exercise 6.3 – 8 points)* Given an n-vertex undirected graph $G = (V, E)$, consider the following method of generating an independent set. Given a permutation σ of the vertices, define a subset $S(\sigma)$ of the vertices as follows: for each vertex i, $i \in S(\sigma)$ if and only if no neighbour j of i precedes i in the permutation σ.

(a) Show that each $S(\sigma)$ is an independent set in G.

(b) Suggest a natural randomized algorithm to produce σ for which you can show that the expected cardinality of $S(\sigma)$ is

$$\sum_{i=1}^{n} \frac{1}{d_i + 1},$$

where d_i is the degree of vertex i.

(c) Prove that G has an independent set of size at least $\sum_{i=1}^{n} 1/(d_i + 1)$.