Problem 1. (Exercise 6.10 from MU – 6 points) A family of subsets \( F \) of \( \{1, 2, \ldots, n\} \) is called an antichain if there is no pair of sets \( A \) and \( B \) in \( F \) satisfying \( A \subset B \).

(a) Given an example of \( F \) where \( |F| = \binom{n}{\lfloor n/2 \rfloor} \).

Solution: Choose every subset of size \( \lfloor n/2 \rfloor \).

(b) Let \( f_k \) be the number of sets in \( F \) with size \( k \). Show that

\[
\sum_{k=0}^{n} \frac{f_k}{\binom{n}{k}} \leq 1.
\]

(Hint: Choose a random permutation of the numbers from 1 to \( n \), and let \( X_k = 1 \) if the first \( k \) numbers in your permutation yield a set in \( F \). If \( X = \sum_{k=0}^{n} X_k \), what can you say about \( X \)?)

Solution: Following the hint, choose a random permutation of \( (1, \ldots, n) \). Let \( X_k = 1 \) if the first \( k \) numbers yield a set in \( F \), and let \( X = \sum_{k=0}^{n} X_k \).

Note that \( \Pr(X_k = 1) = \frac{f_k}{\binom{n}{k}} \). Furthermore, for only one value of \( k \) can \( X_k = 1 \), which means that \( \mathbb{E}[X] \leq 1 \). Therefore,

\[
\mathbb{E}[X] = \sum_{k=0}^{n} \mathbb{E}[X_k] = \sum_{k=0}^{n} \frac{f_k}{\binom{n}{k}} \leq 1
\]

(c) Argue that \( |F| \leq \binom{n}{\lfloor n/2 \rfloor} \) for any antichain \( F \).

Solution: For a fixed \( n \), the binomial coefficient is maximized at \( \binom{n}{\lfloor n/2 \rfloor} \). Therefore,

\[
\frac{1}{\binom{n}{\lfloor n/2 \rfloor}} \sum_{k=0}^{n} f_k \leq \sum_{k=0}^{n} \frac{f_k}{\binom{n}{k}} \leq 1
\]

which implies that

\[
\sum_{k=0}^{n} f_k \leq \binom{n}{\lfloor n/2 \rfloor}
\]

The result follows since \( |F| = \sum_{k=0}^{n} f_k \).
Problem 2. (Exercise 6.14 from MU – 6 points) Consider a graph in $G_{n,p}$, with $p = 1/n$. Let $X$ be the number of triangles in the graph, where a triangle is a clique with three edges. Show that

$$Pr(X \geq 1) \leq \frac{1}{6}$$

and that

$$\lim_{n \to \infty} Pr(X \geq 1) \geq \frac{1}{7}$$

(Hint: Use the conditional expectation inequality.)

Solution: Let $C_1, \ldots, C_{\binom{n}{3}}$ be an enumeration of all subsets of 3 vertices in the graph. Let $X = \sum_{i=1}^{\binom{n}{3}} X_i$, where $X_i$ is 1 if $C_i$ is a triangle and 0 otherwise. We have $E[X_i] = Pr[X_i = 1] = p^3$, so $E[X] = \binom{n}{3} p^3$. So by Markov’s inequality,

$$Pr[X \geq 1] \leq \binom{n}{3} p^3 = \binom{n}{3} (1/n)^3 \leq 1/6$$

To use the conditional expectation inequality, we mimic the argument from the previous part to get

$$E[X | X_i = 1] = 1 + \binom{n-3}{3} p^3 + \binom{n-3}{2} p^3 + \binom{n-3}{1} p^2$$

and then show

$$Pr[X \geq 1] \geq \sum_{i=1}^{\binom{n}{3}} \frac{Pr[X_i = 1]}{E[X | X_i = 1]}$$

$$= \frac{(\binom{n}{3}) p^3}{1 + (\binom{n-3}{3}) p^3 + (\binom{n-3}{2}) p^3 + (\binom{n-3}{1}) p^2}$$

$$= \frac{(\binom{n}{3}) (1/n)^3}{1 + (\binom{n-3}{3}) (1/n)^3 + (\binom{n-3}{2}) (1/n)^3 + (\binom{n-3}{1}) (1/n)^2}$$

$$\to \frac{1}{1 + 1/6 + 0 + 0} as n \to \infty$$

$$= \frac{1}{7}$$

Problem 3 (Exercise 6.18 from MU – 6 points) Let $G = (V, E)$ be an undirected graph and suppose each $v \in V$ is associated with a set $S(v)$ of $8r$ colours, where $r \geq 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$ there are at most $r$ neighbours $u$ of $v$ such that $c$ lies in $S(u)$. Prove that there is a proper colouring of $G$ assigning to each vertex $v$ a colour from its class $S(v)$ such that, for any edge $(u, v) \in E$, the colours assigned to $u$ and $v$ are different. You may want to let $A_{u,v,c}$ be the event that $u$ and $v$ are both coloured with colour $c$ and then consider the family of such events.

Solution: The solution to this problem is an application of the Lovaxa local lemma. Let us check the three requirements necessary to apply the lemma:

(a) $Pr(A_{u,v,c}) \leq Pr(u \text{ is color } c \mid v \text{ is color } c) Pr(v \text{ is color } c) \leq \frac{1}{8r} \cdot \frac{1}{8r} = \frac{1}{64r^2}$
(b) $A_{u,v,c}$ may only depend on events $A_{u',v',c'}$ where either $u = u'$ or $v = v'$. Note that $u$ has at most $8r$ colours, and hence at most $8r^2$ neighbours with which it can share a colour (this includes $v$). A symmetric argument holds for $v$. However, this means that the total number of events that $A_{u,v,c}$ can be dependent on is at most $16r^2$ (note that this is despite the double counting $A_{u,v,c'}$).

(c) $4dp \leq 4 \cdot 16r^2 \cdot \frac{1}{64r^2} = 1$.

So applying the Lovasz local lemma, we have:

$$\Pr\left(\bigcap_{u,v,c} \bar{A}_{u,v,c}\right) > 0$$

In other words, there exists a coloring such that no neighboring vertices have the same color.

**Problem 4 (12 points)** In this problem we will see that the value $p = \ln(n)/n$ is a threshold property that a random graph in the $G_{n,p}$ model has an isolated vertex, i.e. a vertex with no adjacent edges. That is, we will prove that

$$\lim_{n \to \infty} \Pr[G \text{ has an isolated vertex}] = \begin{cases} 0 & \text{if } p = \omega\left(\frac{\ln(n)}{n}\right) \\ 1 & \text{if } p = o\left(\frac{\ln(n)}{n}\right) \end{cases}.$$ 

(a) Let $X$ be the random variable denoting the number of isolated vertices in $G$. Write down the expectation of $X$ as a function of $n$ and $p$.

Solution: Let $X_i$ be the r.v. indicating whether vertex $i$ is isolated. Then

$$E[X_i] = (1 - p)^{n-1}$$

and by linearity of expectation, $E[X] = n(1 - p)^{n-1}$.

(b) Show that $E[X] \to 0$ for $p = \omega\left(\frac{\ln(n)}{n}\right)$, and that $E[X] \to \infty$ for $p = o\left(\frac{\ln(n)}{n}\right)$.

Solution: Write $p = a \cdot \frac{\ln n}{n}$. Note that

$$E[X] = n(1 - p)^{n-1} = n \left(1 - a \cdot \frac{\ln n}{n}\right)^{n-1} \to ne^{-a \ln n} = n^{1-a}$$

The case $p = o\left(\frac{\ln n}{n}\right)$ is equivalent to $a = o(1)$, and thus

$$E[X] \sim n^{1-o(1)} \to \infty$$

The second case, $p = \omega\left(\frac{\ln n}{n}\right)$ is equivalent to $a = \omega(1)$, and thus

$$E[X] \sim n^{-\omega(1)-1} \to 0$$
(c) Deduce from part (b) that Pr\[G\] has an isolated vertex \(\rightarrow 0\) for \(p = \omega(\ln(n)/n)\).

**Solution:** By Markov’s inequality we have \(\Pr[X \geq 1] \leq E[X]\), which by part (b) goes to zero in the case \(p = \omega(\ln(n)/n)\). Hence \(\Pr[X > 0] \rightarrow 0\) as required.

(d) Compute \(\text{Var}(X)\) as a function of \(n\) and \(p\).

**Solution:** For any \(i \neq j\), \(E[X_iX_j] = (1 - p)^{2n-3}\) (there are \(2n - 3\) possible edges adjacent to either \(i\) or \(j\)). Hence

\[
E[X^2] = \sum_{i,j} E[X_iX_j] = n(1 - p)^{n-1} + n(n - 1)(1 - p)^{2n-3}
\]

Therefore

\[
\text{Var}[X] = E[X^2] - E[X]^2 = n(1 - p)^{n-1} + n(n - 1)(1 - p)^{2n-3}(np - 1)
\]

(e) Deduce from parts (b) and (d) that \(\Pr[G\] has an isolated vertex \(\rightarrow 1\) for \(p = o(\ln(n)/n)\).

**Solution:** By Chebyshev’s inequality,

\[
\Pr[X = 0] \leq \Pr[|X - E[X]| \geq E[X]] \leq \frac{\text{Var}[X]}{E[X]^2} = \frac{1}{E[X]} + \frac{np - 1}{n(1 - p)}
\]

Now for \(p = o(\frac{\ln(n)}{n})\) we know from part (b) that \(E[X] \rightarrow \infty\), so the first term here goes to zero. And the second term is \(\frac{np - 1}{n(1 - p)} \leq \frac{p}{1 - p}\), which certainly goes to zero for \(p = o(\frac{\ln(n)}{n})\). Hence we have \(\Pr[X = 0] \rightarrow 0\), i.e. \(\Pr[X > 0] \rightarrow 1\) as required.