Distributed Probabilistic Strategies

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Generalise domain theory. An aim in developing distributed games and strategies is to tackle anomalies like non-deterministic dataflow, and repair the divide between denotational and operational semantics.

Distributed games, with behaviour based on event structures, rather than trees. The extra generality reveals new structure and a mathematical robustness to the concept of strategy—showing strategies are (special) profunctors.
$\rightsquigarrow$ a language for strategies.

Extension with probability to probabilistic strategies, and quantum strategies?

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Event structures

An **event structure** comprises $(E, \leq, \mathrm{Con})$, consisting of a set of *events* E

- partially ordered by $\leq$, the **causal dependency relation**, and

- a nonempty family Con of finite subsets of E, the **consistency relation**,

which satisfy

$$\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,$$

$$\{e\} \in \mathrm{Con} \text{ for all } e \in E,$$

$$Y \subseteq X \in \mathrm{Con} \Rightarrow Y \in \mathrm{Con}, \quad \text{and}$$

$$X \in \mathrm{Con} \ \& \ e \leq e' \in X \Rightarrow X \cup \{e\} \in \mathrm{Con}.$$

Say e, e' are **concurrent** if $\{e, e'\} \in \mathrm{Con} \ \& \ e \not\leq e' \ \& \ e' \not\leq e$.
In games the relation of **immediate dependency** $e \rightarrowtriangle e'$, meaning e and e' are distinct with $e \leq e'$ and no event in between, will play an important role.

Configurations of an event structure

The **configurations**, $\mathcal{C}^\infty(E)$, of an event structure E consist of those subsets $x \subseteq E$ which are

Consistent: $\forall X \subseteq_{\mathrm{fin}} x.\ X \in \mathrm{Con}$ and

Down-closed: $\forall e, e'.\ e' \leq e \in x \Rightarrow e' \in x$.

Often concentrate on the **finite configurations** $\mathcal{C}(E)$.

Structural maps of event structures

A **map** of event structures $f : E \to E'$
is a partial function on events $f : E \rightharpoonup E'$ such that for all $x \in \mathcal{C}(E)$

$fx \in \mathcal{C}(E')$ and

if $e_1, e_2 \in x$ and $f(e_1) = f(e_2),$ then $e_1 = e_2$. *(local injectivity)*

An **affine map** $f : E \to_a E'$ is defined similarly but now $f\emptyset$ maybe a nonempty configuration of E'.
It comprises a configuration $y \in \mathcal{C}(E')$ and a map of event structures $f_1 : E \to E'/y$ where E'/y denotes the event structure after y.
By definition $fx =_{\mathrm{def}} y \cup f_1 x$.

Distributed games

Games and strategies are represented by **event structures with polarity**, an event structure $(E, \leq, \mathrm{Con})$ where events E carry a polarity $+/-$ (Player/Opponent), respected by maps.

(Simple) Parallel composition: $A \| B$, by juxtaposition.

Dual, $B^{\perp}$, of an event structure with polarity B is a copy of the event structure B with a reversal of polarities; this switches the roles of Player and Opponent.

Distributed plays and strategies

A **nondeterministic play** in a game A is represented by a total map

$$\sigma : S \to A$$

preserving polarity; S is the event structure with polarity describing the moves played.

A **strategy in** a game A is a (**special**) nondeterministic play $\sigma : S \to A$.

A **strategy from** A **to** B is a strategy in $A^\perp \parallel B$, so $\sigma : S \to A^\perp \parallel B$.

[Conway, Joyal]

NB: A strategy in a game A is a strategy for Player;
a strategy for Opponent - a counter-strategy - is a strategy in $A^\perp$*.*

Example of a strategy: copy-cat strategy from A to A

$$\mathbb{C}\mathbb{C}_A$$

$$\begin{tikzcd}
A^{\perp} & & & A \\
\overline{a}_2 & \ominus \arrow[r, dashed] & \oplus & a_2 \\
\overline{a}_1 & \oplus \arrow[u] & \ominus \arrow[l, dashed] \arrow[u] & a_1
\end{tikzcd}$$

Composition of strategies $\sigma : S \to A^{\perp} \| B$, $\tau : T \to B^{\perp} \| C$

Via pullback. Ignoring polarities, the composite partial map

$$
\begin{tikzcd}
 & P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45] & \\
S \| C \arrow[dr, "\sigma \| C"'] & & A \| T \arrow[dl, "A \| \tau"] \\
 & A \| B \| C \arrow[d] & \\
 & A \| C &
\end{tikzcd}
$$

has the partial-total map factorization: $P \longrightarrow T \odot S \xrightarrow{\tau \odot \sigma} A \| C$.

For copy-cat to be identity w.r.t. composition

a strategy in a game A has to be $\sigma : S \to A$, a total map of event structures with polarity, such that
(i) whenever $\sigma x \subseteq^{-} y$ in $\mathcal{C}(A)$ there is a unique $x' \in \mathcal{C}(S)$ so that

$x \subseteq x'$ & $\sigma x' = y$, *i.e.*

$$\begin{tikzcd} x \arrow[r, dotted, "\subseteq"] \arrow[d, "\sigma"'] & x' \arrow[d, dotted, "\sigma"] \\ \sigma x \arrow[r, phantom, "\subseteq^{-}"] & y \end{tikzcd}$$

and

(ii) whenever $y \subseteq^{+} \sigma x$ in $\mathcal{C}(A)$ there is a (necessarily unique) $x' \in \mathcal{C}(S)$ so that

$x' \subseteq x$ & $\sigma x' = y$, *i.e.*

$$\begin{tikzcd} x' \arrow[r, dotted, "\subseteq"] \arrow[d, dotted, "\sigma"'] & x \arrow[d, "\sigma"] \\ y \arrow[r, phantom, "\subseteq^{+}"] & \sigma x \end{tikzcd}$$

The only immediate causal dependencies a strategy can introduce: $\ominus \rightarrowtriangle \oplus$

An alternative characterization of strategies

Defining a partial order — *the Scott order* — on configurations of A

$$y \sqsubseteq_A x \quad \text{iff} \quad y \supseteq^- \cdot \subseteq^+ \cdot \supseteq^- \cdots \supseteq^- \cdot \subseteq^+ x$$

we obtain a factorization system $((\mathcal{C}(A), \sqsubseteq_A), \supseteq^-, \subseteq^+)$, $i.e.$ $\exists! z.$

$$\begin{tikzcd} & & x \\ y \arrow[urr, dotted, "\sqsubseteq"] \arrow[rr, phantom, "\supseteq^-"] & & z \arrow[u, phantom, "\cup\!|^{+}"] \end{tikzcd} .$$

Proposition $z \in \mathcal{C}(\mathbb{C}\!\mathbb{C}_A)$ iff $z_2 \sqsubseteq_A z_1$.

Theorem Strategies $\sigma : S \to A$ correspond to discrete fibrations

$$\sigma\text{``} : (\mathcal{C}(S), \sqsubseteq_S) \to (\mathcal{C}(A), \sqsubseteq_A)\,, \quad i.e.\ \exists! x'. \quad \begin{tikzcd} x' \arrow[d, dotted, "\sigma\text{``}"'] \arrow[rr, dotted, phantom, "\sqsubseteq_S"] & & x \arrow[d, "\sigma\text{``}"] \\ y \arrow[rr, phantom, "\sqsubseteq_A"] & & \sigma\text{``}(x) \end{tikzcd} ,$$

which preserve $\supseteq^-$, $\subseteq^+$ and $\emptyset$.

$\rightsquigarrow$ *A lax functor from strategies to profunctors ...*

A bicategory of games

Objects are event structures with polarity—the games, A, B, ... ;

arrows $\sigma : A \dashrightarrow B$ are strategies $\sigma : S \to A^\perp \| B$;

2-cells are (the obvious) maps of strategies.

The vertical composition of 2-cells is the usual composition of maps. Horizontal composition is given by the composition of strategies $\odot$ (which extends to a functor on 2-cells via the universality of pullback).

Duality: $\sigma : A \dashrightarrow B$ *corresponds to* $\sigma^\perp : B^\perp \dashrightarrow A^\perp$, *as* $A^\perp \| B \cong (B^\perp)^\perp \| A^\perp$. *The bicategory of strategies is rich in structure, in particular, it is compact-closed (so has a trace, a feedback operation extending that of nondeterministic dataflow)—though with the addition of the extra features of winning conditions or pay-off, compact closure will weaken to $*$-autonomy.*

From strategies to probabilistic strategies

$$\begin{array}{c} S \\ \big\downarrow \sigma \\ A \end{array}$$

Aim

(1) To endow S with probability, while

(2) taking account of the fact that in a strategy Player can't be aware of the probabilities assigned by Opponent.

Probabilistic event structures

A **probabilistic event structure** comprises an event structure $E = (E, \leq, \mathrm{Con})$ together with a *(normalized) continuous valuation*, *i.e.* a function v from the Scott open subsets of configurations $\mathcal{C}^\infty(E)$ to $[0,1]$ which is

(normalized) $v(\mathcal{C}^\infty(E)) = 1$ **(strict)** $v(\emptyset) = 0$

(monotone) $U \subseteq V \Rightarrow v(U) \leq v(V)$

(modular) $v(U \cup V) + v(U \cap V) = v(U) + v(V)$

(continuous) $v(\bigcup_{i \in I} U_i) = \sup_{i \in I} v(U_i)$ for *directed* unions $\bigcup_{i \in I} U_i$.

Intuition: $v(U)$ is the probability of the result being in U.
A cts valuation extends to a probability measure on Borel sets of configurations.

A workable characterization. A **probabilistic event structure** comprises an event structure E with a *configuration-valuation* $v : \mathcal{C}(E) \to [0,1]$ which satisfies

(normalized) $v(\emptyset) = 1$ and

(non −ve drop) $d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0$, for all $n \in \omega$, and $y \subseteq x_1, \cdots, x_n$ in $\mathcal{C}(E)$.

For $y \subseteq x_1, \cdots, x_n$ in $\mathcal{C}(E)$,

$$d_v^{(n)}[y; x_1, \cdots, x_n] =_{\text{def}} v(y) - \sum_I (-1)^{|I|+1} v(\bigcup_{i \in I} x_i)\,,$$

—the index I ranges over $\emptyset \neq I \subseteq \{1, \cdots, n\}$ s.t. $\{x_i \mid i \in I\}$ is compatible.
(Sufficient to check the 'drop condition' for $y \longrightarrow\!\!\!\!\!\subset x_1, \cdots, x_n$)

Theorem. Continuous valuations restrict to configuration-valuations.
A configuration-valuation extends to a unique continuous valuation on open sets, and that to a unique probabilistic measure on Borel subsets of configurations.
(*The result holds in greater generality, for Scott domains*)

Probabilistic event structure with polarities

Let E be an event structure in which (not necessarily all) events carry $+/-$.
Write $x \subseteq^p y$ if $x \subseteq y$ and no event in $y \setminus x$ has polarity $-$.

Now, a **configuration-valuation** is a function $v : \mathcal{C}(E) \to [0,1]$ for which

$$v(\emptyset) = 1, \qquad\qquad x \subseteq^- y \Rightarrow v(x) = v(y) \text{ , for all } x, y \in \mathcal{C}(E),$$

and the "drop condition"

$$d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0$$

for all $n \in \omega$ and $y \subseteq^p x_1, \cdots, x_n$ in $\mathcal{C}(E)$.

A **probabilistic event structure with polarity** comprises E an event structure with polarity together with a configuration-valuation $v_E : \mathcal{C}(E) \to [0,1]$.

Probabilistic strategies

Assume games are *race-free*, $i.e.$ there is no immediate conflict between events of opposite polarity. $E.g.\ ban\ \ \ominus \rightsquigarrow \oplus$.

A **probabilistic strategy** in A comprises S, v_S, a probabilistic event structure with polarity, and a strategy $\sigma : S \to A$.

A race-free game A has a **probabilistic copy-cat** by taking $v_{\mathbb{C}_A}$ constantly 1 —this is a configuration-valuation as $\mathbb{C}_A$ is deterministic for race-free A.

For the **composition** $\tau \odot \sigma$ endow the pb P with configuration-valuation $v(x) = v_S(\pi_1 x) \times v_T(\pi_2 x)$. This forms a configuration-valuation because assuming $\pi_1 y {-\!\!\subset}^+ \pi_1 x_i$ when $1 \leq i \leq m$ and $\pi_2 y {-\!\!\subset}^+ \pi_2 x_i$ when $m+1 \leq i \leq n$,

$$d_v^{(n)}[y; x_1, \cdots, x_n] = d_v^{(m)}[\pi_1 y; \pi_1 x_1, \cdots, \pi_1 x_m] \times d_v^{(n-m)}[\pi_2 y; \pi_2 x_{m+1}, \cdots, \pi_2 x_n]\,.$$

$\rightsquigarrow$ A bicategory of games and probabilistic strategies

Objects are race-free games A, B, C, ... ;

arrows $\sigma : A \mathrel{+\!\!\!\rightarrow} B$ are probabilistic strategies $\sigma : S \to A^{\perp} \| B$ with configuration valuation $v : \mathcal{C}(S) \to [0, 1]$;

2-cells
$$
\begin{tikzcd}
A \arrow[r, bend left, "{\sigma', v'}" , "|" marking] \arrow[r, bend right, "{\sigma, v}"', "|" marking] \arrow[r, phantom, "\Uparrow f"] & B
\end{tikzcd}
$$
are total maps $f : S \to S'$ making
$$
\begin{tikzcd}
S \arrow[r, "f"] \arrow[dr, "\sigma"'] & S' \arrow[d, "\sigma'"] \\
& A^{\perp} \| B
\end{tikzcd}
$$
commute which *reflect +-compatibility, i.e.*

$$x \subseteq^{+} x_1 \;\&\; x \subseteq^{+} x_2 \;\&\; f x_1 \uparrow f x_2 \Rightarrow x_1 \uparrow x_2 \,,$$

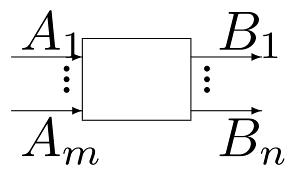
and where $v(x) = v'(fx)$ for all $x \in \mathcal{C}(S)$. *Includes rigid embeddings and* $\trianglelefteq$.
(*Currently I'm working on more general two-cells.*)

Constructions on probabilistic strategies

Types: Race-free games A, B, C, . . . with operations $A^\perp$, $A\|B$, sums $\sum_{i\in I} A_i$, recursively-defined types, . . .

A **term**

$$x_1 : A_1, \cdots, x_m : A_m \vdash \ t \ \dashv y_1 : B_1, \cdots, y_n : B_n \ ,$$



denotes a strategy $A_1\| \cdots \|A_m \;\not\to\; B_1\| \cdots \|B_n$.

Idea: t denotes a strategy $S \to \vec{A}^\perp \| \vec{B}$.
The term t describes witnesses, finite configurations of S, to a relation between finite configurations $\vec{x}$ of $\vec{A}$ and $\vec{y}$ of $\vec{B}$. Cf. profunctors.

Duality and Composition

Duality of input and output:

$$\frac{\Gamma, x : A \vdash t \dashv \Delta}{\Gamma \vdash t \dashv x : A^{\perp}, \Delta}$$

because t denotes a strategy in $(\Gamma^{\perp} \| A^{\perp}) \| \Delta \cong \Gamma^{\perp} \| (A^{\perp} \| \Delta)$.

Composition of strategies:

$$\frac{\Gamma \vdash t \dashv \Delta \qquad \Delta \vdash u \dashv \mathrm{H}}{\Gamma \vdash \exists \Delta. [\, t \parallel u \,] \dashv \mathrm{H}}$$

When Δ is empty, this yields simple parallel composition $t \| u$.

Hom-set terms

Copy-cat on A, $\quad x : A \vdash \; y \sqsubseteq_A x \; \dashv y : A \qquad$ or $\qquad \vdash y \sqsubseteq_A x \dashv x : A^\perp, y : A\,.$

$$\begin{array}{ccc} \oplus & \leftarrow & \ominus \\ \ominus & \rightarrow & \oplus \end{array}$$

Generally, when $f : A \to_a C$ and $g : B \to_a C$ are affine maps s.t. $g\emptyset \sqsubseteq_C f\emptyset$

$$x : A \vdash \; gy \sqsubseteq_C fx \; \dashv y : B$$

denotes a deterministic strategy—with configuration valuation constantly 1—provided f *reflects* $-$*-compatibility* and g *reflects* $+$*-compatibility.*

Lifting maps and shifting strategies

An affine map $f : A \to B$ of games which reflects $-$-compatibility lifts to a deterministic strategy $f_! : A \mathrel{+\!\!\!\to} B$:

$$x : A \vdash \; y \sqsubseteq_B fx \; \dashv y : B\,.$$

An affine map $f : A \to B$ of games which reflects $+$-compatibility lifts to a deterministic strategy $f^* : B \mathrel{+\!\!\!\to} A$:

$$y : B \vdash \; fx \sqsubseteq_B y \; \dashv \; x : A\,.$$

$f^* \odot t$ denotes the **pullback** of a strategy t in B across the map $f : A \to B$. It can introduce extra events and dependencies in the strategy. *It subsumes prefixing.*

Pullback of strategies

$$\frac{\Gamma \vdash t_1 \dashv \Delta \quad \Gamma \vdash t_2 \dashv \Delta}{\Gamma \vdash t_1 \wedge t_2 \dashv \Delta}$$

In the case where t_1 and t_2 denote the respective probabilistic strategies v_1, $\sigma_1 : S_1 \to \Gamma^\perp \| \Delta$ and v_2, $\sigma_2 : S_2 \to \Gamma^\perp \| \Delta$ the strategy $t_1 \wedge t_2$ denotes the pullback

$$\begin{tikzcd}
& S_1 \wedge S_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, dotted, "\sigma_1 \wedge \sigma_2"] & \\
S_1 \arrow[dr, "\sigma_1"'] & & S_2 \arrow[dl, "\sigma_2"] \\
& \Gamma^\perp \| \Delta\,. &
\end{tikzcd}$$

with configuration valuation $x \mapsto v_1(\pi_1 x) \times v_2(\pi_2 x)$ for $x \in \mathcal{C}(S_1 \wedge S_2)$.

Probabilistic sum of strategies

In the **probabilistic sum of strategies**, in the same game, the strategies are glued together on their initial Opponent moves (to maintain receptivity) and only commit to a component with the occurrence of a Player move. For I countable and a sub-probability distribution $p_i, i \in I$,

$$\frac{\Gamma \vdash t_i \dashv \Delta \quad i \in I}{\Gamma \vdash \Sigma_{i \in I} p_i t_i \dashv \Delta\,.}$$

We use $\perp$ for the **empty probabilistic sum**, when the rule above specialises to

$$\Gamma \vdash \perp \dashv \Delta\,,$$

which denotes the minimum strategy in the game $\Gamma^\perp \| \Delta$—it comprises the initial segment of the game $\Gamma^\perp \| \Delta$ consisting of all the initial Opponent events of A.

Duplication

We duplicate arguments through a probabilistic strategy $\delta_A : A \nrightarrow A \| A$.
In the absence of probability δ_A *forms a comonoid with counit* $\perp : A \nrightarrow \emptyset$.
The general defn is involved, but *e.g.*,

if $A = \oplus$, δ_A is

$$
\begin{tikzcd}
 & \oplus\ 1 \\
\ominus \arrow[ur] \arrow[dr] & \\
 & \oplus\ 1\,,
\end{tikzcd}
$$

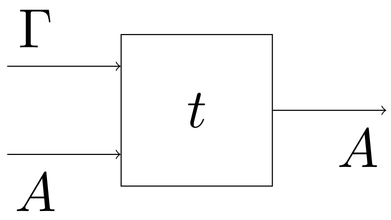
if $A = \ominus$, δ_A is

$$
\begin{tikzcd}
1/2\ \oplus \arrow[d, no head, squiggly] & \ominus \arrow[l] \\
1/2\ \oplus & \ominus\,. \arrow[l]
\end{tikzcd}
$$

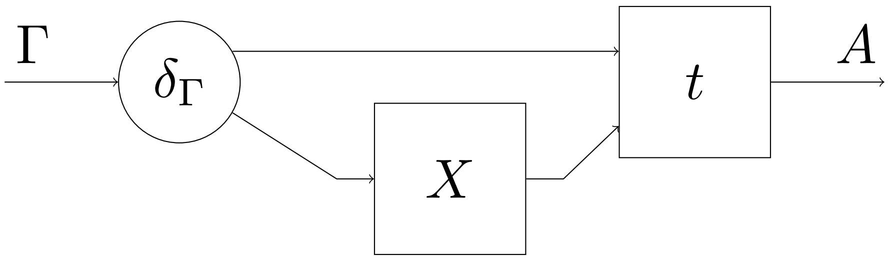
$\rightsquigarrow$ **duplication terms** such as $\quad x : A \vdash \ \delta(x, y_1, y_2) \ \dashv y_1 : A, y_2 : A.$

Recursion

Given $x : A, \Gamma \vdash\ t\ \dashv y : A$,



the term $\Gamma \vdash\ \mu\, x\!:\!A.\, t\ \dashv y : A$ denotes the $\trianglelefteq$-least fixed point amongst strategies $X : \Gamma \twoheadrightarrow A$ of $F(X) = t \odot (\mathrm{id}_\Gamma \| X) \odot \delta_\Gamma$:



As δ_Γ is not a comonoid - it is not associative - not all the "usual" laws of recursion will hold.

Quantum strategies—sketch

A **quantum game** comprises a *quantum event structure A, Q with initial state ρ*, where each event of A has a polarity Player/Opponent.
A **quantum strategy** is a probabilistic strategy $v_s,\ \sigma : S \to A$.

The **play of a strategy** $v_S, \sigma : S \to A$ **against a counter-strategy** $v_T, \tau : T \to A^\perp$ results in a probabilistic quantum experiment P, v_P with $f : P \to A$ where P the pullback of σ, τ and $f =_{\mathrm{def}} \sigma\pi_1 = \tau\pi_2$ in

$$\begin{tikzcd} & P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\ S \arrow[dr, "\sigma"'] & & T \arrow[dl, "\tau"] \\ & A\,. & \end{tikzcd}$$

Quantum strategies inherit types and operations from probabilistic strategies, though need extra type constructions to introduce new *entanglement.*

Welcome to 60 Prakash!

Stay young at heart!