Local orthogonality: a multipartite principle for (quantum) correlations

Antonio Acín
ICREA Professor at ICFO-Institut de Ciencies Fotoniques, Barcelona

Quantum Information Sciences Workshop, Oxford, UK, March 2012

ICFO Collaborators

Remigiusz Augusiak

Jonatan Bohr Brask

Rafael Chaves

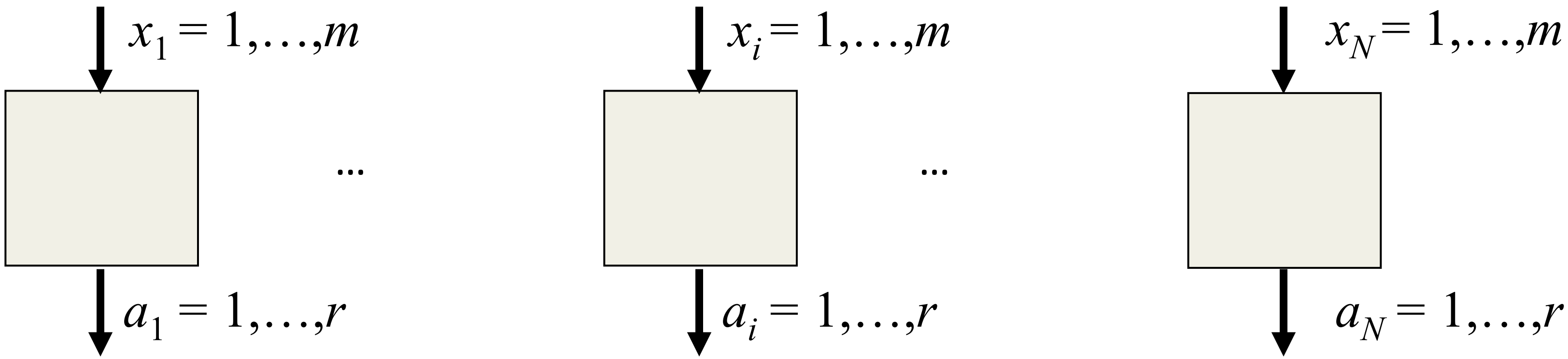
Tobias Fritz

Anthony Leverrier

Belén Saínz

Box-World Scenario

N distant parties performing m different measurements of r outcomes.



$$p(a_1,\ldots,a_N \mid x_1,\ldots,x_N)$$

Physical Correlations

Physical principles translate into limits on correlations.

No-signalling correlations: correlations compatible with the no-signalling principle, i.e. the impossibility of instantaneous communication.

$$\sum_{a_{k+1},\ldots,a_N} p(a_1,\ldots,a_N | x_1,\ldots,x_N) = p(a_1,\ldots,a_k | x_1,\ldots,x_k)$$

Physical Correlations

Physical principles translate into limits on correlations.

No-signalling correlations: correlations compatible with the no-signalling principle, i.e. the impossibility of instantaneous communication.

$$\sum_{a_{k+1},\ldots,a_N} p(a_1,\ldots,a_N | x_1,\ldots,x_N) = p(a_1,\ldots,a_k | x_1,\ldots,x_k)$$

For a finite number of measurements and results, these correlations define a polytope, a convex set with a finite number of extreme points.

Physical Correlations

The set of no-signalling correlations defines again a polytope.

Physical Correlations

Classical correlations: correlations established by classical means.

$$p(a_1,\ldots,a_N|x_1,\ldots,x_N) = \sum_\lambda p(\lambda) D(a_1|x_1,\lambda)\ldots D(a_N|x_N,\lambda)$$

These are the standard "EPR" correlations. Independently of fundamental issues, these are the correlations achievable by classical resources. Bell inequalities define the limits on these correlations.

The set of no-signalling correlations defines again a polytope.

Physical Correlations

Quantum correlations: correlations established by quantum means.

$$p(a_1,\ldots,a_N|x_1,\ldots,x_N) = \mathrm{tr}\left(\rho\, M_{a_1}^{x_1} \otimes \cdots \otimes M_{a_N}^{x_N}\right)$$

$$\sum_{a_i} M_{a_i}^{x_i} = 1 \qquad M_{a'_i}^{x_i} M_{a_i}^{x_i} = \delta_{a_i a'_i} M_{a_i}^{x_i}$$

The set of quantum correlations is again convex, but not a polytope, even if the number of measurements and results is finite.

Physical Correlations

Bell

$$\boxed{C \subset Q \subset NS}$$

Tsirelson

Popescu-Rohrlich

Physical Correlations

Bell

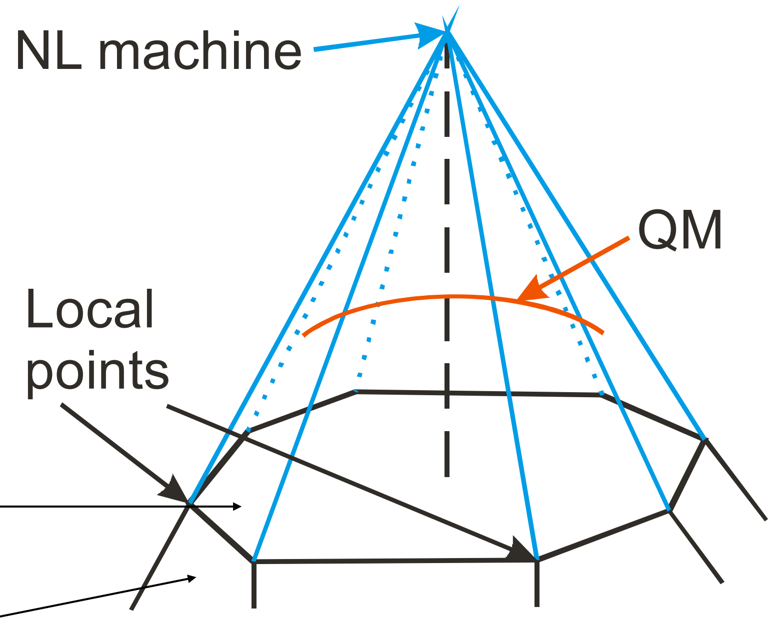
$$C \subset Q \subset NS$$

Tsirelson

Popescu-Rohrlich

Example: 2 inputs
of 2 outputs

$$a, b, x, y = 0,1$$



CHSH inequality

Trivial facets

$\beta_N = 4$

$\beta_Q = 2\sqrt{2}$

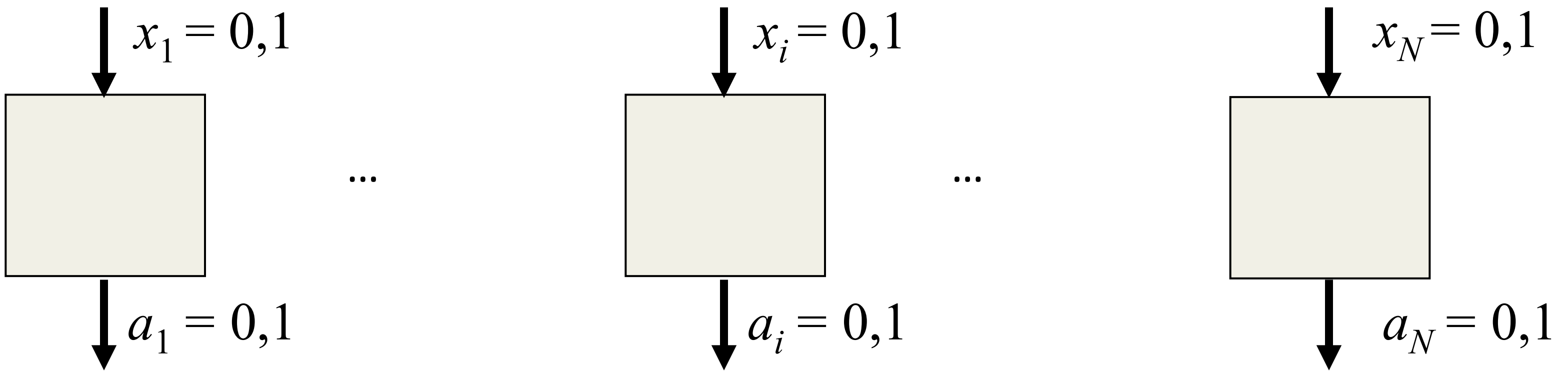
$\beta_L = 2$

Why quantum correlations?

- Information principles have been proposed as the mechanism to bound quantum correlations.
- No-trivial communication complexity (Van Dam): the existence of some supra-quantum correlations would make communication complexity trivial. Some supra-quantum correlations contradict this principle. Open for general supra-quantum correlations.
- Information Causality (Pawloswski et al.): “by sending one bit one cannot send more information than one bit”. It gives the quantum violation of the CHSH Bell inequality (Tsirelson bound). Open for general supra-quantum correlations.
- Macroscopic locality (Navascués and Wunderlich): correlations cannot lead to violation of Bell inequalities in the macroscopic limit. It also reproduces the Tsirelson bound. Larger than quantum correlations.
- Hierarchy of SDP (Navascués-Pironio-Acín): it gives quantum correlations. No operational meaning.

Why multipartite principles?

Guess Your Neighbour's Input (GYNI)



The outcome of party *i* should be equal to the input of party *i*+1: $a_i = x_{i+1}$

$$\beta = \sum_{x_1,\ldots,x_N} p\left(a_1 = x_2, \ldots, a_N = x_1 \middle| x_1, \ldots, x_N\right)$$

One can see that: $\beta_{\mathrm{L}} = \beta_{\mathrm{Q}} = \beta_{\mathrm{NS}} = 2$

Guess Your Neighbour's Input (GYNI)

The picture becomes more interesting if a promise on the inputs is considered. This promise is given by a binary function of the inputs *f*.

$$\beta = \sum_{x_1,\ldots,x_N} f(x_1,\ldots,x_N)\, p(a_1 = x_2,\ldots,a_N = x_1 | x_1,\ldots,x_N)$$

The local and quantum bound are equal for all promises. For some promises, this bound is equal to one, while non-signalling correlations provide a larger value.

Guess Your Neighbour's Input (GYNI)

The picture becomes more interesting if a promise on the inputs is considered. This promise is given by a binary function of the inputs *f*.

$$\beta = \sum_{x_1,\ldots,x_N} f(x_1,\ldots,x_N)\, p(a_1 = x_2,\ldots,a_N = x_1 | x_1,\ldots,x_N)$$

The local and quantum bound are equal for all promises. For some promises, this bound is equal to one, while non-signalling correlations provide a larger value.

Example: 3 parties, *f* is taken to be one whenever $x_1 \oplus x_2 \oplus x_3 = 0$

$$p(000|000) + p(110|011) + p(011|101) + p(101|110) \leq 1$$

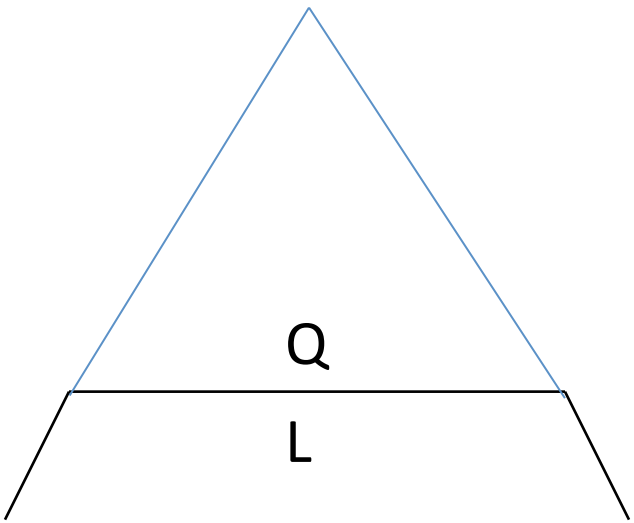
The maximal value of this expression for non-signalling correlations is 4/3.

Guess Your Neighbour's Input (GYNI)

Generalization: N parties, f is one whenever $\begin{cases} x_1 \oplus \ldots \oplus x_N = 0 & \text{Odd } N \\ x_1 \oplus \ldots \oplus x_{N-1} = 0 & \text{Even } N \end{cases}$

- **We observe that the inequalities are tight for N = 3,...,7.** $\beta_{\mathrm{L}} = \beta_{\mathrm{Q}} = 1 < \beta_{\mathrm{NS}}$

Almeida et al, PRL'10



First tight task with no quantum violation.

GYNI and Gleason's correlations

Gleason's Theorem: given a Hilbert space of finite dimension d, all maps v such that

- $\forall X > 0, \quad 0 \leq v(X) \leq 1$
- $\text{If } X_i > 0 \text{ and } \sum_i X_i = 1, \text{ then } \sum_i v(X_i) = 1$

have the quantum form $v(X) = \mathrm{tr}(\tilde{n}\, X)$.

The Theorem shows that, if the quantum structure for the measurements is assumed, in terms of positive operators summing up to the identity, the Born rule follows.

Gleason's correlations

The theorem was generalized to a multipartite scenario. All the maps from measurement operators on Alice and Bob's side to joint probabilities compatible with the no-signalling principle, i.e., such that:

Gleason's correlations

The theorem was generalized to a multipartite scenario. All the maps from measurement operators on Alice and Bob's side to joint probabilities compatible with the no-signalling principle, i.e., such that:

- $\forall A, B > 0, \quad 0 \leq v(A, B) \leq 1$
- If $A_i, B_i > 0$ and $\sum_i A_i = \sum_i B_i = 1$, then $\sum_{i,j} v(A_i, B_j) = 1$

Gleason's correlations

The theorem was generalized to a multipartite scenario. All the maps from measurement operators on Alice and Bob's side to joint probabilities compatible with the no-signalling principle, i.e., such that:

- $\forall A, B > 0, \quad 0 \leq v(A, B) \leq 1$
- If $A_i, B_i > 0$ and $\sum_i A_i = \sum_i B_i = 1$, then $\sum_{i,j} v(A_i, B_j) = 1$
- If $A, B_i, \widetilde{B}_i > 0$ and $\sum_i B_i = \sum_i \widetilde{B}_i = 1$, then

$$\sum_i v(A, B_i) = \sum_i v(A, \widetilde{B}_i)$$

Gleason's correlations

The theorem was generalized to a multipartite scenario. All the maps from measurement operators on Alice and Bob's side to joint probabilities compatible with the no-signalling principle, i.e., such that:

- $\forall A, B > 0, \quad 0 \leq v(A, B) \leq 1$
- If $A_i, B_i > 0$ and $\sum_i A_i = \sum_i B_i = 1$, then $\sum_{i,j} v(A_i, B_j) = 1$
- If $A, B_i, \widetilde{B}_i > 0$ and $\sum_i B_i = \sum_i \widetilde{B}_i = 1$, then

$$\sum_i v(A, B_i) = \sum_i v(A, \widetilde{B}_i)$$

have the form $v(A, B) = \mathrm{tr}(W \; A \otimes B)$, where W is a normalized entanglement witness, i.e., an operator positive on product states of trace one.

Klay et al., Barnum et al.

Gleason’s correlations

Gleason’s correlations are identical to quantum correlations for two parties (see also **Barnum et al, Acín et al, PRL’10**). In the general mutipartite case, the set of Gleason’s correlations is strictly larger than the quantum set.

Proof: the set of Gleason’s correlations violate GYNI!

Gleason's correlations

Gleason's correlations are identical to quantum correlations for two parties (see also **Barnum et al, Acín et al, PRL'10**). In the general mutipartite case, the set of Gleason's correlations is strictly larger than the quantum set.

Proof: the set of Gleason's correlations violate GYNI!

UPB $$|000\rangle, |1f^{\perp}g\rangle, |e1g^{\perp}\rangle, |e^{\perp}f1\rangle$$

GYNI $$p(000|000)+p(110|011)+p(011|101)+p(101|110)\leq 1$$

Because of this coincidence, the witness built from this UPB violates GYNI.

Acin et al, PRL'10

GYNI and information principles

Information causality: Alice, has a string of n bits. Alice is then allowed to send m classical bits to Bob. Information causality states that Bob cannot get more than m bits of information on Alice's string of bits, even if the parties have access to some pre-established correlations.

Non-trivial communication complexity: it refers to a bipartite scenario where Alice and Bob have to compute a function in a distributed manner.

Generalization to an arbitrary number of parties: any bipartite object built from the correlations $p(a_1,\ldots,a_N|x_1,\ldots,x_N)$ should be compatible with the principle.

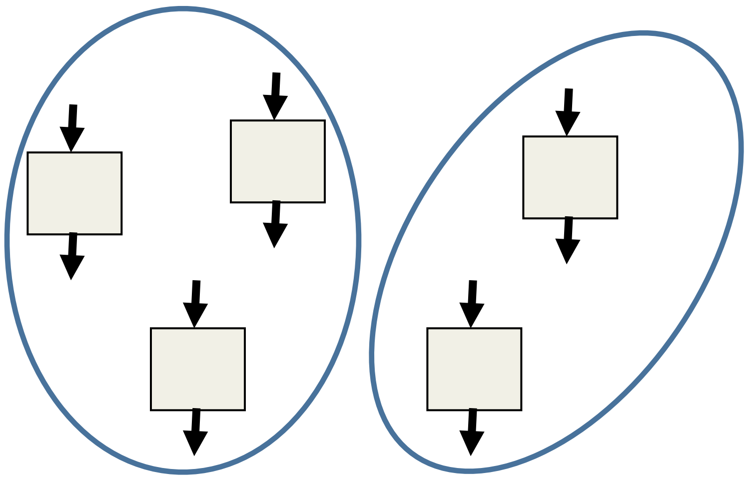
GYNI and information principles

Information causality: Alice, has a string of n bits. Alice is then allowed to send m classical bits to Bob. Information causality states that Bob cannot get more than m bits of information on Alice's string of bits, even if the parties have access to some pre-established correlations.

Non-trivial communication complexity: it refers to a bipartite scenario where Alice and Bob have to compute a function in a distributed manner.

Generalization to an arbitrary number of parties: any bipartite object built from the correlations $p(a_1,\ldots,a_N|x_1,\ldots,x_N)$ should be compatible with the principle.

No-signalling principle is generalized in the same manner to the mutipartite scenario.



GYNI and information principles

Result: we have identified tripartite correlations such that

(i) any bipartite object that can be derived from them is classical but

(ii) they violate GYNI.

Gallego et al, PRL'11

Intrinsically multipartite concepts are required to bound the set of quantum correlations in a mutipartite scenario and, in particular, to understand the limitations arising from GYNI.

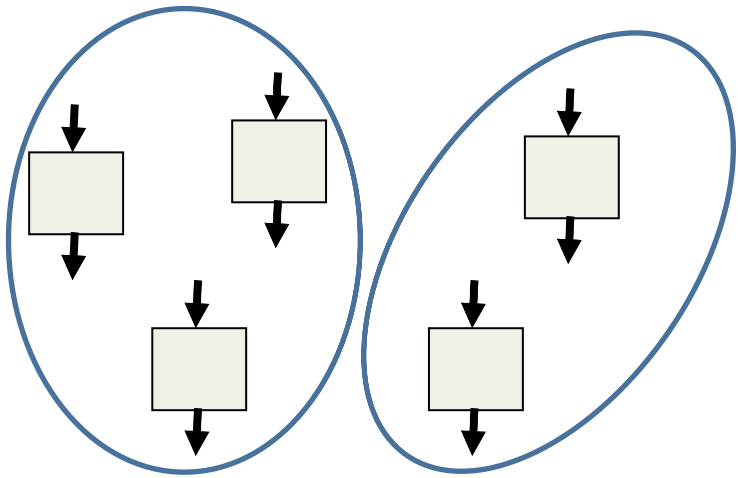
GYNI and information principles

Result: we have identified tripartite correlations such that
(i) any bipartite object that can be derived from them is classical but
(ii) they violate GYNI.

Gallego et al, PRL'11

Intrinsically multipartite concepts are required to bound the set of quantum correlations in a mutipartite scenario and, in particular, to understand the limitations arising from GYNI.

The no-signalling principle is intrinsically bipartite.



Local orhogonality: a multipartite principle

Local orthogonality

Local orthogonality: different outcomes of the same measurement by one of the observers define orthogonal event, independently of the rest of measurements.

Event			Input			Output		
Event	Input	Output	Event	Input	Output	Event	Input	Output
e_1	$x_1 \ldots \boldsymbol{x_i} \ldots x_N$	$a_1 \ldots \boldsymbol{a_i} \ldots a_N$	e_1	$x_1 \ldots \boldsymbol{x_i} \ldots x_N$	$a_1 \ldots \boldsymbol{a_i} \ldots a_N$	e_1	$x_1 \ldots \boldsymbol{x_i} \ldots x_N$	$a_1 \ldots \boldsymbol{a_i} \ldots a_N$
e_2	$x'_1 \ldots \boldsymbol{x_i} \ldots x'_N$	$a'_1 \ldots \boldsymbol{\bar{a}_i} \ldots a'_N$	e_2	$x'_1 \ldots \boldsymbol{x_i} \ldots x'_N$	$a'_1 \ldots \boldsymbol{\bar{a}_i} \ldots a'_N$	e_2	$x'_1 \ldots \boldsymbol{x_i} \ldots x'_N$	$a'_1 \ldots \boldsymbol{\bar{a}_i} \ldots a'_N$
Event	Input	Output	Event	Input	Output	Event	Input	Output
e_1	$x_1 \ldots \boldsymbol{x_i} \ldots x_N$	$a_1 \ldots \boldsymbol{a_i} \ldots a_N$	e_1	$x_1 \ldots \boldsymbol{x_i} \ldots x_N$	$a_1 \ldots \boldsymbol{a_i} \ldots a_N$	e_1	$x_1 \ldots \boldsymbol{x_i} \ldots x_N$	$a_1 \ldots \boldsymbol{a_i} \ldots a_N$
e_2	$x'_1 \ldots \boldsymbol{x_i} \ldots x'_N$	$a'_1 \ldots \boldsymbol{\bar{a}_i} \ldots a'_N$	e_2	$x'_1 \ldots \boldsymbol{x_i} \ldots x'_N$	$a'_1 \ldots \boldsymbol{\bar{a}_i} \ldots a'_N$	e_2	$x'_1 \ldots \boldsymbol{x_i} \ldots x'_N$	$a'_1 \ldots \boldsymbol{\bar{a}_i} \ldots a'_N$

Operationally: the sum of probabilities of pairwise orthogonal events is bounded by 1.

Local orthogonality is satisfied both by classical and quantum theory.
Indeed, while quantum physics breaks the orthogonality of preparations, it keeps the orthogonality of measurement outcomes .
Measurement outcomes are always of classical nature.

LO and the no-signalling principle

For two parties: compatibility with LO $\leftrightarrow$ non-signalling correlations.

For more parties: LO is strictly more restrictive than no-signalling.

LO and the no-signalling principle

For two parties: compatibility with LO $\leftrightarrow$ non-signalling correlations.

For more parties: LO is strictly more restrictive than no-signalling.

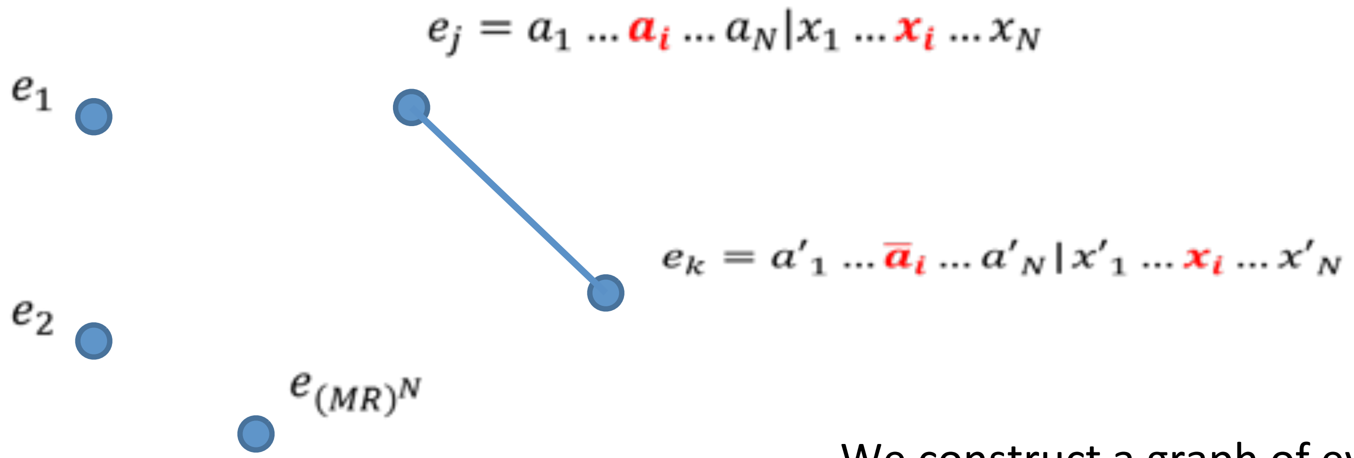
Example: GYNI.

$$p(000|000) + p(110|011) + p(011|101) + p(101|110) \leq 1$$

All events in GYNI are pairwise orthogonal.

LO and graph theory

How to get LO inequalities in a general scenario consisting of *N* parties making *M* measurements of *R* possible outcomes?
There are M^N possible combination of inputs. For each of them, there are R^N possible results. This makes $(MR)^N$ different events.

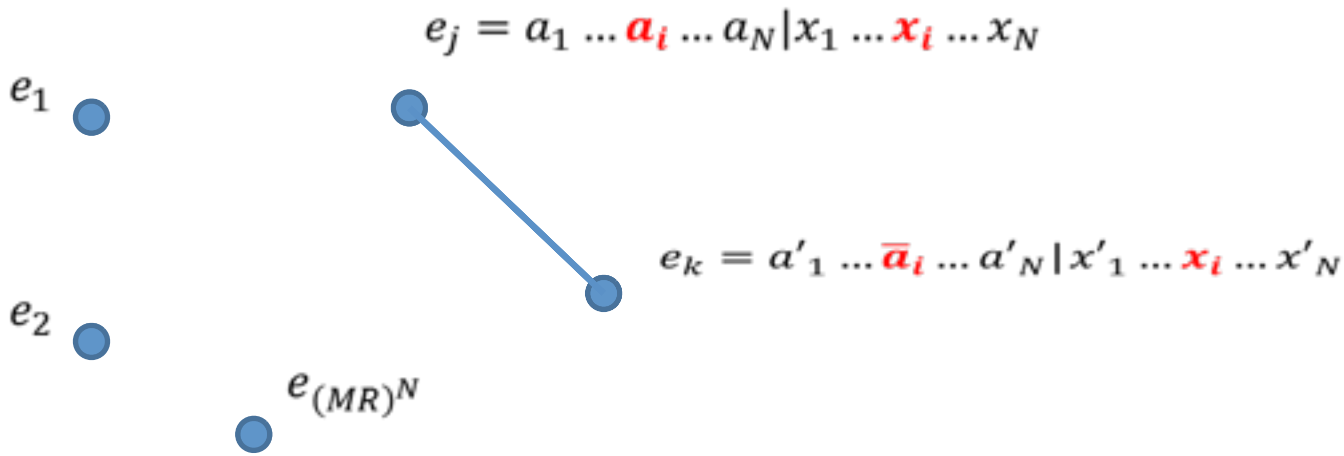


Cabello, Severini and Winter

We construct a graph of events:

- Nodes: events.
- Edges: orthogonality condition.

LO and graph theory



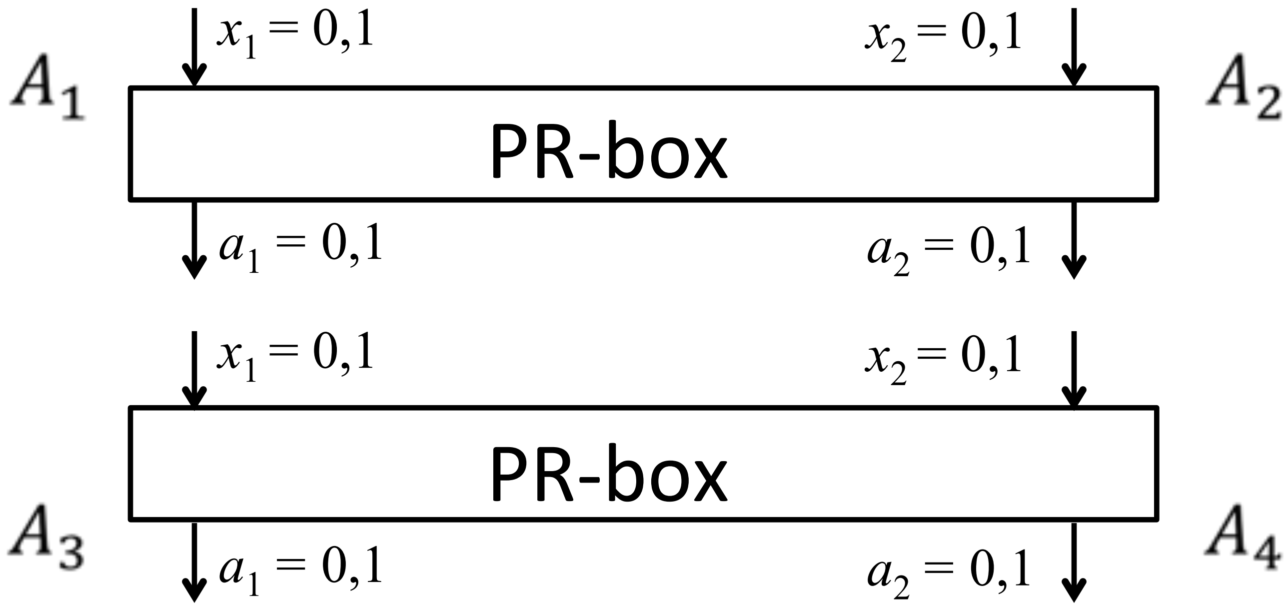
Clique: fully connected subgraph → set of pairwise orthogonal events.
Maximum clique → optimal LO inequality.
There exist algorithm to find cliques of a graph. Recall that finding the maximum clique of an arbitrary graph is an NP-hard problem. These graphs are not arbitrary.

LO and extremal tripartite correlations

- All extremal non-signalling correlations for 3 observers performing 2 measurements of 2 outcomes were listed in **S. Pironio et al, JPA'11**. They can be classified into 46 classes (one of them corresponding to local points).
- All but one of the 45 classes of non-local correlations can be ruled out by information causality (**Tzyh Haur et al, NJP'12**).
- The remaining point, box 4, is an example of a point that cannot be falsified by bipartite principles.
- All the tripartite boxes contradict LO and, thus, do not have a quantum realization. In particular, it rules out box 4 because of its intrinsically multipartite formulation.

LO and bipartite correlations

Despite the equivalence with NS for two parties, LO can be used to rule out supra-quantum bipartite correlations. How? Use **networks**.



Check now for violation of LO inequalities for 4 parties.

LO and bipartite correlations

Two PR-boxes distributed among 4 observers violate the LO inequality:

$$p(0000|0000) + p(1110|0011) + p(0011|0110) + p(1101|1011) + p(0111|1101) \leq 1$$

A noisy PR-box, $q\mathrm{PR}+(1-q)1/4$, violates the inequality up to $q = \frac{4\sqrt{5}-5}{5} \approx 0.79$.

A more complex LO inequality for 4 parties allows one to reach a critical noise $q \approx 0.72$ significantly close to the Tsirelson bound $1/\sqrt{2} \approx 0.707$.

Using similar tricks, we can rule out bipartite correlations for 2 measurements of 2 outcomes for which no information principle is known to work.

Conjectures

Conjecture 1: Local orthogonality defines the quantum set.

Conjectures

Conjecture 1: Local orthogonality defines the quantum set.

Principle: there is always someone smarter than you!

Conjectures

Conjecture 1: Local orthogonality defines the quantum set.

Principle: there is always someone smarter than you!

Navascués: there are supra-quantum correlations compatible with LO.
The proof exploits the SDP hierarchy for quantum correlations.

Conjectures

Conjecture 2: Local orthogonality defines the quantum set if local quantum measurements are assumed (à la Gleason).

Conjecture 3: all tight Bell inequalities that are not violated by quantum correlations are LO inequalities.

- Evidence 1: all known examples (not many) are LO inequalities.
- Evidence 2: all bipartite tight Bell inequalities have a quantum violation.

Conclusions

- Multipartite principle are needed for our understanding of quantum correlations.
- Local orthogonality represents an intrinsically multipartite principle.
- It captures the classical nature of local measurement outcomes: two outcomes of the same measurement define incompatible events.
- It is equivalent to the no-signalling principle for 2 parties, but more restrictive for more parties.
- It is a powerful method when combined with graph-theory concepts and network geometries.
- It rules out supra-quantum bipartite correlations.
- The principle alone does not specify the set of quantum correlations.
- It rules out supra-quantum correlations that could not be detected by any bipartite principle.

We have openings for post-docs.
Contact: antonio.acin@icfo.es

ICFOnest post-doctoral program: it aims at providing high-level training and support for outstanding international researchers in the early stages of their careers. See: http://nestpostdocs.icfo.es/