Pseudospectral Bound on the Norm of a Matrix Function

THEOREM. Let f be a function that is analytic on $\Lambda_\varepsilon(A)$ for a fixed $\varepsilon > 0$. Then provided the boundary $\partial\Lambda_\varepsilon(A)$ of $\Lambda_\varepsilon(A)$ consists of a finite union of Jordan curves,

$$\|f(A)\| \le \frac{\mathcal{L}(\partial\Lambda_\varepsilon(A))}{2\pi\varepsilon} \max_{z\in\Lambda_\varepsilon(A)} |f(z)|,$$

where $\mathcal{L}(\partial\Lambda_\varepsilon(A))$ denotes the arc length of $\partial\Lambda_\varepsilon(A)$.

Notation. This result holds for any norm.

Proof. Given our assumptions on f and $\partial\Lambda_\varepsilon(A)$, we can write $f(A)$ as the Dunford integral,

$$f(A) = \frac{1}{2\pi i}\int_{\partial} \Lambda_\varepsilon(A)(zI - A)^{-1} f(z)\, dz;$$

see Theorem VII.9.4 of [DS58]. The result follows by coarsely bounding this integral,

$$\begin{aligned}
\|f(A)\| &= \frac{1}{2\pi}\left\|\int_{\partial\Lambda_\varepsilon(A)} (zI - A)^{-1} f(z)\, dz\right\| \\
&\le \frac{1}{2\pi}\int_{\partial\Lambda_\varepsilon(A)} \|(zI - A)^{-1}\|\, |f(z)|\, |dz| \\
&= \frac{1}{2\pi\varepsilon}\int_{\partial\Lambda_\varepsilon(A)} |f(z)|\, |dz| \\
&\le \frac{\mathcal{L}(\partial\Lambda_\varepsilon(A))}{2\pi\varepsilon} \max_{z\in\Lambda_\varepsilon(A)} |f(z)|. \qquad \blacksquare
\end{aligned}$$

History. This result (with f taken to be a polynomial), was first presented by Trefethen as part of a pseudospectral bound for convergence of the GMRES algorithm for solving linear systems [Tre90]. Greenbaum has further analyzed this bound for arbitrary analytic functions f when each connected component of $\Lambda_\varepsilon(A)$ contains no more than one (possibly repeated) eigenvalue [Gre00].

Bibliography.

[DS58] N. Dunford and J. T. Schwarz. *Linear Operators: Part I: General Theory.* Wiley-Interscience, New York, 1958.

[Gre00] A. Greenbaum. *Using the Cauchy integral formula and the partial fractions decomposition of the resolvent to estimate* $\|f(A)\|$. Manuscript, 2000.

[Tre90] L. N. Trefethen. *Approximation theory and numerical linear algebra.* In *Algorithms for Approximation II*, J. C. Mason and M. G. Cox, eds., Chapman and Hall, London, 1990.