Random Measurable Selections

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PrakashFest —May 25, 2014

What can you do on Sunday, May 25, 2014?

- Mother's day
- European parliamentary elections*
- celebrate Prakash

* Polls in France predict: 60% abstention, and a tie between extreme right wing (Front National, 21%) and ordinary right wing (UMP, 21%); socialists 17%, center 11%, ecologists 9%, etc.

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I hope you'll be pleased with the choice I made, Prakash (but am not sure).

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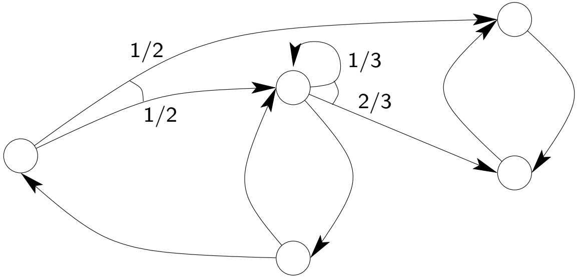
- so I chose him as managing editor for my next MSCS submission.
- He has not tried further.

Outline

Probabilistic Automata

Probabilistic + Non-deterministic choice:



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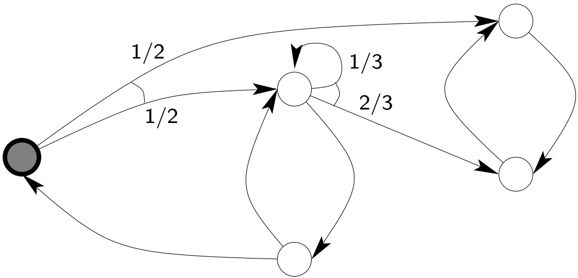

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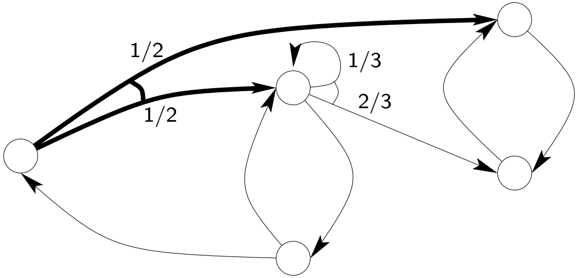

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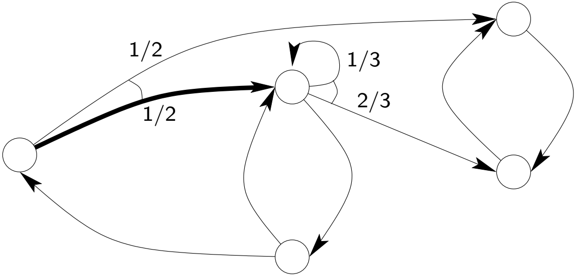

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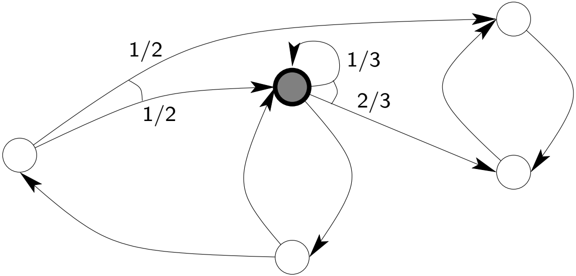

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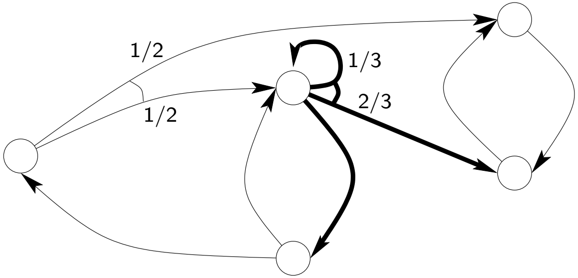

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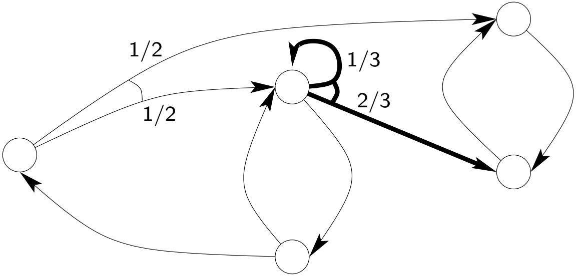

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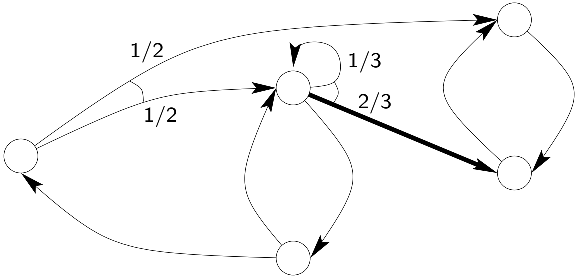

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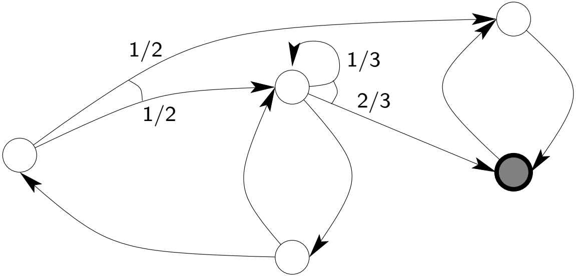

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Segala Automata

In general, transition function θ:

- incorporates *actions* from L:

$$\theta\colon Q \to \mathbb{P}(\text{Transitions})$$

where $\text{Transitions} = L \times \mathcal{P}(Q)$.

- Q and L not assumed finite
  . . . but need to be measurable spaces.

Schedulers

$$\theta\colon Q \to \mathbb{P}(\text{Transitions})$$

Schedulers specify how one can resolve non-determinism.

Definition (Pure schedulers)

σ : Histories $\to$ Transitions
such that $\sigma(\cdots q) \in \theta(q)$

More general:

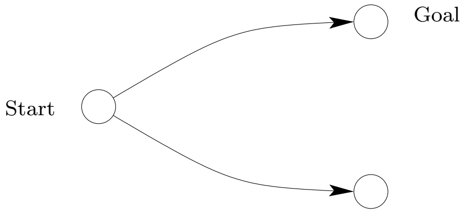
Definition (Randomized schedulers)

η: Histories $\to \mathcal{P}$(Transitions)
such that $\eta(\cdots q)$ prob. concentrated on set $\theta(q)$.

Once scheduler is fixed, system behaves in a **purely probabilistic** way.

Randomized schedulers are needed

What is the prob. of reaching Goal from Start?



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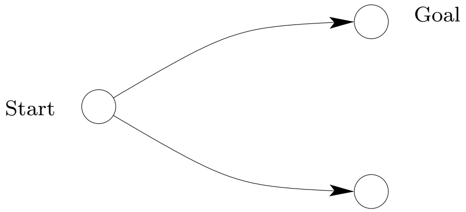

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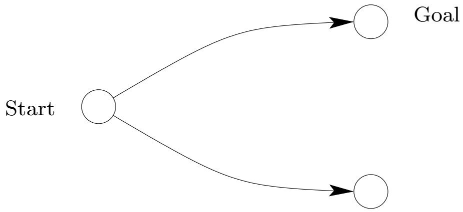

- Using pure schedulers: either 0 or 1
- Randomized schedulers: any $\alpha \in [0, 1]$

Our goal

Conjecture (Randomness for Free)

For every measurable set of paths $\mathcal{E}$,
If $Pr_\eta[\text{trajectory} \in \mathcal{E}] = \alpha$ for some **randomized** scheduler η,
then there are two **pure** schedulers σ^-, σ^+ such that:

$$Pr_{\sigma^-}[\text{trajectory} \in \mathcal{E}] \leq \alpha \leq Pr_{\sigma^+}[\text{trajectory} \in \mathcal{E}]$$

(Under, hopefully reasonable, measure-theoretic assumptions.)

Our goal

Randomness for free = can bound any scheduler by two **pure** ones.

- Algorithms: reduces space of schedulers to search for
- Applies to any $\mathcal{E}$: reachability, ω-reachability, and more
- Known for **finite** state and action spaces [CDGH, MFCS'10]
- We will show that this can be considerably relaxed.

The Chatterjee-Doyen-Gimbert-Henzinger argument

Apply Erdös' principle:

- Given randomized scheduler η,
  draw pure scheduler σ **at random**
  so that $\sigma(h)$ is drawn with probability $\eta(h)$, $h \in$ Histories
- This defines a measure ϖ. Show that:

$$\int_\sigma Pr_\sigma[\text{trajectory} \in \mathcal{E}]d\varpi = Pr_\eta[\text{trajectory} \in \mathcal{E}]$$

- Conclude that σ^-, σ^+ exists so that:

$$Pr_{\sigma^-}[\text{trajectory} \in \mathcal{E}] \leq \alpha \leq Pr_{\sigma^+}[\text{trajectory} \in \mathcal{E}]$$

Oops... what is the σ-**algebra** on the space of pure schedulers?

σ-Algebras on spaces of schedulers

For finite state and action spaces, as in [CDGH10]:

- can take **product** σ-algebra, indexed by histories
- there is a unique measure ϖ as above by standard extension theorems (Carathéodory)

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For general measurable state and action spaces:

- Need to find suitable σ-algebra on a **set of functions** (pure schedulers)
- Turns out to be the one and only difficulty

Outline

Measurable selections

Pure schedulers are essentially the same thing as the following standard notion:

Definition (Selection)

Given a function $\theta\colon Q \to \mathbb{P}^*(T)$, a **selection** of θ is a map $\sigma\colon Q \to T$ such that $\sigma(q) \in \theta(q)$ for every $q \in Q$.

- Selections always exist, by the Axiom of Choice
- We must require **measurable** selections σ, otherwise

$$Pr_\sigma[\text{trajectory} \in \mathcal{E}]$$

makes no sense.
(I hid a lot in this notation. This is defined on specific sets $\mathcal{E}$ by iterated integrals, then extended to all by Carathéodory's extension theorem. See [Cattani, Segala, Kwiatkowska FoSSaCS'05].)

Measurable selections

Pure schedulers are essentially the same thing as the following standard notion:

Definition (Measurable Selection)

Given a function $\theta \colon Q \to \mathbb{P}^*(T)$, a **measurable selection** of θ is a measurable map $\sigma \colon Q \to T$ such that $\sigma(q) \in \theta(q)$ for every $q \in Q$.

- Many, many, many existing measurable selection existence theorems

The measurable space of measurable selections

For $\theta\colon Q \to \mathbb{P}^*(T)$,

Definition (Weak σ-algebra)

Let $Sel(\theta)$ be the space of all measurable selections of θ, with the smallest σ-algebra containing the sets:

$$[q \to E] \stackrel{\text{def}}{=} \{\sigma \in Sel(\theta) \mid \sigma(q) \in E\}$$

for $q \in Q$, E measurable in T.

The theorem

Let Q, T be measurable spaces (sets with a σ-algebra).

Theorem

Assume $\theta\colon Q \to \mathbb{P}^*(T)$ *measurable,* $Sel(\theta) \neq \emptyset$*,*
$\eta\colon Q \to \mathcal{P}(T)$ *measurable with* $\eta(q)$ *supported on* $\theta(q)$*.*
There is a unique probability measure ϖ *on* $Sel(\theta)$ *such that:*

$$\varpi(\bigcap_{i=1}^{n}[q_i \to E_i]) = \prod_{i=1}^{n} \eta(q_i)(E_i)$$

In particular, $\varpi([q \to E]) = \eta(q)(E)$, i.e.,

$$Pr[\sigma(q) \in E] = \eta(q)(E).$$

The proof (1/3)

Goal: find ϖ on $Sel(\theta)$ such that:

$$\varpi(\bigcap_{i=1}^{n}[q_i \to E_i]) = \prod_{i=1}^{n} \eta(q_i)(E_i)$$

- Define ϖ as above.
- Collection $\mathcal{A}$ of sets $\bigcap_{i=1}^{n}[q_i \to E_i]$ is a *semiring*
- By Carathéodory, enough to check σ-additivity **on** $\mathcal{A}$:

$$\varpi(\biguplus_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \varpi(A_n)$$

  for all A_n **in** $\mathcal{A}$ whose union **is in** $\mathcal{A}$.
- How do you do it? Any guess?

The proof (2/3): the Łomnick-Ulam trick

The Łomnick-Ulam trick

To check:

$$\varpi(\biguplus_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \varpi(A_n),$$

find a measure μ that coincides with ϖ on A_n, $n \in \mathbb{N}$ and their union.

- μ **not** required to coincide with ϖ on whole σ-algebra
- Here each A_n is a finite intersection of sets $[q_{ni} \to E_{ni}]$... so only **countably many** states q_{ni} involved...

is that enough of a hint?

The proof (3/3)

- Build product measure $\mu_0 = \bigotimes_{n,i} \eta(q_{ni})$ on $T^{\mathbb{N}}$

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  . . . **countably independent** choice
- Pick fixed measurable selection $\sigma_0 \in Sel(\theta)$
- Defines measurable map: $$\begin{array}{rcl} patch\colon T^{\mathbb{N}} & \to & Sel(\theta) \\ \vec{t_{ni}} & \mapsto & \sigma_0[\overrightarrow{q_{ni} \mapsto t_{ni}}] \end{array}$$

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- Image measure $\mu = patch[\mu_0]$ implements random choice of $\sigma \in Sel(\theta)$ such that:
  - $\sigma(q_{ni})$ chosen **independently**
  - with probability $\eta(q_{ni})$

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- Image measure $\mu = patch[\mu_0]$ implements random choice of $\sigma \in Sel(\theta)$ such that:
  - $\sigma(q_{ni})$ chosen **independently**
  - with probability $\eta(q_{ni})$
- Recall $A_n = \bigcap_i [q_{ni} \to E_{ni}]$. By construction:

$$\varpi(\biguplus_{n\in\mathbb{N}} A_n) = \mu(\biguplus_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n) = \sum_{n\in\mathbb{N}} \varpi(A_n),$$

- $\Rightarrow \varpi$ σ-additive on $\mathcal{A}$.

Outline

Summary

- One can pick measurable selections σ of measurable $\theta\colon Q \to \mathbb{P}^*(T)$ **at random** so that $Pr[\sigma(q) \in E] = \eta(q)(E)$ for every $q \in Q$
- such that any **countable** collection of values $\sigma(q)$ is **independent**—yet σ is **measurable**.
- Corollary: can sample integrals by drawing σ at random:

$$\int_\sigma h(\sigma(q))d\varpi = \int_{t \in T} h(t)d\eta(q)$$

for every measurable map h.

Conclusion

Extension (see paper—I tried hard to make it readable!)

- One can pick measurable schedulers σ of probabilistic automata **at random**
  so that

$$Pr_\eta[\text{trajectory} \in \mathcal{E}] = \int_\sigma Pr_\sigma[\text{trajectory} \in \mathcal{E}] d\varpi$$

- In particular, **randomness for free**: there are σ^-, σ^+ such that:

$$Pr_{\sigma^-}[\text{trajectory} \in \mathcal{E}] \leq Pr_\eta[\text{trajectory} \in \mathcal{E}] \leq Pr_{\sigma^+}[\text{trajectory} \in \mathcal{E}]$$

- Proof = elaboration of previous proof
- Need a more complicated σ-algebra,
  and Q, T should have *measurable diagonals*.