CS 174: Combinatorics and Discrete Probability Fall 2012

Homework 9 Solutions

Problem 1. *(Exercise 7.12 from MU – 6 points)* Let X_n be the sum of n independent rolls of a fair die. Show that, for any $k \geq 2$,

$$\lim_{n\to\infty} \Pr[X_n \text{ is divisible by } k] = \frac{1}{k}$$

Solution: Let $Y_t = X_t \pmod{k}$, meaning that Y_t is the remainder of X_t divided by k. Then $Y_0, Y_1, ...$ is a Markov chain with k states, and $Y_0 = 0$. The transition probabilities are

$$P_{i,j} = \frac{1}{6}\sum_{a=1}^{6} \mathbb{I}[i + a = j \pmod{k}].$$

X_t is divisible by k if and only if $Y_t = 0$. And we know that $\Pr[Y_t = 0] = P^t_{0,0}$. This is a finite Markov chain. This is clearly irreducible, since you can reach from state i to state j by rolling $j - i \pmod{k}$ 1s on the die. It is also aperiodic because from any state i, there is a path to i of length k (k rolls of 1) and $k - 1$ ($k - 2$ rolls of 1, and 1 roll of 2). Thus, the chain is ergodic and hence has a unique stationary distribution. It is easy to check that the uniform distribution is indeed a stationary distribution. Therefore,

$$\lim_{n\to\infty} \Pr[X_n \text{ is divisible by } k] = \frac{1}{k}$$

Problem 2. *(Exercise 7.13 from MU – 6 points)* Consider a finite Markov chain on n states with stationary distribution $\bar{\pi}$ and transition probabilities $P_{i,j}$. Imagine starting the chain at time 0 and running it for m steps, obtaining the sequence of states $X_0, X_1, \ldots, X_m$. Here X_0 is chosen according to distribution $\bar{\pi}$. Consider the states in reverse order, $X_m, X_{m-1}, \ldots, X_0$.

(a) Argue that given X_{k+1}, the state X_k is independent of $X_{k+2}, X_{k+3}, \ldots, X_m$. Thus, the reverse sequence is Markovian.

Solution: We begin by simply writing out the definition of conditional expectation:

$$\begin{aligned}
\Pr[X_k|X_{k+1}, ..., X_m] &= \frac{\Pr[X_k, X_{k+1}, ..., X_m]}{\Pr[X_{k+1}, ..., X_m]} \\
&= \frac{\Pr[X_k]\Pr[X_{k+1}|X_k]\Pr[X_{k+2}, ..., X_m|X_k, X_{k+1}]}{\Pr[X_{k+1}]\Pr[X_{k+2}, ..., X_m|X_{k+1}]} \\
&= \frac{\Pr[X_k]\Pr[X_{k+1}|X_k]\Pr[X_{k+2}, ..., X_m|X_{k+1}]}{\Pr[X_{k+1}]\Pr[X_{k+2}, ..., X_m|X_{k+1}]} \\
&= \frac{\Pr[X_k]\Pr[X_{k+1}|X_k]}{\Pr[X_{k+1}]}
\end{aligned}$$

Since this is a function only of X_k and X_{k+1}, we have the desired Markovian dependency on only the previous state.

(b) Argue that for the reverse sequence, the transition probabilities, $Q_{i,j}$, are given by

$$Q_{i,j} = \frac{\pi_j P_{j,i}}{\pi_i}.$$

Solution: Using the result for part (a), we substitute the stationary distribution in for the marginals $\Pr[X_k = j] = \pi_j$ and $\Pr[X_{k+1} = j] = \pi_j$:

$$\Pr[X_k = j | X_{k+1} = i] = \frac{\pi_j \Pr[X_{k+1} = j | X_k = i]}{\pi_i} = \frac{\pi_j P_{j,i}}{\pi_i}.$$

(c) Prove that if the original Markov chain is time reversible, so that $\pi_i P_{i,j} = \pi_j P_{j,i}$, then $Q_{i,j} = P_{i,j}$. That is, the states follow the same transition probabilities whether viewed in forward or reverse order.

Solution: This follows directly from part (b), where we obtain

$$\pi_i Q_{i,j} = \pi_j P_{j,i},$$

which can only be true if $Q_{i,j} = P_{i,j}$.

Problem 3. *(Exercise 7.20 from MU – 6 points)* We have considered the gambler's ruin problem in the case where the game is fair. Consider the case where the game is not fair; instead, the probability of losing a dollar each game is $2/3$ and the probability of winning a dollar each game is $1/3$. Suppose you start with i dollars and finish either when you reach n or lose it all. Let W_t be the amount you have gained after t rounds of play.

(a) Show that $\mathbb{E}[2^{W_{t+1}}] = \mathbb{E}[2^{W_t}]$.

Solution: We need to consider two cases: (i) When $W_t = 0$ or $W_t = n$. In this case, $W_{t+1} = W_t$, and hence, $\mathbb{E}[2^{W_{t+1}}] = 2^{W_t}$. (ii) When $0 < W_t < n$, we have the following.

$$\begin{aligned}
\mathbb{E}[2^{W_{t+1}} | W_t] &= \frac{2}{3} 2^{W_t - 1} + \frac{1}{3} 2^{W_t + 1} \\
&= \frac{1}{3} 2^{W_t} + \frac{2}{3} 2^{W_t} \\
&= 2^{W_t}
\end{aligned}$$

Thus, in either case we have

$$\Rightarrow \mathbb{E}[2^{W_{t+1}}] = \mathbb{E}[\mathbb{E}[2^{W_{t+1}} | W_t]] = \mathbb{E}[2^{W_t}]$$

(b) Use part (a) to determine the probability of finishing with 0 dollars and the probability of finishing with n dollars when starting at position i.

Solution: Let p be the probability of finishing with n dollars and $1 - p$ the probability of finishing with 0. Let T be the stopping time (i.e. the first time we reach 0 or n dollars). By the strong Markov property,

$$\begin{aligned}\mathbb{E}[2^{W_T}] &= E[2^{W_0}] \\ \Rightarrow p2^n + (1-p)2^0 &= 2^i \\ \Rightarrow p(2^n - 1) &= 2^i - 1 \\ \Rightarrow p &= \frac{2^i - 1}{2^n - 1}\end{aligned}$$

Problem 4. *(Exercise 7.22 from MU – 6 points)* A cat and a mouse take a random walk on a connected, undirected, non-bipartite graph G. They start at the same time on different nodes, and each makes one transition at each time step. The cat eats the mouse if they are ever at the same node at some time step. Let n and m denote, respectively, the number of vertices and edges of G. Show an upper bound of $O(m^2 n)$ on the expected time before the cat eats the mouse. (*Hint*: Consider a Markov chain whose states are the ordered pair (a, b), where a is the position of the cat and b is a position of the mouse.)

Solution: Following the hint, we formulate a new Markov chain with n^2 states of the form $(i, j) \in [1, n]^2$. Each node (i, j) in the new chain is connected to $N(i)N(j)$ neighbors, where $N(i)$ denotes the number of neighbors of state i in the old Markov chain. Hence the number of edges in the new chain comes to

$$2|E| = \sum_i \sum_j N((i,j)) = \sum_i \sum_j N(i)N(j) = \left(\sum_i N(i)\right)\left(\sum_j N(j)\right) = 4m^2$$

By Lemma 16, if an edge exists between nodes $u = (i_1, j_1)$ and $v = (i_2, j_2)$, then $h_{u,v} \leq 2|E| = 4m^2$.

In order to obtain the $O(m^2 n)$ upper bound, we need to show that for any node (i, j), there exists a path of length $O(n)$ connecting it to some node of the form (v, v). In fact, we show that there exists a length $O(n)$ path between (i, j) and (i, i). Since the graph is undirected, the cat can always go back to node i in two steps. At the same time, because the graph is connected, theres a path of length $k < n$ from j to i. If k is even, then the mouse will run into the cat. If k is odd, then the mouse will get to node i when the cat is away. But since the chain is non-bipartite, there must be a path of odd length from i back to itself; let the mouse follow this path, and it will run into the cat on the next return to i. Thus the total length of this path from (i, j) to (i, i) is at most $3n$. Each edge on this path requires at most $4m^2$ steps, thus the desired upper bound on the time to collision is $O(m^2 n)$ steps.

Problem 5. *(Exercise 7.24 from MU – 6 points)* The *lollipop* graph on n vertices is a clique on $n/2$ vertices connected to a path on $n/2$ vertices. (See Figure 7.3 on pg. 186 of the text book.) The node u is a part of both the clique and the path. Let v denote the other end of the path.

(a) Show that the expected covering time of a random walk starting at v is $\Theta(n^2)$.

Solution: We need the expected time it takes to travel the stick part of the lollipop from v to u ($h_{v,u}$), and the expected cover time of the clique part of the lollipop starting from node u (c_u). Say there are k nodes in the stick part (excluding u), and k nodes in the ball part of the graph (including u), so that the total number of nodes is $n = 2k$. $h_{v,u}$ is just the time it takes to reach the k^{th} node on a chain starting from 0, i.e., h_0 in the chain for 2-SAT. So

$$h_{v,u} = k^2.$$

c_u, on the other hand, is upper bounded by the expected time it takes to travel to each of the nodes in the clique and return to u.

$$c_u \leq \sum_{w \in \text{clique}} h_{u,w} + h_{w,u}.$$

Let w and x denote nodes in the clique other than u, and let $i \in \{1, 2, \ldots, k\}$ denote the nodes on the stick, with 1 being u's neighbor and k being synonymous with v. We can write $h_{u,w}$ in terms of the following system of equations:

$$\begin{aligned}
h_{u,w} &= \frac{1}{k} \cdot 0 + \frac{k-2}{k} h_{x,w} + \frac{1}{k} h_{1,w} + 1 \\
h_{x,w} &= \frac{1}{k} \cdot 0 + \frac{k-3}{k-1} h_{x,w} + \frac{1}{k-1} h_{u,w} + 1 \\
h_{1,w} &= \frac{1}{2} h_{u,w} + \frac{1}{2} h_{2,w} + 1 \\
h_{2,w} &= \frac{1}{2} h_{1,w} + \frac{1}{2} h_{3,w} + 1 \\
&\cdots \\
h_{k-1,w} &= \frac{1}{2} h_{k-2,w} + \frac{1}{2} h_{k,w} + 1 \\
h_{k,w} &= \frac{1}{2} h_{k-1,w} + 1
\end{aligned}$$

We obtain $h_{k-i,w} = h_{k-i-1,w} + (2i+1)$, and hence $h_{1,w} = h_{u,w} + 2k - 1$. Solving the equations, we get

$$h_{u,w} = \frac{k^2 + 9k - 2}{2k}$$

To calculate $\sum h_{w,u}$, use the same proof technique as in Lemma 16.

$$\frac{2|E|}{d(u)} = h_{u,u} = \frac{1}{d(u)} \sum_{w \in N(u)} (1 + h_{w,u}),$$

hence

$$\sum_{w \in N(u)} h_{w,u} = 2|E| - k = 2(k(k-1) + k) - k = 2k^2 - k.$$

Therefore, we have

$$c_u \le \sum_{w \in \text{clique}} h_{u,w} + h_{w,u} \le (k-1)\frac{k^2+9k-2}{2k} + 2k^2 - k.$$

Combining everything, we get

$$k^2 = h_{v,u} \le \text{Cover time starting from } v \le h_{v,u} + c_u = O(k^2)$$

Hence the cover time starting from v is $\Theta(k^2) = \Theta(n^2)$.

(b) Show that the expected covering time for a random walk starting at u is $\Theta(n^3)$.

Solution: We've shown that it takes time $\Theta(k^2)$ to cover the clique part of the graph starting from u. We now show that $h_{u,v} = \Theta(k^3)$, which gives us the required result. We can write down the following system of equations, using the same node naming convention as before:

$$\begin{aligned} h_{u,v} &= \frac{k-1}{k}h_{w,v} + \frac{1}{k}h_{1,v} + 1 \\ h_{w,v} &= \frac{k-2}{k-1}h_{w,v} + \frac{1}{k-1}h_{u,v} + 1 \\ h_{i,v} &= \frac{1}{2}h_{i-1,v} + \frac{1}{2}h_{i+1,v} + 1 \end{aligned}$$

We derive that $h_{w,v} = h_{u,v} + k - 1$ and $h_{k-i,v} = \frac{i}{i+1}h_{k-i-1,v} + i$, and hence $h_{1,v} = k - 1kh_{u,v} + (k-1)$. From this we get $h_{u,v} = k^3$.