The Complexity of Computing a Bisimilarity Pseudometric on Probabilistic Automata

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Joint work with James Worrell

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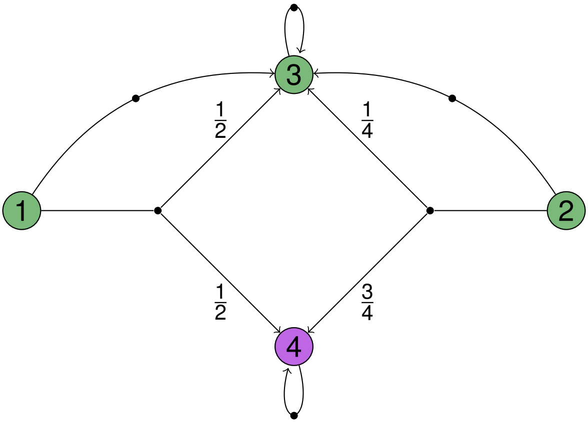
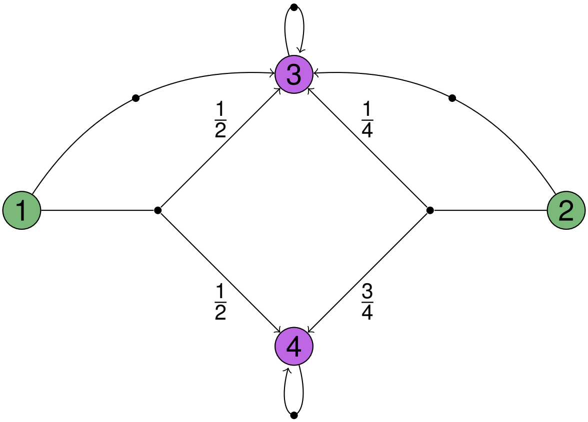


A probabilistic automaton contains nondeterministic and probabilistic choices.

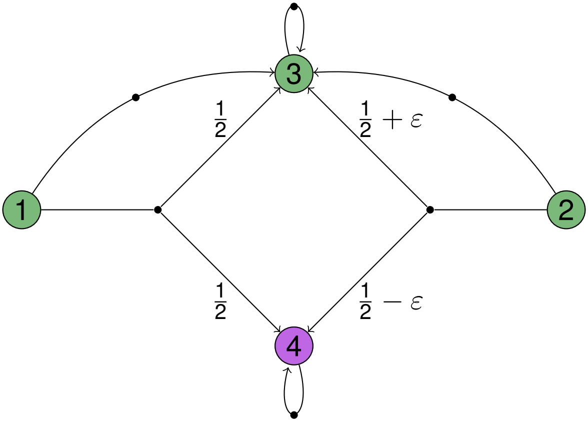
Probabilistic Bisimilarity



Probabilistic bisimilarity captures which states of the automaton behave the same.

Roberto Segala, in collaboration with Nancy Lynch, introduced probabilistic bisimilarity for probabilistic automata.

Probabilistic Bisimilarity is not Robust



States 1 and 2 are not bisimilar for all $\varepsilon > 0$.

Scott Smolka

Scott Smolka, in collaboration with Alessandro Giacalone and Chi-chang Jou, first suggested to use pseudometrics instead of equivalence relations.

From Equivalence Relations to Pseudometrics

An equivalence relation on a set S can be viewed as function in

$$S \times S \to \mathbb{B}$$

A (1-bounded) pseudometric on a set S is a function in

$$S \times S \to [0, 1]$$

Equivalence is captured by distance zero.

A Metric for Nondeterministic Choices

Nondeterministic choices can be modelled as subsets of a set.

The distance of the subsets A and B is defined by

$$d(A, B) = \max \left\{ \max_{a \in A} \min_{b \in B} d(a, b), \max_{b \in B} \min_{a \in A} d(b, a) \right\}$$

This can be seen as a quantitative generalization of

$$\wedge \left(\forall_{a \in A} \exists_{b \in B} \ldots, \forall_{b \in B} \exists_{a \in A} \ldots \right)$$

which should remind you of bisimilarity.

Felix Hausdorff

Felix Hausdorff introduced the metric on subsets. This metric is known as the Hausdorff metric.

A Metric for Probabilistic Choices

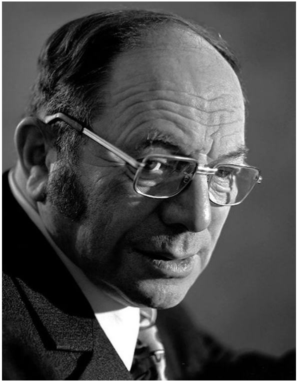
Probabilistic choices can be modelled as probability distributions on a set.

The distance of the probability distributions μ and ν is defined by

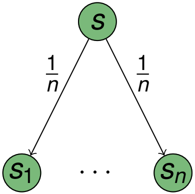
$$d(\mu, \nu) = \max \left\{ \sum_{x \in X} f(x)(\mu(x) - \nu(x)) \;\middle|\; f \in (X, d) \to_{\preceq} [0, 1] \right\}$$

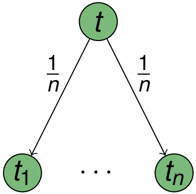
Leonid Kantorovich

Leonid Kantorovich introduced the metric on probability distributions. This metric is known as the Kantorovich metric.



Probabilistic Bisimilarity





States s and t are probabilistic bisimilar if and only if

$$\exists_{\pi \text{ is a permutation}} \forall_{1 \leq i \leq n}\ s_i \text{ and } t_{\pi(i)} \text{ are probabilistic bisimilar}$$

This is generalized by

$$d(s, t) = \min \left\{ \sum_{i=1}^{n} \frac{1}{n} \cdot d(s_i, t_{\pi(i)}) \;\middle|\; \pi \text{ is a permutation} \right\}.$$

Catuscia Palamidessi

Catuscia Palamidessi, in collaboration with Yuxin Deng, Tom Chothia and Jun Pang, combined the Hausdorff metric and the Kantorovich metric to obtain a pseudometric on the state space of a probabilistic automaton and showed

States s and t are probabilistic bisimilar if and only if $d(s, t) = 0$.

Our Main Result

Theorem

The problem of computing the bisimilarity pseudometric introduced by Palamidessi et al. is in **PPAD**.

- Computing Nash equilibria of two player games is **PPAD**-complete.
- Computing values of simple stochastic games is in **PPAD**.
- Computing fixed points of discretized Brouwer functions is in **PPAD**.

Characterizations of Bisimilarity

Bisimilarity for labelled transition systems has been characterized in terms of

- a logic (Hennessy and Milner, 1980),
- a fixed point (Milner, 1980), and
- a game (Stirling, 1993).

Characterizations of Probabilistic Bisimilarity

Probabilistic bisimilarity for probabilistic automata has been characterized in terms of

- a logic (Parma and Segala, 2007), and
- a fixed point (Segala, 1995).

The bisimilarity pseudometric for probabilistic automata has been characterized in terms of

- a logic (De Alfaro et al, 2007),
- a fixed point (Deng et al, 2005).

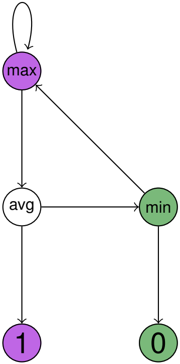
Another Result

Theorem

The bisimilarity distance of two states is the value of a simple stochastic game.

This provides a game theoretic characterization of the bisimilarity pseudometric and also of probabilistic bisimilarity.

Simple Stochastic Game



A Characterization of the Bisimilarity Pseudometric

$$d(s,t) = \max\left\{ \max_{s \to \mu} \min_{t \to \nu} d(\mu,\nu), \max_{t \to \nu} \min_{s \to \mu} d(\nu,\mu) \right\}$$

where

$$\begin{aligned} & d(\mu,\nu) \\ = \;& \max\left\{ \sum_{s \in S} f(s)(\mu(s) - \nu(s)) \;\middle|\; f \in (S,d) \rightarrow\!\!\!\!\!\!\nleftarrow [0,1] \right\} \\ = \;& \min\left\{ \sum_{u,v \in S} \omega(u,v) d(u,v) \;\middle|\; \omega \in \Omega_{\mu,\nu} \right\} \end{aligned}$$

Couplings

The set $\Omega_{\mu,\nu}$ consists of the couplings of μ and ν.

A probability distribution ω on $S \times S$ is a coupling of μ and ν if for all u, $v \in S$,

$$\sum_{v \in S} \omega(u, v) = \mu(u) \text{ and } \sum_{u \in S} \omega(u, v) = \nu(v)$$

The set $\Omega_{\mu,\nu}$ is a convex polytope. We denote its set of vertices by $V(\Omega_{\mu,\nu})$.

A Characterization of the Bisimilarity Pseudometric

$$d(s,t) = \max \left\{ \max_{s \to \mu} \min_{t \to \nu} d(\mu, \nu), \max_{t \to \nu} \min_{s \to \mu} d(\nu, \mu) \right\}$$

where

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