On the Decidability of the Continuous Infinite Zeros Problem

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Abstract. We study the Continuous Infinite Zeros Problem, which asks whether a real-valued function f satisfying a given ordinary linear differential equation has infinitely many zeros on $\mathbb{R}_{\geq 0}$. We consider also the closely related Unbounded Continuous Skolem Problem, which asks whether f has a zero in a given unbounded subinterval of $\mathbb{R}_{\geq 0}$. These are fundamental reachability problems arising in the analysis of continuous linear dynamical systems, including linear hybrid automata and continuous-time Markov chains.

Our main decidability result is that if the ordinary differential equation satisfied by f is of order at most 7 or if the imaginary parts of its characteristic roots are all rational multiples of one another, then the Infinite Zeros Problem is decidable, and moreover, if f has only finitely many zeros, then an upper bound T may be found such that $f(t) = 0$ entails $t \leq T$. On the other hand, our main hardness results is that if the Infinite Zeros Problem is decidable for ordinary differential equations of order at least 9, then this would entail a major breakthrough in Diophantine Approximation, specifically, the computability of the Lagrange constant $L_\infty(x)$ for all real algebraic x.

1 Introduction

The Continuous Skolem Problem is a fundamental decision problem concerning reachability in continuous-time linear dynamical systems. The problem asks whether a real-valued function satisfying an ordinary linear differential equation has a zero in a given interval of real numbers. More precisely, an instance of the problem comprises an interval $I \subseteq \mathbb{R}_{\geq 0}$ with rational endpoints and an ordinary differential equation (ODE)

$$f^{(n)} + a_{n-1} f^{(n-1)} + \ldots + a_0 f = 0 \tag{1}$$

with the coefficients $a_0, \ldots, a_{n-1}$ and initial conditions $f(0), \ldots, f^{(n-1)}(0)$ being real algebraic numbers. Writing $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ for the unique solution of the differential equation that satisfies the initial conditions, the question is whether there exists $t \in I$ such that $f(t) = 0$. It is natural to partition this problem into two sub-problems based on the boundedness of I: the Bounded Continuous Skolem Problem and the Unbounded Continuous Skolem Problem, respectively. The nomenclature *Continuous Skolem Problem* is based on viewing the problem as a continuous analog of the Skolem Problem for linear recurrence sequences, which asks whether a given linear recurrence sequence has a zero term [10]. Whether the latter problem is decidable is an outstanding question in number theory and theoretical computer science; see, e.g., the exposition of Tao [20, Section 3.9].

The *characteristic polynomial* of the linear differential equation (1) is

$$\chi(x) := x^n + a_{n-1} x^{n-1} + \ldots + a_0 \,.$$

Let $\lambda_1, \ldots, \lambda_m$ be the distinct roots of χ. Any solution of (1) has the form $f(t) = \sum_{i=1}^{m} P_i(t) e^{\lambda_i t}$, where the P_i are polynomials with algebraic coefficients that are determined by (and computable from) the initial conditions of the differential equation, see [2]. We call a function f in this form an *exponential polynomial*. The Continuous Skolem Problem can equivalently be formulated in terms of whether an exponential polynomial has a zero in a given interval of non-negative reals.

Another natural question is whether a given exponential polynomial has *infinitely many* zeros on an interval $I \subseteq \mathbb{R}_{\geq 0}$. Since the analyticity of exponential polynomials prohibits them from having infinitely many zeros on any bounded interval, we may always take the interval of interest to be $[0, \infty)$. We call this the Continuous Infinite Zeros Problem: given an ODE (1) with real algebraic coefficients $a_0, \ldots, a_{n-1}$ and real algebraic initial conditions $f(0), \ldots, f^{(n-1)}(0)$, determine whether its unique solution has infinitely many zeros on $\mathbb{R}_{\geq 0}$. This, too, can be seen as a continuous analog of a problem on linear recurrence sequences, which we call the Discrete Infinite Zeros Problem: given a linear recurrence sequence, determine whether it has infinitely many zero terms. The decidability of the Discrete Infinite Zeros Problem was established in [3].

In reference [4], we showed decidability of the Bounded Continuous Skolem Problem subject to Schanuel's Conjecture, a unifying conjecture in transcendental number theory, generalising both the Lindemann-Weierstrass Theorem and Baker's Theorem on linear independence of logarithms of algebraic numbers. In the unbounded case, by way of hardness, we showed that decidability of the Continuous Skolem Problem for ODEs of order 9 would entail major new effectiveness results in Diophantine approximation.

In the present paper, we study the Continuous Infinite Zeros Problem and establish both decidability and hardness results. On the decidability front, we will restrict our attention to exponential polynomials f such that the order of f is at most 7 or the span of $\{\Im(\lambda_j) : j = 1, \ldots, m\}$ is a one-dimensional $\mathbb{Q}$-vector space. We show that for f satisfying this assumption, the Continuous Infinite Zeros Problem is decidable, and moreover, if f has only finitely many zeros, then there exists an effective upper bound T such that all zeros of f are contained in $[0, T]$. In the context of the Continuous Skolem Problem, this unconditional reduction from the unbounded case to the bounded case, together with our conditional decidability result in [4], immediately yields decidability for the Unbounded Continuous Skolem Problem of order at most 7, subject to Schanuel's Conjecture. Finally, with regards to hardness, we exhibit a reduction to show that decidability of the Continuous Infinite Zeros Problem for instances of order at least 9 would entail major advancements in the field of Diophantine Approximation, analogously to our hardness result for the Unbounded Continuous Skolem Problem in [4].

2 Mathematical Background

2.1 General Form of a Solution

We recall some facts about the general form of solutions of ordinary linear differential equations. Consider a homogeneous linear differential equation

$$f^{(n)} + c_{n-1}f^{(n-1)} + \ldots + c_0 f = 0 \tag{2}$$

of order n with characteristic polynomial χ. If λ is a root of χ of multiplicity m, then the function $f(t) = t^j e^{\lambda t}$ satisfies (2) for $j = 0, 1, \ldots, m-1$. There are n distinct linearly independent solutions of (2) having this form, and these span the space of all solutions.

Let the distinct roots of χ be $\lambda_1, \ldots, \lambda_k$, with respective multiplicities $m_1, \ldots, m_k$. Write $\lambda_j = r_j + ia_j$ for real algebraic numbers r_j, a_j, $j = 1, \ldots, k$. It follows from the discussion above that, given real algebraic initial values of $f(0), f'(0), \ldots, f^{(n-1)}(0)$, the uniquely defined solution f of (2) can be written in one of the following three equivalent forms.

1. As an *exponential polynomial*
$$f(t) = \sum_{j=1}^{k} P_j(t) e^{\lambda_j t}$$
where each P_j is a polynomial with (complex) algebraic coefficients and degree at most $m_j - 1$.
2. As a function of the form
$$f(t) = \sum_{j=1}^{k} e^{r_j t}(P_j(t)\cos(a_j t) + Q_j(t)\sin(a_j t))$$
where the polynomials P_j, Q_j have real algebraic coefficients and degrees at most $m_j - 1$.
3. As a function of the form
$$f(t) = \sum_{j=1}^{k} e^{r_j t} \sum_{l=0}^{m_l - 1} b_{j,l} t^l \cos(a_j t + \varphi_{j,l})$$
where $b_{j,l}$ is real algebraic and $e^{i\varphi_{j,l}}$ algebraic for each j, l.

We refer the reader to [2, Theorem 7] for details.

2.2 Number-theoretic tools

Recall that a standard way to represent an algebraic number α is by its minimal polynomial M and a numerical approximation of sufficient accuracy to distinguish α from the other roots of M [5, Section 4.2.1]. Given two algebraic numbers α and β under this representation, the *Field Membership Problem* is to determine whether $\beta \in \mathbb{Q}(\alpha)$ and, if so, to return a polynomial P with rational coefficients such that $\beta = P(\alpha)$. This problem can be decided using the LLL algorithm, see [5, Section 4.5.4].

Given the characteristic polynomial χ of a linear differential equation we can compute approximations to each of its roots $\lambda_1, \ldots, \lambda_n$ to within an arbitrarily small additive error [16]. Moreover, by repeatedly using an algorithm for the Field Membership Problem we can compute a primitive element θ for the splitting field of χ and representations of $\lambda_1, \ldots, \lambda_n$ as polynomials in θ. Thereby we can determine maximal $\mathbb{Q}$-linearly independent subsets of $\{\Re(\lambda_j) : 1 \le j \le n\}$ and $\{\Im(\lambda_j) : 1 \le j \le n\}$.

We now move to some techniques from Transcendental Number Theory on which our results depend in a critical way. The following theorem was originally proven in 1934 by A. Gelfond [6,7] and independently by T. Schneider [18,19], settling Hilbert's seventh problem in the affirmative.

Theorem 1. *(Gelfond-Schneider) If a and b are algebraic numbers with $a \neq 0, 1$ and $b \notin \mathbb{Q}$, then a^b is transcendental.*

Next we state a powerful result due to Baker on linear forms of logarithms of algebraic numbers.

Theorem 2. *(Baker [1, Theorem 3.1]) Let $\alpha_1, \ldots, \alpha_m$ be non-zero algebraic numbers with degrees at most d and heights at most H. Further, let $\beta_0, \ldots, \beta_m$ be algebraic numbers with degrees at most d and heights at most $B \geq 2$. Write*

$$\Lambda = \beta_0 + \beta_1 \log(\alpha_1) + \ldots + \beta_m \log(\alpha_m)\,.$$

Then either $\Lambda = 0$ or $|\Lambda| > B^{-C}$, where C is an effectively computable number depending only on m, d, H and the chosen branch of the complex logarithm.

The following lemma, proven in [2], is a useful consequence of Baker's Theorem:

Lemma 3. *([2, Lemma 13]) Let $a, b \in \mathbb{R} \cap \mathbb{A}$ be linearly independent over $\mathbb{Q}$ and let φ_1, φ_2 be logarithms of algebraic numbers, that is, $e^{i\varphi_1}, e^{i\varphi_2} \in \mathbb{A}$. There exist effective constants $C, N, T > 0$ such that for all $t \geq T$, at least one of $1 - \cos(at + \varphi_1) > C/t^N$ and $1 - \cos(bt + \varphi_2) > C/t^N$ holds.*

Another necessary tool is a version of Kronecker's well-known Theorem in Diophantine Approximation.

Theorem 4. *(Kronecker, appears in [11]) Let $\lambda_1, \ldots, \lambda_m$ and $x_1, \ldots, x_m$ be real numbers. Suppose that for all integers $u_1, \ldots, u_m$ such that $u_1\lambda_1 + \cdots + u_m\lambda_m \in \mathbb{Z}$, we also have $u_1 x_1 + \cdots + u_m \lambda_m \in \mathbb{Z}$, that is, all integer relations among the λ_j also hold among the x_j (modulo $\mathbb{Z}$). Then for all $\epsilon > 0$, there exist $p \in \mathbb{Z}^m$ and $n \in \mathbb{N}$ such that $|n\lambda_j - x_j - p_j| < \epsilon$ for all $1 \leq j \leq m$. In particular, if $1, \lambda_1, \ldots, \lambda_m$ are linearly independent over $\mathbb{Z}$, then there exist such $n \in \mathbb{N}$ and $p \in \mathbb{Z}^m$ for all $x \in \mathbb{R}^m$ and $\epsilon > 0$.*

A direct consequence is the following:

Lemma 5. *Let $a_1, \ldots, a_m \in \mathbb{R} \cap \mathbb{A}$ be linearly independent over $\mathbb{Q}$ and let $\varphi_1, \ldots, \varphi_m \in \mathbb{R}$. Write $x \bmod 2\pi$ to denote $\min_{k \in \mathbb{Z}} |x + 2k\pi|$ for any $x \in \mathbb{R}$. Then the image of the mapping $h(t) : \mathbb{R}_{\geq 0} \to [0, 2\pi)^m$ given by*

$$h(t) = ((a_1 t + \varphi_1) \bmod 2\pi, \ldots, (a_m t + \varphi_m) \bmod 2\pi)$$

is dense in $[0, 2\pi)^m$. Moreover, the set

$$\{h(t) \,|\, (a_1 t + \varphi_1) \bmod 2\pi = 0\}$$

is dense in $\{0\} \times [0, 2\pi)^{m-1}$.

Proof. For the first part of the claim, note that the linear independence of $1, a_1/2\pi, \ldots, a_m/2\pi$ follows from the linear independence of $a_1, \ldots, a_m$ and the transcendence of π. Then by Kronecker's Theorem, the restriction $\{h(t) \,|\, t \in \mathbb{N}\}$ is dense in $[0, 2\pi)^m$, so certainly the whole image of $h(t)$ must also be dense in $[0, 2\pi)^m$. For the second part, the trajectory $h(t)$ has zero first coordinate precisely when $t = -\varphi_1/a_1 + 2n\pi$ for some $n \in \mathbb{Z}$, at which times the trajectory is

$$g(n) \stackrel{\text{def}}{=} h\left(\frac{-\varphi_1}{a_1} + 2n\pi\right) = \{0\} \times \left(n\frac{2\pi a_j}{a_1} + \frac{a_1\varphi_j - \varphi_1 a_j}{a_1} \bmod 2\pi\right)_{2 \leq j \leq m}$$

As before, we have that $\{1, 2\pi a_2/a_1, \ldots, 2\pi a_m/a_1\}$ are linearly independent over $\mathbb{Q}$ from the linear independence of $a_1, \ldots, a_m$ and the transcendence of π, so applying Kronecker's Theorem to the last $m - 1$ components of this discrete trajectory yields the second part of the claim. □

2.3 First-Order Theory of the Reals

We denote by $\mathcal{L}$ the first-order language $\mathbb{R}\langle +, \times, 0, 1, <, = \rangle$. Atomic formulas in this language are of the form $P(x_1, \ldots, x_n) = 0$ and $P(x_1, \ldots, x_n) > 0$ for $P \in \mathbb{Z}[x_1, \ldots, x_n]$ a polynomial with integer coefficients. A set $X \subseteq \mathbb{R}^n$ is *definable* in $\mathcal{L}$ if there exists some $\mathcal{L}$-formula $\phi(\bar{x})$ with free variables $\bar{x}$ which holds precisely for valuations in X. Analogously, a function is definable if its graph is a definable set.

We denote by $Th(\mathbb{R})$ *the first-order theory of the reals*, that is, the set of all valid sentences in the language $\mathcal{L}$. It is worth remarking that any real algebraic number is readily definable within $\mathcal{L}$ using its minimal polynomial and a rational approximation to distinguish it from the other roots. Thus, we can treat real algebraic numbers constants as built into the language and use them freely in the construction of formulas. A celebrated result due to Tarski [21] is that the first-order theory of the reals admits quantifier elimination: that each formula $\phi_1(\bar{x})$ in $\mathcal{L}$ is equivalent to some effectively computable formula $\phi_2(\bar{x})$ which uses no quantifiers. This immediately entails the decidability of $Th(\mathbb{R})$. It also follows that sets definable in $\mathcal{L}$ are precisely the semialgebraic sets. Tarski's original result had non-elementary complexity, but improvements followed, culminating in the detailed analysis of Renegar [17].

Decidability and geometrical properties of definable sets in the first-order theory of the structure $\mathcal{L}_{exp} = \mathbb{R}\langle +, \times, 0, 1, <, =, \exp\rangle$, the reals with exponentiation, have been explored by a number of authors. Most notably, Wilkie [22] showed that the theory is *o-minimal* and Macintyre and Wilkie [13] showed that if Schanuel's conjecture is true then the theory is decidable. We will not need the above two results in this paper, however we use the following, which is very straightforward to establish directly.

Proposition 6. *There is a procedure that, given a semi-algebraic set $S \subseteq \mathbb{R}^k$ and real algebraic numbers $a_1, \ldots, a_k$, returns an integer T such that $\{t \geq 0 : (e^{a_1 t}, \ldots, e^{a_k t}) \in S\}$ either contains the interval (T, ∞) or is disjoint from (T, ∞). The procedure also decides which of these two eventualities is the case.*

Proof. Consider a polynomial $P \in \mathbb{Z}[u_1, \ldots, u_k]$. For suitably large t the sign of $P(e^{a_1 t}, \ldots, a^{a_k t})$ is identical to the sign of the coefficient of the dominant term in the expansion of $P(e^{a_1 t}, \ldots, a^{a_k t})$ as an exponential polynomial. It follows that the sign of $P(e^{a_1 t}, \ldots, a^{a_k t})$ is eventually constant. It is moreover clear that one can effectively compute a threshold beyond which the sign $P(e^{a_1 t}, \ldots, a^{a_k t})$ remains the same. Since the set S is a defined by a Boolean combination of inequalities $P(u_1, \ldots, u_k) \sim 0$, for $\sim \in \{<, =\}$, the proposition immediately follows.

2.4 Useful Results About Exponential Polynomials

We restate two useful theorems due to Bell et al. [2].

Theorem 7. *([2, Theorem 12]) Exponential polynomials $f(t)$ with no real dominant characteristic roots have infinitely many zeros.*

Theorem 8. *([2, Theorem 15]) Suppose we are given an exponential polynomial whose dominant characteristic roots are simple, at least four in number and have imaginary parts linearly independent over $\mathbb{Q}$. Then the existence of infinitely many zeros is decidable. Moreover, if there are finitely many zeros, there exists an effective threshold T such that all zeros are in $[0, T]$.*

3 One Linearly Independent Oscillation

In this section we consider exponential polynomials $f(t) = \sum_{j=1}^{k} P_j(t) e^{\lambda_j t}$ under the assumption that the span of $\{\Im(\lambda_j) : j = 1, \ldots, k\}$ is a one-dimensional $\mathbb{Q}$-vector space. In this case we can use fundamental geometric properties of semi-algebraic sets to decide whether or not f has finitely many zeros and, if so, to compute an interval $[0, T]$ that contains all zeros of f.

Theorem 9. *Consider an exponential polynomial $f(t) = \sum_{j=1}^{k} P_j(t) e^{\lambda_j t}$, where the span of $\{\Im(\lambda_j) : j = 1, \ldots, k\}$ is a one-dimensional $\mathbb{Q}$-vector space. Then the existence of infinitely many zeros of f is decidable and, if there are only finitely many zeros, then there exists a computable bound T such that all zeros of f lie in the interval $[0, T]$.*

Proof. Write $\lambda_j = a_j + i b_j$, where a_j, b_j are real algebraic numbers for $j = 1, \ldots, k$. By assumption there is a single real algebraic number b such that each b_j is an integer multiple of b. Recall that for each integer n, both $\cos(nbt)$ and $\sin(nbt)$ can be written as polynomials in $\sin(bt)$ and $\cos(bt)$ with integer coefficients. Using this fact we can write f in the form

$$f(t) = Q(t, e^{a_1 t}, \ldots, e^{a_k t}, \cos(bt), \sin(bt)),$$

for some multivariate polynomial Q with algebraic coefficients.

Now consider the semi-algebraic set

$$E := \left\{ (\boldsymbol{u}, s) \in \mathbb{R}^{k+2} : Q\left(u_0, \ldots, u_k, \tfrac{1-s^2}{1+s^2}, \tfrac{2s}{1+s^2}\right) = 0 \right\} .$$

Recall that $\left\{ \left(\frac{1-s^2}{1+s^2}, \frac{2s}{1+s^2} \right) : s \in \mathbb{R} \right\}$ comprises all points in the unit circle in $\mathbb{R}^2$ except $(-1, 0)$. Indeed, given $\theta \in (-\pi, \pi)$, setting $s := \tan(\theta/2)$ we have $\cos(\theta) = \frac{1-s^2}{1+s^2}$ and $\sin(\theta) = \frac{2s}{1+s^2}$. It follows that $f(t) = 0$ and $\cos(bt) \neq -1$ imply that $(t, e^{a_1 t}, \ldots, e^{a_k t}, \tan(bt/2)) \in E$.

By the Cell Decomposition Theorem for semi-algebraic sets [14], there are semi-algebraic sets $C_1, \ldots, C_m \subseteq \mathbb{R}^{k+2}$, $D_1, \ldots, D_m \subseteq \mathbb{R}^{k+1}$, and continuous semi-algebraic functions $\xi_j, \xi_j^{(1)}, \xi_j^{(2)} : D_j \to \mathbb{R}$ such that E can be written as a disjoint union $E = C_1 \cup \ldots \cup C_m$, where either

$$C_j = \{ (\boldsymbol{u}, s) \in \mathbb{R}^{k+2} : \boldsymbol{u} \in D_j \wedge s = \xi_j(\boldsymbol{u}) \} \tag{3}$$

or

$$C_j = \{ (\boldsymbol{u}, s) \in \mathbb{R}^{k+2} : \boldsymbol{u} \in D_j \wedge \xi_j^{(1)}(\boldsymbol{u}) < s < \xi_j^{(2)}(\boldsymbol{u}) \} \tag{4}$$

Moreover such a decomposition is computable from E. Clearly then

$$\{ t \in \mathbb{R} : f(t) = 0 \} \subseteq \bigcup_{j=1}^{m} \{ t \in \mathbb{R} : (t, e^{a_1 t}, \ldots, e^{a_k t}) \in D_j \} \cup Z ,$$

where $Z := \{ t \in \mathbb{R} : \cos(bt) = -1 \}$.

The restriction of f to Z is given by $f(t) = Q(t, e^{a_1 t}, \ldots, e^{a_k t}, -1, 0)$. Since this expression is a linear combination of terms of the form $t^j e^{rt}$ for real algebraic r, for sufficiently large t the sign of $f(t)$ is determined by the sign of the coefficient of the dominant term. Thus f is either identically zero on Z (in which case f has infinitely many zeros) or we can compute a threshold T such that all zeros of f in Z lie in the interval $[0, T]$.

We now consider zeros of f that do not lie in Z. There are two cases. First suppose that each set $\{ t \in \mathbb{R} : (t, e^{a_1 t}, \ldots, e^{a_k t}) \in D_j \}$ is bounded for $j = 1, \ldots, m$. In this situation, using Proposition 6, we can compute an upper bound T such that if $f(t) = 0$ then $t < T$. On the other hand, if some set $\{ t \in \mathbb{R}_{\geq 0} : (t, e^{a_1 t}, \ldots, e^{a_k t}) \in D_j \}$ is unbounded then, by Proposition 6, it contains an infinite interval (T, ∞). We claim that in this case f must have infinitely many zeros $t \geq 0$. We give the argument in the case C_j satisfies (3). The argument in case C_j satisifes (4) is similar.

Define $\eta_j(t) = \xi_j(t, e^{a_1 t}, \ldots, e^{a_k t})$ for $t \in (T, \infty)$. Then for $t \notin (T, \infty) \setminus Z$,

$$\begin{aligned} f(t) = 0 \iff & (t, e^{a_1 t}, \ldots, e^{a_k t}, \tan(bt/2)) \in C_j \\ \iff & (t, e^{a_1 t}, \ldots, e^{a_k t}) \in D_j \text{ and } \eta_j(t) = \tan(bt/2) . \end{aligned}$$

In other words, f has a zero at each point $t \in (T, \infty) \setminus Z$ at which the graph of η_j intersects the graph of $\tan(bt/2)$. Since η_j is continuous there are clearly infinitely many such intersection points, see Figure 1. This completes the proof. $\square$

4 Decidability Results up to Order 7

We now shift our attention to instances of the Infinite Zeros Problem of low order. Given an exponential polynomial $f(t)$, we will once again be interested in two questions: does f have infinitely many zeros, and if not, can we derive a bound T such that all zeros of f lie in the interval $[0, T]$? In particular, for exponential polynomials corresponding to differential equations of order at most 7, we settle both questions, establishing decidability of the Infinite Zeros Problem and a reduction from the Unbounded Skolem Problem to the Bounded Skolem Problem. Both of these results are independent of Schanuel's Conjecture. The latter result, combined with our results on the Bounded Skolem Problem in [4], immediately yields decidability, conditional on Schanuel's Conjecture, for the Unbounded Skolem Problem of order up to 7.

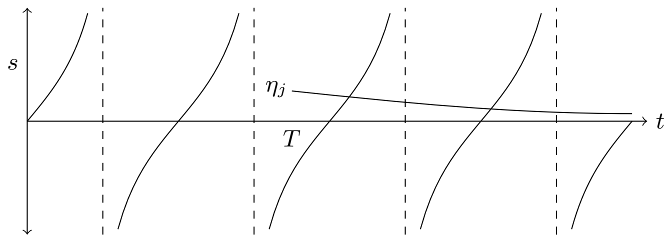

Fig. 1. Intersection points of $\eta_j(t)$ and $\tan(bt/2)$.

Theorem 10. *For differential equations of order at most* 7*, the Unbounded Skolem Problem reduces to the Bounded Skolem Problem, and the Infinite Zeros Problem is decidable.*

Proof. Sort the characteristic roots of the input matrix according to their real parts, and let r_j denote throughout the j-th largest real part of a characteristic root. We will refer to the characteristic roots of maximum real part as the *dominant characteristic roots.* Let also $mul(\lambda)$ denote the multiplicity of λ as a root of the characteristic polynomial of the given ODE.

We will now perform a case analysis on the number of dominant characteristic roots. By Theorem 7, it is sufficient to confine our attention to exponential polynomials with an odd number of dominant characteristic roots. Throughout, we rely on known general forms of solutions to ordinary linear differential equations, outlined in Section 2.1.

Case I. Suppose first that there is only one dominant, necessarily real, root r. Then if we divide $f(t)$ by e^{rt}, we have:

$$\frac{f(t)}{e^{rt}} = P_1(t) + \mathcal{O}\left(e^{(r_2-r)t}\right),$$

as the contribution of the non-dominant roots shrinks exponentially, relative to that of the dominant root. Thus, for large $t \geq 0$, the sign of $f(t)$ matches the sign of the leading coefficient of $P_1(t)$, so $f(t)$ cannot have infinitely many zeros. Further, a bound T on the zeros of $f(t)$ can be found easily from the description of $f(t)$.

Case II. We now move to the case of three dominant characteristic roots: r and $r \pm ia$, so that

$$\frac{f(t)}{e^{rt}} = P_1(t) + P_2(t)\cos(at) + P_3(t)\sin(at) + \mathcal{O}\left(e^{(r_2-r)t}\right),$$

where $P_1, P_2, P_3 \in (\mathbb{R} \cap \mathbb{A})[x]$ have degrees $d_1 \stackrel{\text{def}}{=} \deg(P_1) \leq mul(r) - 1$ and $d_2 \stackrel{\text{def}}{=} \deg(P_2) = \deg(P_3) \leq mul(r \pm ai)$.

Case IIa. Suppose $d_1 > d_2$. Now, it is easy to see that for large t the sign of $f(t)$ matches the sign of the leading coefficient p_1 of $P_1(t)$:

$$\frac{f(t)}{e^{rt}t^{d_1}} = p_1 + \mathcal{O}(1/t) + \mathcal{O}\left(e^{(r_2-r)t}\right),$$

so a bound T follows such that $f(t) = 0 \Rightarrow t \leq T$. Similarly, if $d_2 > d_1$, then $f(t)$ clearly has infinitely many zeros. Indeed, if p_2, p_3 are the leading coefficients of P_2, P_3, respectively, then we have:

$$\begin{aligned}\frac{f(t)}{e^{rt}t^{d_2}} &= p_2\cos(at) + p_3\sin(at) + \mathcal{O}(1/t) + \mathcal{O}\left(e^{(r_2-r)t}\right)\\ &= \frac{\cos(at+\varphi)}{\sqrt{p_2^2+p_3^2}} + \mathcal{O}(1/t) + \mathcal{O}\left(e^{(r_2-r)t}\right)\end{aligned}$$

where $\varphi \in [0, 2\pi)$ with $\tan(\varphi) = -p_3/p_2$, so $f(t)$ is infinitely often positive and infinitely often negative.

Thus, we can now assume $d_1 = d_2$. Notice that since the order of our exponential polynomial is no greater than 7, we must have $d_1 = d_2 \leq 2$.

Case IIb. Suppose that $d_1 = d_2 = 2$. Then our function is of the form

$$\frac{f(t)}{e^{rt}} = t(A\cos(at+\varphi_1)+B) + (C\cos(at+\varphi_2)+D) + e^{(r_2-r)t}F,$$

for constants $A, B, C, D, F, a \in \mathbb{R}\cap\mathbb{A}$ with $a > 0$ and $e^{i\varphi_1}, e^{i\varphi_2} \in \mathbb{A}$. In this case, Theorem 10 follows from Lemma 21 in Appendix D.

Case IIc. Suppose that $d_1 = d_2 = 1$, so that

$$\frac{f(t)}{e^{rt}} = A_1\cos(at+\varphi_1) + A_2 + e^{(r_2-r)t}F_1(t),$$

where $A_1, A_2, a \in \mathbb{R}\cap\mathbb{A}$, $a > 0$, $e^{i\varphi_1} \in \mathbb{A}$ and $F_1(t)$ is an exponential polynomial with dominant characteristic root whose real part is 0. Consider first the magnitudes of A_1 and A_2. If $|A_1| > |A_2|$, then the term $A_1\cos(at+\varphi_1)$ makes $f(t)$ change sign infinitely often, so $f(t)$ must have infinitely many zeros. On the other hand, if $|A_1| < |A_2|$, then $f(t)$ is clearly ultimately positive or ultimately negative, depending on the sign of A_2, with an effective threshold beyond which $f(t) \neq 0$. The remaining case is that $|A_1| = |A_2|$. Dividing $f(t)$ by A_2, replacing φ_1 by $\varphi_1 + \pi$ if needed and scaling constants by A_2 as necessary, we can assume the function has the form:

$$\frac{f(t)}{e^{rt}} = 1 - \cos(at+\varphi_1) + e^{(r_2-r)t}F_1(t).$$

We now enumerate the possibilities for the dominant characteristic roots of the exponential polynomial $F_1(t)$, that is, the characteristic roots of $f(t)$ with second-largest real part. Since $f(t)$ has order at most 7, there are the following cases to consider:

- $F_1(t)$ has four simple, necessarily complex, dominant roots, so that

  $$\frac{f(t)}{e^{rt}} = 1 - \cos(at+\varphi_1) + e^{(r_2-r)t}(B\cos(bt+\varphi_2) + C\cos(ct+\varphi_3)),$$

  where $B, C, b, c \in \mathbb{R}\cap\mathbb{A}$ with $b, c > 0$ and $e^{i\varphi_2}, e^{i\varphi_3} \in \mathbb{A}$. In this case, Theorem 10 follows from Lemma 17 in Appendix A.
- $F_1(t)$ has some subset of one real and two complex numbers as dominant roots, all simple, so that

  $$\frac{f(t)}{e^{rt}} = 1 - \cos(at+\varphi_1) + e^{(r_2-r)t}(B\cos(bt+\varphi_2) + C) + e^{(r_3-r)t}F_2(t),$$

  where $B, C, b \in \mathbb{R}\cap\mathbb{A}$, $b > 0$, $e^{i\varphi_2} \in \mathbb{A}$ and $F_2(t)$ is an exponential polynomial with dominant characteristic root whose real part is 0. In this case, Theorem 10 follows from Lemma 16 in Appendix A.
- $F_1(t)$ has a repeated real and possibly two simple complex dominant roots, so that

  $$\frac{f(t)}{e^{rt}} = 1 - \cos(at+\varphi_1) + e^{(r_2-r)t}(B\cos(bt+\varphi_2) + P(t)) + e^{(r_3-r)t}F_2(t),$$

  where $B, b \in \mathbb{R}\cap\mathbb{A}$, $b > 0$, $e^{i\varphi_2} \in \mathbb{A}$, and $P(t) \in (\mathbb{R}\cap\mathbb{A})[x]$ is non-constant. Now, if the leading coefficient of $P(t)$ is negative, then $f(t)$ will be infinitely often negative (consider large times t such that $\cos(at+\varphi_1) = 1$) and infinitely often positive (consider large times t such that $\cos(at+\varphi_1) = 0$), so $f(t)$ must have infinitely many zeros. On the other hand, if the leading coefficient of $P(t)$ is positive, then it is easy to see that $f(t)$ is ultimately positive, with an effective threshold.
- $F_1(t)$ has a repeated pair of complex roots, so that

  $$\frac{f(t)}{e^{rt}} = 1 - \cos(at+\varphi_1) + e^{(r_2-r)t}(Bt\cos(bt+\varphi_2) + C\cos(bt+\varphi_3)),$$

  where $B, C, b \in \mathbb{R}\cap\mathbb{A}$, $b > 0$ and $e^{i\varphi_2}, e^{i\varphi_3} \in \mathbb{A}$. In this case, Theorem 10 follows from Lemma 18 in Appendix A.

Case III. We now consider the case of five dominant characteristic roots. Let these be r, $r \pm ai$ and $r \pm bi$. If $r \pm ai$ are repeated, i.e., $mul(r \pm ai) \geq 2$, then we must have $mul(r) = mul(r \pm bi) = 1$, since otherwise the order of our exponential polynomial exceeds 7. Then by an argument analogous to *Case IIa* above, $f(t)$ must have infinitely many zeros. The situation is symmetric when $mul(r \pm bi) \geq 2$. Similarly, if $mul(r) \geq 2$, then $mul(r \pm ai) = mul(r \pm bi) = 1$, since otherwise the instance exceeds order 7. Then by the same argument as in *Case IIa*, $f(t)$ is ultimately positive or ultimately negative, with an effectively computable threshold T. Thus, we may assume that all the dominant roots are simple, so the exponential polynomial is of the form:

$$\frac{f(t)}{e^{rt}} = A\cos(at + \varphi_1) + B\cos(bt + \varphi_2) + C + e^{(r_2 - r)t}F(t),$$

where $A, B, C, a, b \in \mathbb{R} \cap \mathbb{A}$, $a, b > 0$, $e^{i\varphi_1}, e^{i\varphi_2} \in \mathbb{A}$ and $F(t)$ is an exponential polynomial of order at most 2 whose dominant characteristic roots have real part equal to 0. In this case, Theorem 10 follows from Lemma 19 in Appendix B.

Case IV. Finally, suppose there are seven dominant characteristic roots: r, $r \pm ai$, $r \pm bi$ and $r \pm ci$. Since we are limiting ourselves to instances of order 7, these roots must all be simple, and there can be no other characteristic roots. Thus, the exponential polynomial has the form

$$\frac{f(t)}{e^{rt}} = A\cos(at + \varphi_1) + B\cos(bt + \varphi_2) + C\cos(ct + \varphi_3) + D,$$

with $A, B, C, D, a, b, c \in \mathbb{R} \cap \mathbb{A}$ with $a, b, c > 0$ and $e^{i\varphi_1}, \ldots, e^{i\varphi_3} \in \mathbb{A}$. In this case, Theorem 10 follows from Lemma 20 in Appendix C. $\square$

Corollary 11. *For differential equations of order at most* 7*, the Continuous Skolem Problem is decidable subject to Schanuel's conjecture and the Infinite Zeros Problem is decidable unconditionally.*

5 Hardness at Order 9

Diophantine approximation is a branch of number theory concerned with approximating real numbers by rationals. A central role is played in this theory by the notion of *continued fraction expansion*, which allows to compute a sequence of rational approximations to a given real number that is optimal in a certain well-defined sense. For our purposes it suffices to note that the behaviour of the continued fraction expansion of a real number a is closely related to the following two constants associated with a. The *Lagrange constant* (or *homogeneous Diophantine approximation constant*) of a is defined by

$$L_\infty(a) = \inf\left\{c : \left|a - \frac{n}{m}\right| < \frac{c}{m^2} \text{ for infinitely many } m, n \in \mathbb{Z}\right\}.$$

Following the terminology of Lagarias and Shallit [12], the *(homogeneous Diophantine approximation) type* of a is defined by

$$L(a) = \inf\left\{c : \left|a - \frac{n}{m}\right| < \frac{c}{m^2} \text{ for some } m, n \in \mathbb{Z}\right\}.$$

A real number a is called *badly approximable* if $L_\infty(a) > 0$ (or equivalently, $L(a) > 0$). The badly approximable numbers are precisely those whose continued fraction expansions have bounded partial quotients.

Khinchin showed in 1926 that almost all real numbers (in the measure-theoretic sense) have Lagrange constant and type equal to zero. However, information on the Lagrange constants and types of specific numbers or classes of numbers has proven to be elusive. In particular, concerning algebraic numbers, Guy [8] asks

> Is there an algebraic number of degree greater than two whose simple continued fraction expansion has unbounded partial quotients? Does every such number have unbounded partial quotients?

The above question can equivalently be formulated in terms of whether any algebraic number of degree greater than two has stricly positive type or whether all such numbers have type 0.

Recall that a real number a is *computable* if there is an algorithm which, given any rational $\varepsilon > 0$ as input, returns a rational q such that $|q - x| < \varepsilon$. We can now state the main result of the section.

In this section, we will show that a decision procedure for the Infinite Zeros Problem would yield the computability of $L_\infty(a)$ for all $a \in \mathbb{R} \cap \mathbb{A}$.

Fix positive $a \in \mathbb{R} \cap \mathbb{A}$, $c \in \mathbb{Q}$ and define the functions:

$$\begin{aligned} f_1(t) &\stackrel{\text{def}}{=} e^t(1-\cos(t)) + t(1-\cos(at)) - c\sin(at), \\ f_2(t) &\stackrel{\text{def}}{=} e^t(1-\cos(t)) + t(1-\cos(at)) + c\sin(at), \\ f(t) &\stackrel{\text{def}}{=} e^t(1-\cos(t)) + t(1-\cos(at)) - c|\sin(at)| = \min\{f_1(t), f_2(t)\}. \end{aligned}$$

It is easy to see that $f_1(t)$ and $f_2(t)$ are exponential polynomials of order 9, with six characteristic roots: three simple (1 and $1 \pm i$) and three repeated (0 and $\pm ai$). Thus, the problem of determining whether $f_j(t)$ has infinitely many zeros is an instance of the Infinite Zeros Problem. Moreover, it is easy to check that $f(t)$ has infinitely many zeros if and only if at least one of $f_1(t)$ and $f_2(t)$ has infinitely many zeros.

We will first state two lemmas which show a connection between the existence of infinitely many zeros of $f(t)$ and the Lagrange constant of a. We defer the proofs to Appendix E.

Lemma 12. *Fix $a \in \mathbb{R} \cap \mathbb{A}$ and $\varepsilon, c \in \mathbb{Q}$ with $a, c > 0$ and $\varepsilon \in (0,1)$. If $f(t) = 0$ for infinitely many $t \geq 0$, then $L_\infty(a) \leq c/2\pi^2(1-\varepsilon)$.*

Lemma 13. *Fix $a \in \mathbb{R} \cap \mathbb{A}$ and $\varepsilon, c \in \mathbb{Q}$ with $a, c > 0$ and $\varepsilon \in (0,1)$. If $L_\infty(a) \leq c(1-\varepsilon)/2\pi^2$, then $f(t) = 0$ for infinitely many $t \geq 0$.*

We now use the above lemmas to derive an algorithm to compute $L_\infty(a)$ using an oracle for the Infinite Zeros Problem, establishing our central hardness result:

Theorem 14. *Fix a positive real algebraic number a. If the Infinite Zeros Problem is decidable for instances of order 9, then $L_\infty(a)$ may be computed to within arbitrary precision.*

Proof. Suppose we know $L_\infty(a) \in [p,q]$ for non-negative $p, q \in \mathbb{Q}$. Choose $c \in \mathbb{Q}$ with $c > 0$ and $\varepsilon \in \mathbb{Q}$ with $\varepsilon \in (0,1)$ such that

$$p < \frac{c(1-\varepsilon)}{2\pi^2} < \frac{c}{2\pi^2(1-\varepsilon)} < q.$$

Write $A \stackrel{\text{def}}{=} c(1-\varepsilon)/2\pi^2$ and $B \stackrel{\text{def}}{=} c/2\pi^2(1-\varepsilon)$. Use the oracle for the Infinite Zeros Problem to determine whether at least one of $f_1(t), f_2(t)$ has infinitely many zeros. If this is the case, then $f(t)$ also has infinitely many zeros, so by Lemma 12, $L_\infty(a) \leq B$ and we continue the approximation recursively on the interval $[p, B]$. If not, then $L(a) \geq A$ by Lemma 13, so we continue on the interval $[A, q]$. Notice that in this procedure, one can choose c, ε at each stage in such a way that the confidence interval shrinks by at least a fixed factor, whatever the outcome of the oracle invocations. It follows therefore that $L_\infty(a)$ can be approximated to within arbitrary precision. ◻

A One dominant oscillation

Lemma 15. *Let $A, B, a, b, r \in \mathbb{R} \cap \mathbb{A}$ where $a, b, r > 0$. Let $\varphi_1, \varphi_2 \in \mathbb{R}$ be such that $e^{i\varphi_1}, e^{i\varphi_2} \in \mathbb{A}$. Suppose also that a, b are linearly dependent over $\mathbb{Q}$ and that whenever $1 - \cos(at + \varphi_1) = 0$, it holds that $A\cos(bt + \varphi_2) + B > 0$. Define the function*

$$f(t) = 1 - \cos(at + \varphi_1) + e^{-rt}(A\cos(bt + \varphi_2) + B).$$

Then $f(t) = \Omega(e^{-rt})$, that is, there exist effective constants $T \geq 0$ and $c > 0$ such that for $t \geq T$, we have $f(t) \geq ce^{-rt}$.

Proof. The case of $A = 0$ is easy: by the premise of the Lemma, we have $B > 0$ and then $f(t) \geq Be^{-rt}$ for all t. Thus, assume $A \neq 0$ throughout. Let the linear dependence between a, b be given by $an_1 - bn_2 = 0$ for $n_1, n_2 \in \mathbb{N}$ coprime and let $\mathcal{C}$ be the equivalence class of $-\varphi_1/a$ modulo $2\pi/a$, that is,

$$\mathcal{C} \stackrel{\text{def}}{=} \left\{ \frac{-\varphi_1 + 2k\pi}{a} \,\middle|\, k \in \mathbb{Z} \right\}.$$

We will refer to $\mathcal{C}$ as the set of *critical points* throughout.

It is clear that at critical points, we have $1 - \cos(at + \varphi_1) = 0$. Moreover, the linear dependence of a, b entails that for each fixed value of $(\cos(at), \sin(at))$, there are only finitely many possible values for $(\cos(bt), \sin(bt))$. Indeed, we have

$$e^{ibt} \in \{\omega e^{iatn_1} \mid \omega \text{ an } n_2\text{-th root of unity}\},$$

so in particular, for $t \in \mathcal{C}$, we have

$$e^{ibt} \in \{\omega e^{-in_1\varphi_1} \mid \omega \text{ an } n_2\text{-th root of unity}\}.$$

Thus, the possible values of $(\cos(bt), \sin(bt))$ for t critical are algebraic and effectively computable. Let $M \stackrel{\text{def}}{=} \min\{A\cos(bt + \varphi_2) + B \mid t \in \mathcal{C}\}$. By the premise of the Lemma, we have $M > 0$.

Let $t_1, t_2, \ldots, t_j, \ldots$ be the non-negative critical points. Note that by construction we have $|t_j - t_{j-1}| = 2\pi/a$. For each t_j, define the *critical region* to be the interval $[t_j - \delta, t_j + \delta]$, where

$$\delta \stackrel{\text{def}}{=} \frac{M}{2|A|b}.$$

Let $g(t) \stackrel{\text{def}}{=} A\cos(bt + \varphi_2) + B$ and notice that $g'(t) \leq |A|b$ everywhere. We first prove the claim for t inside critical regions: suppose t lies in a critical region and let j minimise $|t - t_j| \leq \delta$. Then by the Mean Value Theorem, we have

$$|g(t) - g(t_j)| \leq |t - t_j||A|b \leq \delta|A|b = \frac{M}{2},$$

so

$$g(t) \geq g(t_j) - \frac{M}{2} \geq \frac{M}{2},$$

whence $f(t) \geq e^{-rt}g(t) \geq Me^{-rt}/2 = \Omega(e^{-rt})$.

Now suppose t is outside all critical regions and let j minimise $|t - t_j|$. Since the distance between critical points is $2\pi/a$ by construction, we have $a|t - t_j| \leq \pi$. Therefore,

$$1 - \cos(at + \varphi_1) = 1 - \cos(at - at_j) \geq \frac{|a(t - t_j)|^2}{2} > \frac{(a\delta)^2}{2} = \frac{a^2M^2}{8|A|^2b^2} > 0.$$

Thus, there exists a computable constant $D > 0$ such that $f(t) = 1 - \cos(at + \varphi_1) + e^{-rt}g(t) \geq D$ for all large enough t outside critical regions.

Combining the two results, we have $f(t) = \Omega(e^{-rt})$ everywhere. $\square$

Lemma 16. *Let C, D, a, b, r_1, r_2 be real algebraic numbers such that $a, b, r_1, r_2 > 0$ and C, D are not both 0. Let also $\varphi_1, \varphi_2 \in \mathbb{R}$ be such that $e^{i\varphi_1}, e^{i\varphi_2} \in \mathbb{A}$. Define the exponential polynomial $f(t)$ by*

$$f(t) = 1 - \cos(at + \varphi_1) + e^{-r_1t}(C\cos(bt + \varphi_2) + D) + e^{-(r_1+r_2)t}F(t).$$

Here $F(t)$ is an exponential polynomial whose dominant characteristic roots are purely imaginary. Suppose also that $f(t)$ has order at most 7. Then it is decidable whether $f(t)$ has infinitely many zeros. Moreover, if $f(t)$ has only finitely many zeros, then there exists an effectively computable threshold T such that all zeros of $f(t)$ are contained in $[0, T]$.

Proof. Notice that the dominant term of $f(t)$ is always non-negative, so the function is positive for arbitrarily large t. Thus, $f(t) = 0$ for some t if and only if $f(t) \leq 0$ for some t, and analogously, $f(t)$ has infinitely many zeros if and only if $f(t) \leq 0$ infinitely often. We can eliminate the case $|D| > |C|$, since then $f(t)$ is clearly ultimately positive or oscillating, depending on the sign of D. Thus, we can assume $|D| \leq |C|$.

We now consider two cases, depending on whether $a/b \in \mathbb{Q}$.

Case I. Suppose first that a, b are linearly independent over $\mathbb{Q}$. By Lemma 5, the trajectory $(at + \varphi_1 \bmod 2\pi, bt + \varphi_2 \bmod 2\pi)$ is dense in $[0, 2\pi)^2$, and moreover the restriction of this trajectory to $at + \varphi_1 \bmod 2\pi = 0$ is dense in $\{0\} \times [0, 2\pi)$.

If $|D| < |C|$, then we argue that $f(t)$ is infinitely often negative, and hence has infinitely many zeros. Indeed, $|D| < |C|$ entails the existence of a non-trivial interval $I \subseteq [0, 2\pi)$ such that

$$t \bmod 2\pi \in I \Rightarrow C \cos(bt + \varphi_2) + D < 0.$$

What is more, we can in fact find $\epsilon > 0$ and a subinterval $I' \subseteq I$ such that

$$t \bmod 2\pi \in I' \Rightarrow C \cos(bt + \varphi_2) + D < -\epsilon.$$

Thus, by density, $1-\cos(at+\varphi_1) = 0$ and $C \cos(bt+\varphi_2)+D < -\epsilon$ will infinitely often hold simultaneously. Then just take t large enough to ensure, say, $|e^{-r_2 t}F(t)| < \epsilon/2$ at these infinitely many points, and the claim follows.

Thus, suppose now $|C| = |D|$. Replacing φ_2 by $\varphi_2 + \pi$ if necessary, we can write the function as:

$$f(t) = 1 - \cos(at + \varphi_1) + De^{-r_1 t}(1 - \cos(bt + \varphi_2)) + e^{-(r_1+r_2)t}F(t).$$

As a, b are linearly independent, for all t large enough, $1 - \cos(at + \varphi_1)$ and $1 - \cos(bt + \varphi_2)$ cannot simultaneously be 'too small'. More precisely, by Lemma 3, there exist effective constants $E, T, N > 0$ such that for all $t \geq T$, we have

$$1 - \cos(at + \varphi_1) > E/t^N \text{ or } 1 - \cos(bt + \varphi_2) > E/t^N.$$

Now, if $D < 0$, it is easy to show that $f(t)$ has infinitely many zeros. Indeed, consider the times t where the dominant term $1 - \cos(at + \varphi_1)$ vanishes. For all large enough such t, since t^{-N} shrinks more slowly than $e^{-r_2 t}$, we will have

$$\begin{aligned} f(t) &= e^{-r_1 t}D(1 - \cos(bt + \varphi_2)) + e^{-(r_1+r_2)t}F(t) \\ &< e^{-r_1 t}(EDt^{-N} + e^{-r_2 t}F(t)) \\ &\leq e^{-r_1 t}\frac{1}{2}EDt^{-N} \\ &< 0, \end{aligned}$$

so $f(t)$ has infinitely many zeros. Similarly, if $D > 0$, we can show that $f(t)$ is ultimately positive. Indeed, for all t large enough, we have

$$\begin{aligned} f(t) &\geq e^{-r_1 t}D(1 - \cos(bt + \varphi_2)) + e^{-(r_1+r_2)t}F(t) \\ &> e^{-r_1 t}DEt^{-N} + e^{-(r_1+r_2)t}F(t) \\ &> 0, \end{aligned}$$

or

$$\begin{aligned} f(t) &\geq 1 - \cos(at + \varphi_1) + e^{-(r_1+r_2)t}F(t) \\ &> Et^{-N} + e^{-(r_1+r_2)t}F(t) \\ &> 0. \end{aligned}$$

Therefore, $f(t)$ has only finitely many zeros, all occurring up to some effective bound T.

Case II. Now suppose a, b are linearly dependent. By the premise of the Lemma, the order of $F(t)$ is at most 2 (in fact, at most 1 if $D \neq 0$). However, by Theorem 9, the claim follows immediately for all cases in which the characteristic roots of $F(t)$ are all real or complex but with frequencies linearly dependent on a. Thus, the only remaining case to consider is the function

$$f(t) = 1 - \cos(at + \varphi_1) + e^{-r_1 t}C \cos(bt + \varphi_2) + e^{-(r_1+r_2)t}H \cos(ct + \varphi_3),$$

where $H, c \in \mathbb{R} \cap \mathbb{A}$, $c > 0$ and $a/c \notin \mathbb{Q}$.

As explained at the beginning of the proof of Lemma 15, due to the linear dependence of a, b over $\mathbb{Q}$, when $1-\cos(at+\varphi_1)=0$, there are only finitely many possibilities for the value of $C\cos(bt+\varphi_2)$, each algebraic, effectively computable and occurring periodically. If at least one of these values is non-positive, then by the linear independence of a, c over $\mathbb{Q}$, we will simultaneously have $1-\cos(at+\varphi_1)=0$, $C\cos(bt+\varphi_2)\leq 0$ and $H\cos(ct+\varphi_3)<0$ infinitely often, which yields $f(t)<0$ infinitely often and entails the existence of infinitely many zeros. On the other hand, if at the critical points $1-\cos(at+\varphi_1)=0$ we always have $C\cos(bt+\varphi_2)>0$, then by Lemma 15, we have

$$1-\cos(at+\varphi_1)+e^{-r_1t}C\cos(bt+\varphi_2)=\Omega(e^{-r_1t}),$$

whereas obviously

$$\left|e^{-(r_1+r_2)t}H\cos(ct+\varphi_3)\right|=\mathcal{O}(e^{-(r_1+r_2)t}).$$

An effective threshold T follows such that for $t\geq T$, $f(t)$ is ultimately positive. ☐

Lemma 17. *Let A, B, a, b, c, r be real algebraic numbers such that $a, b, c, r>0$, $A, B\neq 0$. Let also $\varphi_1, \varphi_2, \varphi_3 \in \mathbb{R}$ be such that $e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3}\in\mathbb{A}$. Define the exponential polynomial $f(t)$ by*

$$f(t)=1-\cos(ct+\varphi_3)+e^{-rt}(A\cos(at+\varphi_1)+B\cos(bt+\varphi_2)).$$

Then it is decidable whether $f(t)$ has infinitely many zeros. Moreover, if $f(t)$ has only finitely many zeros, then there exists an effective threshold T such that all zeros of $f(t)$ are contained in $[0, T]$.

Proof. We argue the function is infinitely often positive and infinitely often negative by looking at the values of t for which the dominant term $1-\cos(ct+\varphi_3)$ vanishes. This happens precisely at the times $t=-(\varphi_3+2k\pi)/c$ for $k\in\mathbb{Z}$, giving rise to a discrete restriction of f:

$$g(k)\stackrel{\text{def}}{=}e^{r\varphi_3}\left(e^{2\pi r}\right)^k\left(A\cos\left(k\frac{2\pi a}{c}-\frac{a\varphi_3}{c}+\varphi_1\right)+B\cos\left(k\frac{2\pi b}{c}-\frac{b\varphi_3}{c}+\varphi_2\right)\right)$$

This is a linear recurrence sequence over $\mathbb{R}$ of order 4, with characteristic roots $e^{2\pi(r\pm ia/c)}$ and $e^{2\pi(r\pm ib/c)}$. In particular, it has no real dominant characteristic root. It is well-known that real-valued linear recurrence sequences with no dominant real characteristic root are infinitely often positive and infinitely often negative: see for example [9, Theorem 7.1.1]. Therefore, by continuity, $f(t)$ must have infinitely many zeros. ☐

Lemma 18. *Let A, B, a, b, r be real algebraic numbers such that $a, b, r>0$, $A\neq 0$. Let also $\varphi_1, \varphi_2, \varphi_3\in\mathbb{R}$ be such that $e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3}\in\mathbb{A}$. Define the exponential polynomial $f(t)$ by*

$$f(t)=1-\cos(at+\varphi_1)+e^{-rt}(At\cos(bt+\varphi_2)+B\cos(bt+\varphi_3)).$$

Then it is decidable whether $f(t)$ has infinitely many zeros. Moreover, if $f(t)$ has only finitely many zeros, then there exists an effective threshold T such that all zeros of $f(t)$ are contained in $[0, T]$.

Proof. If $a/b\in\mathbb{Q}$, then the claim follows immediately from Theorem 9. If $a/b\notin\mathbb{Q}$, then by Lemma 5, it will happen infinitely often that $1-\cos(at+\varphi_1)=0$ and $At\cos(bt+\varphi_2)<-|A|t/2$. Then clearly $f(t)<0$ infinitely often. Since $f(t)>0$ infinitely often as well, due to the non-negative dominant term $1-\cos(at+\varphi_1)$, it follows that $f(t)$ has infinitely many zeros. ☐

B Two dominant oscillations

Lemma 19. *Let A, B, C, a, b, r be real algebraic numbers such that $a, b, r>0$, $a\neq b$ and $A, B, C\neq 0$. Let also $\varphi_1, \varphi_2\in\mathbb{R}$ be such that $e^{i\varphi_1}, e^{i\varphi_2}\in\mathbb{A}$. Define the exponential polynomial $f(t)$ by*

$$f(t)=A\cos(at+\varphi_1)+B\cos(bt+\varphi_2)+C+e^{-rt}F(t).$$

where $F(t)$ is an exponential polynomial whose dominant characteristic roots are purely imaginary. Suppose also $f(t)$ has order at most 8. It is decidable whether $f(t)$ has infinitely many zeros, and moreover, if $f(t)$ has only finitely many zeros, then there exists an effective threshold T such that all zeros of $f(t)$ are contained in $[0, T]$.

Proof. If the frequencies a, b of the dominant term's oscillations are linearly independent over $\mathbb{Q}$, then the claim follows immediately by Theorem 8. Therefore, assume $na - mb = 0$ for some $n, m \in \mathbb{N}^+$. Notice that $a \neq b$ guarantees $n \neq m$. We perform the change of variable $t \to tm/a$, so that:

$$f(t) = A\cos(mt + \varphi_1) + B\cos(nt + \varphi_2) + C + e^{-rmt/a}F(tm/a).$$

Using the standard trigonometric identities, we express the dominant term as a polynomial in $\sin(t), \cos(t)$:

$$f(t) = P(\sin(t), \cos(t)) + e^{-rmt/a}F(tm/a),$$

where $P \in (\mathbb{R} \cap \mathbb{A})[x, y]$ has effectively computable coefficients. It is clear that the dominant term is periodic. It is immediate from the definition of exponential polynomials and the premise of the Lemma that $F(tm/a) \stackrel{\text{def}}{=} F_2(t)$ is an exponential polynomial in t, of the same order as $F(t)$, also with purely imaginary dominant characteristic roots. Let $\alpha(t) \stackrel{\text{def}}{=} P(\sin(t), \cos(t))$, $r_2 \stackrel{\text{def}}{=} rm/a > 0$ and $\beta(t) \stackrel{\text{def}}{=} e^{-rmt/a}F(tm/a) = e^{-r_2 t}F_2(t)$.

We are now interested in the extrema of $P(\sin(t), \cos(t))$. Let

$$M_1 \stackrel{\text{def}}{=} \min_{x^2+y^2=1} P(x, y) = \min_{t \geq 0} \alpha(t),$$
$$M_2 \stackrel{\text{def}}{=} \max_{x^2+y^2=1} P(x, y) = \max_{t \geq 0} \alpha(t).$$

We can construct defining formulas $\phi_1(u), \phi_2(u)$ in the first-order language $\mathcal{L}$ of real closed fields for M_1, M_2, so that each $\phi_j(u)$ holds precisely for the valuation $u = M_j$. Then performing quantifier elimination on these formulas using Renegar's algorithm [17], we convert ϕ_1, ϕ_2 into the form

$$\phi_j(u) \equiv \bigvee_l \bigwedge_k P_{l,k}(u) \sim_{l,k} 0,$$

where $P_{l,k}$ are polynomials with integer coefficients and each $\sim_{l,k}$ is either $<$ or $=$. Now $\phi_j(u)$ must have a satisfiable disjunct. Using the decidability of the theory $Th(\mathbb{R})$, we can readily identify this disjunct. Moreover, since $\phi_j(u)$ has a unique satisfying valuation, namely $u = M_j$, this disjunct must contain at least one equality predicate. It follows immediately that M_1, M_2 are algebraic. Moreover, we can effectively compute from $\phi_j(u)$ a representation for M_j consisting of its minimal polynomial and a sufficiently accurate rational approximation to distinguish M_j from its Galois conjugates. By an analogous argument, the pairs $(\sin(t), \cos(t))$ at which $P(\sin(t), \cos(t))$ achieves the extrema M_1, M_2 are also algebraic and effectively computable.

We now perform a case analysis on the signs of M_1 and M_2.

- First, if $0 < M_1 \leq M_2$, then $f(t)$ cannot have infinitely many zeros: if t is large enough to ensure $|\beta(t)| < M_1$, we have $f(t) > 0$.
- Second, if $M_1 \leq M_2 < 0$, then by the same reasoning, the function will ultimately be strictly negative.
- Third, if $M_1 < 0 < M_2$, then $f(t)$ oscillates around 0: for all t such that $\alpha(t) = M_1 < 0$ and large enough to ensure $|\beta(t)| < |M_1|$, we will have $f(t) < 0$, and similarly, for large enough t such that $\alpha(t) = M_2 > 0$, we will have $f(t) > 0$, so the function must have infinitely many zeros.
- Next, we argue that the case $M_1 = M_2 = 0$ is impossible. Indeed, if $M_1 = M_2 = 0$, then $\alpha(t) = P(\sin(t), \cos(t))$ is identically zero, and the same holds for all derivatives of $\alpha(t)$. Thus, from $\alpha'(t) \equiv \alpha'''(t) \equiv 0$, we have

  $$0 \equiv -Am\sin(mt + \varphi_1) - Bn\sin(nt + \varphi_2),$$
  $$0 \equiv Am^3\sin(mt + \varphi_1) + Bn^3\sin(nt + \varphi_2).$$

  Multiplying the first identity through by m^2 and summing, we have

  $$Bn\sin(nt + \varphi_2)(n^2 - m^2) \equiv 0.$$

  By the premise of the Lemma, $B \neq 0$, so $n(n - m)(n + m) = 0$, which is a contradiction.
- Finally, only the symmetric cases $M_1 < M_2 = 0$ and $0 = M_1 < M_2$ remain. Without loss of generality, by replacing $f(t)$ by $-f(t)$ if necessary, we need only consider the case $0 = M_1 < M_2$.

Thus, assume $0 = M_1 < M_2$. We now move our attention to the possible forms of $F_2(t)$. Since $f(t)$ has order at most 8, it follows that $F_2(t)$ has order at most 3. Thus, there are three possibilities for the set of dominant characteristic roots of $F_2(t)$: $\{0\}$, $\{\pm ic\}$, or $\{0, \pm ic\}$, for some positive $c \in \mathbb{R} \cap \mathbb{A}$. We consider each of these cases in turn.

First, if $F_2(t)$ only has the real dominant eigenvalue 0, then $F_2(t)$ is ultimately positive or ultimately negative, depending on the sign of the most significant term of $F_2(t)$, with an effectively computable threshold. Ultimate positivity of $F_2(t)$ entails ultimate positivity of $f(t)$ as well, since $P(\sin(t), \cos(t)) \geq 0$ everywhere, whereas an ultimately negative $F_2(t)$ makes $f(t)$ change sign infinitely often.

Second, assume the dominant characteristic roots of $F_2(t)$ are $\{\pm ic\}$, so that

$$f(t) = P(\sin(t), \cos(t)) + e^{-r_2 t}\left(D\cos(ct + \varphi_3) + Ee^{-r_3 t}\right)$$

for some $r_3 > 0$ and $\varphi_3 \in \mathbb{R}$ such that $e^{i\varphi_3} \in \mathbb{A}$. Without loss of generality, we can assume $c \notin \mathbb{Q}$, since otherwise, we are done by Theorem 9. But by Lemma 5, it will happen infinitely often that $P(\sin(t), \cos(t)) = 0$ and $D\cos(ct + \varphi_3) < -|D|/2$, say. For large enough such t, $|Ee^{-(r_2+r_3)t}| < |D|/4$, so we conclude that $f(t)$ is infinitely often negative, and hence has infinitely many zeros.

Third, assume the dominant characteristic roots of $F_2(t)$ are $\{0, \pm ic\}$, so that

$$f(t) = P(\sin(t), \cos(t)) + e^{-r_2 t}(D\cos(ct + \varphi_3) + E).$$

We again assume $c \notin \mathbb{Q}$, since otherwise the claim follows from Theorem 9. Let $M_3 \stackrel{\text{def}}{=} E - |D| = \min_{t \geq 0} F_2(t)$. If $M_3 > 0$, then $f(t)$ clearly has no zeros. If $M_3 < 0$, then there exists a non-trivial interval $I \subseteq [0, 2\pi)$ such that if $ct + \varphi_3 \bmod 2\pi \in I$, then $F_2(t) < 0$. Since $c \notin \mathbb{Q}$, Lemma 5 guarantees that $F_2(t) < 0 = P(\sin(t), \cos(t))$ happens infinitely often, so $f(t)$ must have infinitely many zeros. Finally, if $M_3 = 0$, we argue that $f(t)$ is ultimately positive. Indeed, since $P(\sin(t), \cos(t))$ and $F_2(t)$ are both non-negative everywhere, $f(t) = 0$ can only happen if $P(\sin(t), \cos(t)) = D\cos(ct + \varphi_3) + E = 0$. This, however, would entail $e^{it} \in \mathbb{A}$ and $e^{ict} \in \mathbb{A}$, which contradicts the Gelfond-Schneider Theorem, since $c \notin \mathbb{Q}$. Thus, we conclude $f(t)$ has no zeros. □

C Three dominant oscillations

Lemma 20. *Let A, B, C, a, b, c be real algebraic numbers such that $a, b, c > 0$ and $A, B, C \neq 0$. Let also $\varphi_1, \varphi_2, \varphi_3 \in \mathbb{R}$ be such that $e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3} \in \mathbb{A}$. Define the exponential polynomial $f(t)$ by*

$$f(t) = A\cos(at + \varphi_1) + B\cos(bt + \varphi_2) + C\cos(ct + \varphi_3) + D.$$

It is decidable whether $f(t)$ has infinitely many zeros, and moreover, if $f(t)$ has only finitely many zeros, then there exists an effective threshold T such that all zeros of $f(t)$ are contained in $[0, T]$.

Proof. The argument consists of three cases, depending on the linear dependencies over $\mathbb{Q}$ satisfied by a, b and c.

Case I. First, if a, b, c are linearly independent over $\mathbb{Q}$, then the claim follows directly from Theorem 8.

Case II. Second, suppose that a, b, c are all rational multiples of one another:

$$b = \frac{n}{m}a,\ c = \frac{k}{l}a \text{ where } n, m, k, l \in \mathbb{N}^+.$$

We make the change of variable $t \to tml$ to obtain:

$$f(t) = A\cos((at)ml + \varphi_1) + B\cos((at)nl + \varphi_2) + C\cos((at)km + \varphi_3) + D = P(\sin(at), \cos(at)),$$

where $P \in \mathbb{A}[x, y]$ is a polynomial obtained using the standard trigonometric identities. It is now clear that $f(t)$ is periodic, so it has either no zeros or infinitely many zeros. Let

$$M_1 \stackrel{\text{def}}{=} \min_{x^2+y^2=1} P(x, y) = \min_{t \geq 0} f(t),$$

$$M_2 \stackrel{\text{def}}{=} \max_{x^2+y^2=1} P(x, y) = \max_{t \geq 0} f(t).$$

Using the same reasoning as in Lemma 19, we see that M_1, M_2 are algebraic and effectively computable: simply construct defining formulas in the first-order language $\mathcal{L}$ of real closed fields, and then perform quantifier elimination using Renegar's algorithm [17]. Then $f(t)$ clearly has infinitely many zeros if and only if $M_1 \leq 0 \leq M_2$.

Case III. Finally, suppose that a, b, c span a $\mathbb{Q}$-vector space of dimension 2, so that a, b, c satisfy a single linear dependence $am + bn + cp = 0$ where $m, n, p \in \mathbb{Z}$ are coprime. At most one of the ratios a/b, a/c and b/c is rational (otherwise we have $\dim \mathit{span}\{a, b, c\} = 1$), so assume without loss of generality that $a/c \notin \mathbb{Q}$ and $b/c \notin \mathbb{Q}$.

Define the set

$$\begin{aligned} \mathbb{T} &\stackrel{\text{def}}{=} \left\{ x \in [0, 2\pi)^3 \,\middle|\, \forall u \in \mathbb{Z}^3 \,.\, u \cdot (a, b, c) \in 2\pi\mathbb{Z} \Rightarrow u \cdot x \in 2\pi\mathbb{Z} \right\} \\ &= \left\{ (x_1, x_2, x_3) \in [0, 2\pi)^3 \,\middle|\, mx_1 + nx_2 + px_3 = 0 \in 2\pi\mathbb{Z} \right\} \end{aligned}$$

Notice that if $mx_1 + nx_2 + px_3 = 2k\pi$ for x_1, x_2, x_3, then $k \leq |m| + |n| + |p|$, so $\mathbb{T}$ partitions naturally into finitely many subsets: $\mathbb{T} = \bigcup_{k=1}^{N} \mathbb{T}_k$, where

$$\mathbb{T}_k \stackrel{\text{def}}{=} \left\{ (x_1, x_2, x_3) \in [0, 2\pi)^3 \,\middle|\, mx_1 + nx_2 - px_3 = 2k\pi \right\}.$$

Consider the trajectory $h(t) \stackrel{\text{def}}{=} \{(at, bt, ct) \bmod 2\pi \,|\, t \geq 0\}$. Define also the sets $R \stackrel{\text{def}}{=} \{h(2k\pi) \,|\, k \in \mathbb{N}\}$ and $H \stackrel{\text{def}}{=} \{h(t) \,|\, t \geq 0\}$. Because of the linear dependence satisfied by a, b, c, it is easy to see that $R \subseteq H \subseteq \mathbb{T}$. By Kronecker's Theorem, R is a dense subset of $\mathbb{T}$, so clearly H must be a dense subset of $\mathbb{T}$ as well.

Now define the function

$$F(x_1, x_2, x_3) \stackrel{\text{def}}{=} A\cos(x_1 + \varphi_1) + B\cos(x_2 + \varphi_2) + C\cos(x_3 + \varphi_3) + D,$$

so that the image of $f(t)$ is exactly $\{F(x_1, x_2, x_3) \,|\, (x_1, x_2, x_3) \in H\}$. Let also the extrema of F over $\mathbb{T}$ be:

$$\begin{aligned} M_1 &\stackrel{\text{def}}{=} \min_{\mathbb{T}} F(x_1, x_2, x_3), \\ M_2 &\stackrel{\text{def}}{=} \max_{\mathbb{T}} F(x_1, x_2, x_3). \end{aligned}$$

Both of these values are algebraic and can be computed using quantifier elimination in the first-order language $\mathcal{L}$ of the real numbers: just use separate variables for $\cos(x_j), \sin(x_j)$ and apply the standard trigonometric identities to convert the linear dependence on x_1, x_2, x_3 into a polynomial dependence between $\cos(x_j), \sin(x_j)$.

Now, by the density of H in $\mathbb{T}$, if $M_1 < 0 < M_2$, then $f(t)$ must clearly be infinitely often positive and infinitely often negative, so it must have infinitely many zeros. The case $M_1 < 0 = M_2$ is symmetric to $0 = M_1 < M_2$ (just replace f and F by $-f$ and $-F$, respectively), so without loss generality, we can assume $0 = M_1 < M_2$. In this case, we argue that $f(t)$ has no zeros, that is, even though F vanishes on some points in $\mathbb{T}$, none of these points appear in the dense subset H. Indeed, consider the set

$$Z \stackrel{\text{def}}{=} \{(\cos(x_1), \sin(x_1), \ldots, \cos(x_3), \sin(x_3)) \,|\, (x_1, x_2, x_3) \in \mathbb{T}, F(x_1, x_2, x_3) = 0\}.$$

Note that Z is clearly semi-algebraic, as one can directly write a defining formula in $\mathcal{L}$ from $F(x_1, x_2, x_3) = 0$ and $mx_1 + nx_2 + px_3 \in 2\pi\mathbb{Z}$. Moreover, by the Zero-Dimensionality Lemma [15, Lemma 10], the function $F(x_1, x_2, x_3)$ achieves its minimum $M_1 = 0$ at only finitely many points in $\mathbb{T}_k$, for each k. Since $\mathbb{T}$ is the union of finitely many $\mathbb{T}_k$, we immediately have that Z is finite. By the Tarski-Seidenberg Theorem, projecting Z to any fixed component will also give a finite, semi-algebraic subset of $\mathbb{R}$, that is, a finite subset of $\mathbb{A}$. Thus, we have shown that if $F(x_1, x_2, x_3) = 0$, then $e^{ix_j} \in \mathbb{A}$ for all $j = 1, 2, 3$. Now if $f(t) = 0$ for some $t \geq 0$, then we must have $e^{ati}, e^{cti} \in \mathbb{A}$, which by the Gelfond-Schneider Theorem entails $a/c \in \mathbb{Q}$, a contradiction. $\square$

D One repeated oscillation

Lemma 21. *Let A, B, C, D, a, r be real algebraic numbers such that $a, r > 0$ and $A \neq 0$. Let also $\varphi_1, \varphi_2 \in \mathbb{R}$ be such that $e^{i\varphi_1}, e^{i\varphi_2} \in \mathbb{A}$. Define the exponential polynomial $f(t)$ by*

$$f(t) = t(A\cos(at + \varphi_1) + B) + (C\cos(at + \varphi_2) + D) + e^{-rt}F(t)$$

where $F(t)$ is an exponential polynomial with purely imaginary dominant characteristic roots. Suppose also that $f(t)$ has order at most 8. It is decidable whether $f(t)$ has infinitely many zeros, and moreover, if $f(t)$ has only finitely many zeros, then there exists an effective threshold T such that all zeros of $f(t)$ are contained in $[0, T]$.

Proof. Since $f(t)$ has order no greater than 8, it follows that $F(t)$ has order at most 2. Therefore, $F(t)$ must be of the form $E\cos(bt + \varphi_3)$ for some $E, b \in \mathbb{R} \cap \mathbb{A}$, $b > 0$, such that $a/b \notin \mathbb{Q}$, and some φ_3 such that $e^{i\varphi_3} \in \mathbb{A}$, since otherwise the imaginary parts of the characteristic roots of $f(t)$ are pairwise linearly dependent over $\mathbb{Q}$, so our claim is proven immediately by Theorem 9.

Consider first the magnitudes of A and B. If $|A| > |B|$, then the term $tA\cos(at + \varphi_1)$ makes $f(t)$ change sign infinitely often, whereas if $|B| > |A|$, then for t large enough, the term tB makes $f(t)$ ultimately positive or ultimately negative, depending on the sign of B. Thus, we can assume $|A| = |B|$. Dividing $f(t)$ by B, and replacing φ_1 by $\varphi_1 + \pi$ if necessary, we can assume the function has the form:

$$f(t) = t(1 - \cos(at + \varphi_1)) + (C\cos(at + \varphi_2) + D) + e^{-rt}E\cos(bt + \varphi_3).$$

Considering the dominant term, it is clear that $f(t)$ is infinitely often positive. Let $\alpha(t) \stackrel{\text{def}}{=} t(1 - \cos(at + \varphi_1))$, $\beta(t) \stackrel{\text{def}}{=} C\cos(at + \varphi_2) + D$ and $\gamma(t) \stackrel{\text{def}}{=} e^{-rt}E\cos(bt + \varphi_3)$.

We now focus on the sign of the term $\beta(t)$ at the positive *critical times* $t_j \stackrel{\text{def}}{=} -\varphi_1/a + 2j\pi/a$ ($j \in \mathbb{Z}$) when $1 - \cos(at + \varphi_1)$ vanishes. Notice that $\beta(t_j) = C\cos(\varphi_2 - \varphi_1) + D \stackrel{\text{def}}{=} M$ is independent of j. First, if $M < 0$, then for all t_j large enough, $f(t_j) < 0$, so the function must have infinitely many zeros. Second, if $M = 0$, then by the linear independence of a, b and Lemma 5, we have $\alpha(t_j) = \beta(t_j) = 0 > \gamma(t_j)$ for infinitely many t_j, so we can conclude $f(t)$ has infinitely many zeros.

Finally, suppose $M > 0$. We will prove that $f(t)$ is ultimately positive. For each t_j, define the *critical region* $[t_j - \delta_j, t_j + \delta_j]$, given by

$$\delta_j \stackrel{\text{def}}{=} \frac{2\sqrt{|C| + |D|}}{a\sqrt{t_{j-1}}}.$$

From here onwards, we only consider t large enough for any two adjacent critical regions to be disjoint. The argument consists of two parts: first we show $f(t) > 0$ for all large enough t outside all critical regions, and then we show $f(t) > 0$ for large enough t in a critical region.

Suppose t is outside all critical regions and let j minimise $|t - t_j|$. Since the distance between critical points is $2\pi/a$ by construction, we have $a|t - t_j| \leq \pi$. Therefore,

$$\frac{|a(t - t_j)|^2}{2} \leq 1 - \cos(at - at_j) = 1 - \cos(at + \varphi_1).$$

On the other hand, we have the following chain of inequalities:

$$
\begin{aligned}
&\frac{|a(t-t_j)|^2}{2}\\
&> \{\ |t-t_j| > \delta_j\ \}\\
&\frac{(a\delta_j)^2}{2}\\
&= \{\ \text{definition of } \delta_j\ \}\\
&\frac{2(|C|+|D|)}{t_{j-1}}\\
&> \{\ \text{by } t > t_{j-1}\ \}\\
&\frac{2(|C|+|D|)}{t}\\
&\geq \{\ \text{triangle inequality and } |\cos(x)| \leq 1\ \}\\
&\frac{|C|+|D|}{t} + \frac{|C\cos(at+\varphi_2)+D|}{t}.
\end{aligned}
$$

Combining, we have

$$\alpha(t)+\beta(t) \geq \alpha(t) - |\beta(t)| = t(1-\cos(at+\varphi_1)) - |C\cos(at+\varphi_2)+D| \geq |C|+|D|.$$

Thus, if t is large enough to ensure $|\gamma(t)| < |C|+|D|$, we have $f(t) > 0$ outside critical regions.

For the second part of the argument, we consider t in critical regions. Notice that the values of $\beta(t)$ on $[t_j-\delta_j, t_j+\delta_j]$ are independent of the choice of t_j. Moreover, we have $\beta(t_j) = M > 0$, so there exists some $\epsilon > 0$ such that for all $t \in [t_j-\epsilon, t_j+\epsilon]$, we have $\beta(t) \geq M/2$, say. Now for any critical point t_j chosen large enough, we will have $[t_j-\delta_j, t_j+\delta_j] \subseteq [t_j-\epsilon, t_j+\epsilon]$, so $\beta(t) > M/2$ on the entire critical region. Let also t_j be large enough so that for any t in the critical region, we have $|\gamma(t)| < M/2$. Then we have $f(t) = \alpha(t)+\beta(t)+\gamma(t) \geq \beta(t) - |\gamma(t)| > 0$, completing the claim. □

E Proofs of Hardness Lemmas

Throughout this section, let

$$f(t) \stackrel{\text{def}}{=} e^t(1-\cos(t)) + t(1-\cos(at)) - c|\sin(at)|.$$

Lemma 12. *Fix $a \in \mathbb{R}\cap\mathbb{A}$ and $\varepsilon, c \in \mathbb{Q}$ with $a, c > 0$ and $\varepsilon \in (0,1)$. If $f(t) = 0$ for infinitely many $t \geq 0$, then $L_\infty(a) \leq c/2\pi^2(1-\varepsilon)$.*

Proof. Suppose $f(t) = 0$ for infinitely many t. Clearly, this also entails $f(t) = 0$ for infinitely many $t \geq T$, for any particular threshold $T \geq 0$. (Indeed, $f(t) = \min\{f_1(t), f_2(t)\}$ for exponential polynomials f_1 and f_2 given at the beginning of Section 5. Thus, on any bounded interval, f has no more zeros than f_1 and f_2 combined, i.e., only finitely many, by the analiticity of f_1 and f_2.) We will show that T can be chosen in such a way that every zero of $f(t)$ on $[T,\infty)$ yields a pair $(n,m) \in \mathbb{N}^2$ which satisfies the inequality

$$\left|a - \frac{n}{m}\right| < \frac{c}{2\pi^2 m^2(1-\varepsilon)}.$$

This is sufficient, since infinitely many zeros of f yield infinitely many solutions, and therefore witness $L_\infty(a) \leq c/2\pi^2(1-\varepsilon)$.

Thus, consider some t such that $f(t) = 0$ and $t \geq T$ for some threshold T to be specified later. Let $t = 2\pi m + \delta_1$ and $at = 2\pi n + \delta_2$, where $m, n \in \mathbb{N}$ and $\delta_1, \delta_2 \in [-\pi, \pi)$. Then we have

$$\left|a - \frac{n}{m}\right| = \frac{|\delta_2 - a\delta_1|}{2\pi m}.$$

We will show that for T large enough, $f(t) = 0$ for $t \geq T$ allows us to bound $|\delta_2|$ and $|a\delta_1|$ separately from above and then apply the triangle inequality to bound $|\delta_2 - a\delta_1|$.

First, choose $\varphi_1, \varphi_2 \in (0, 1)$ such that $1-\varphi_2 > 1-\varphi_1 > 1-\varepsilon$. Let T be large enough for the following property to hold:

$$\frac{t+\pi}{t-2\pi} \leq \frac{1-\varphi_2}{1-\varphi_1} \text{ for all } t \geq T.$$

In particular, since $m = (t - \delta_1)/2\pi$ and $|\delta_1| \leq \pi$, we have

$$\frac{2m}{2m-1} \leq \frac{t+\pi}{t-2\pi} \leq \frac{1-\varphi_2}{1-\varphi_1}. \tag{5}$$

Let also T be large enough to make the following property valid:

$$\text{if } 1 - \cos(x) \leq c|x|/T \text{ and } |x| \leq \pi, \text{ then } (1-\varphi_2)x^2/2 \leq 1 - \cos(x). \tag{6}$$

Now we have the following chain of inequalities:

$$\begin{aligned}
& 1 - \cos(\delta_2) \\
\leq & \ \{ \ f(t) = 0, \text{ noting } e^t(1-\cos(t)) \geq 0 \ \} \\
& \frac{c|\sin(\delta_2)|}{t} \\
\leq & \ \{ \text{ by } |\sin(x)| \leq |x| \ \} \\
& \frac{c|\delta_2|}{t}.
\end{aligned}$$

Then by (6), we have

$$1 - \cos(\delta_2) \geq \frac{(1-\varphi_2)\delta_2^2}{2}.$$

Thus, combining the upper and lower bounds on $1 - \cos(\delta_2)$ and using (5) on the last step, we have

$$|\delta_2| \leq \frac{2c}{t(1-\varphi_2)} \leq \frac{2c}{(2m-1)\pi(1-\varphi_2)} \leq \frac{c}{m\pi(1-\varphi_1)}.$$

Second, let $\alpha \stackrel{\text{def}}{=} (1-\varepsilon)^{-1} - (1-\varphi_1)^{-1} > 0$. Let the threshold T be large enough so that

$$e^{-t} \leq \frac{c\alpha^2}{4\pi^2 a^2}\left(\frac{2\pi}{t+\pi}\right)^2 \text{ for } t \geq T \tag{7}$$

and

$$\text{if } 1 - \cos(x) \leq c/e^T \text{ and } |x| \leq \pi, \text{ then } x^2/4 \leq 1 - \cos(x). \tag{8}$$

The following chain of inequalities holds:

$$\begin{aligned}
& 1 - \cos(\delta_1) \\
= & \ \{ \text{ by } f(t) = 0 \ \} \\
& \frac{c|\sin(\delta_2)| - t(1-\cos(\delta_2))}{e^t} \\
\leq & \ \{ \text{ by } |\sin(\delta_2)|, |\cos(\delta_2)| \leq 1 \} \\
& \frac{c}{e^t} \\
\leq & \ \{ \text{ by (7) } \} \\
& \frac{c^2\alpha^2}{4\pi^2 a^2}\left(\frac{2\pi}{t+\pi}\right)^2 \\
\leq & \ \{ \text{ by } |\delta_1| \leq \pi \ \} \\
& \frac{c^2\alpha^2}{4\pi^2 a^2}\left(\frac{2\pi}{t-\delta_1}\right)^2 \\
= & \ \{ \ t = 2\pi m + \delta_1 \ \} \\
& \frac{c^2\alpha^2}{4\pi^2 a^2 m^2}.
\end{aligned}$$

Moreover, as $1-\cos(\delta_1) \le ce^{-t} \le ce^{-T}$, by (8), we have

$$1-\cos(\delta_1) \ge \frac{\delta_1^2}{4},$$

so combining the lower and upper bound on $1-\cos(\delta_1)$, we can conclude

$$|a\delta_1| \le \frac{c\alpha}{\pi m}.$$

Finally, by the triangle inequality and the bounds on $|a\delta_1|$ and $|\delta_2|$, we have

$$\left|a-\frac{n}{m}\right| = \frac{|\delta_2 - a\delta_1|}{2\pi m} \le \frac{|\delta_2|+|a\delta_1|}{2\pi m} \le \frac{c}{2\pi^2 m^2}\left(\alpha + \frac{1}{1-\varphi_1}\right) = \frac{c}{2\pi^2 m^2(1-\varepsilon)}.$$

Now, by the premise of the Lemma, there are infinitely many $t \ge T$ such that $f(t)=0$, each yielding a pair $(n,m) \in \mathbb{N}^2$ which satisfies the above inequality. These infinitely many pairs (n,m) witness $L_\infty(a) \le c/2\pi^2(1-\varepsilon)$, as required. □

Lemma 13. *Fix $a \in \mathbb{R}\cap\mathbb{A}$ and $\varepsilon, c \in \mathbb{Q}$ with $a, c > 0$ and $\varepsilon \in (0,1)$. If $L_\infty(a) \le c(1-\varepsilon)/2\pi^2$, then $f(t)=0$ for infinitely many t.*

Proof. We will show that there exists an effective threshold M, dependent on a, c, ε, such that if

$$\left|a-\frac{n}{m}\right| \le \frac{c(1-\varepsilon)}{2\pi^2 m^2} \tag{9}$$

for natural numbers n, m with $m \ge M$, then $f(2\pi m) \le 0$. Note that this is sufficient to prove the Lemma: the premise guarantees infinitely many solutions $(n,m) \in \mathbb{N}^2$ of (9), so there must be infinitely many solutions with $m \ge M$, each yielding $f(2\pi m) \le 0$. Since $f(t)$ is continuous and moreover is positive for arbitrarily large times, it must have infinitely many zeros on $[2\pi M, \infty)$.

Now let M be large enough, so that $c(1-\varepsilon)/\pi M < \pi$ and

$$\text{if } |x| < c(1-\varepsilon)/\pi M, \text{ then } (1-\varepsilon)|x| \le |\sin(x)|. \tag{10}$$

Suppose that (9) holds for $n, m \in \mathbb{N}$ with $m \ge M$ and write $t \overset{\text{def}}{=} 2\pi m$. We will show that $f(t) \le 0$. By (9), we have $|am-n| \le c(1-\varepsilon)/2\pi^2 m$. Therefore, $at = 2\pi a m = 2\pi n + \delta$ where $|\delta| \le c(1-\varepsilon)/\pi m < \pi$. We have

$$\begin{aligned}
&f(t)\\
= &\{ \text{ as } \cos(t)=1 \ \}\\
&t(1-\cos(\delta)) - c|\sin(\delta)|\\
\le &\{ \text{ by (10) and } 1-\cos(x) \le x^2/2 \ \}\\
&\pi m\delta^2 - c(1-\varepsilon)|\delta|\\
\le &\{ \text{ by } |\delta| \le c(1-\varepsilon)/\pi m \ \}\\
&0.
\end{aligned}$$

□

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