A remarkable representation of the Clifford group

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Outline

- Useful groups in physics
- The Zak basis for finite systems
- An application to SIC-POVMs

Heisenberg Groups

Heisenberg groups can be defined in terms of upper triangular matrices

$$\begin{pmatrix} 1 & x & \phi \\ 0 & 1 & p \\ 0 & 0 & 1 \end{pmatrix}$$

where x, p, ϕ are elements of a ring, R.

- $R = \mathbb{R}$ - a three dimensional Lie group whose Lie algebra includes the position and momentum commutator
- $R = \mathbb{Z}_N$ - a finite dimensional Heisenberg group $H(N)$
- $R = \mathbb{F}_{p^k}$ - an alternative finite dimensional group

Finite and CV systems

Finite systems	CV systems
Heisenberg group $H(N)$ elements are translations in discrete phase space ($N = p$)	Heisenberg group $H(\mathbb{R})$ elements are translations in phase space
	$H(\mathbb{R})\|0\rangle \rightarrow$ Coherent states
normalizer $H(N) = C(N)$ The Clifford group $C(N) \cong H(N) \rtimes SL(2, N)$	normalizer $H(\mathbb{R}) = HSp$ The Affine Symplectic group $HSp \cong H(\mathbb{R}) \rtimes SL(2, \mathbb{R})$
	$HSp\|0\rangle \rightarrow$ Gaussian states
Our new basis	Zak basis

The Clifford group

Write elements of H(N) as

$$D_{ij} = \tau^{ij} X^i Z^j$$

where $\tau = -e^{\frac{i\pi}{N}}$, $X|j\rangle = |j+1\rangle$ and $Z|j\rangle = \omega^j|j\rangle$.
Then the *composition law* is

$$D_{ij} D_{kl} = \tau^{kj-il} D_{i+k,j+l}$$

The **Clifford group** is the normalizer of $H(N)$ i.e. all unitary operators U such that

$$U D_{ij} U^\dagger = \tau^{k'} D_{i',j'}$$

Definition of the basis

The Heisenberg group $H(N)$ is defined by $\omega = e^{2\pi i/N}$ and generators X, Z with relations

$$ZX = \omega XZ, \qquad X^N = Z^N = 1$$

- There is a unique unitary representation [Weyl]
- The *standard* representation is to choose Z to be diagonal.

But suppose $N = n^2$, then

$$Z^n X^n = X^n Z^n$$

- There is a (maximal) abelian subgroup $\langle Z^n, X^n \rangle$ of order $n^2 = N$
- So choose a basis in which this special subgroup is diagonal

The entire Clifford group is monomial in the new basis

Armchair argument:

- The Clifford group permutes the maximal abelian subgroups of $H(N)$
- It preserves the order of any element
- In dimension $N = n^2$ there is a *unique* maximal abelian subgroup where all of the group elements have order αn
- Hence the Clifford group maps $\langle Z^n, X^n \rangle$ to itself
- The basis elements are permuted and multiplied by phases

Theorem:
There exists a monomial representation of the Clifford group with $H(N)$ as a subgroup if and only if the dimension is a square, $N = n^2$.

The new basis

In the Hilbert space $\mathcal{H}_N = \mathcal{H}_n \otimes \mathcal{H}_n$, the new basis is

$$X|r,s\rangle = \begin{cases} |r,s+1\rangle & \text{if } s+1 \neq 0 \bmod n \\ \sigma^r|r,0\rangle & \text{otherwise} \end{cases}$$

$$Z|r,s\rangle = \omega^s|r-1,s\rangle$$

where the phases are $\omega = e^{\frac{2\pi i}{N}}$ and $\sigma = e^{\frac{2\pi i}{n}}$. Indeed,

$$X^n|r,s\rangle = \sigma^r|r,s\rangle \qquad Z^n|r,s\rangle = \sigma^s|r,s\rangle$$

We have gone "half-way" to the Fourier basis

$$|r,s\rangle = \frac{1}{\sqrt{n}}\sum_{t=0}^{n-1}\omega^{-ntr}|nt+s\rangle\,.$$

i.e. apply $F_n \otimes \mathbb{I}$, where F_n is the $n \times n$ Fourier matrix.

An Application to SIC POVMs

A SIC is a set of N^2 vectors $\{|\psi_i\rangle \in \mathbb{C}^N\}$ such that

$$|\langle\psi_i|\psi_j\rangle|^2 = \frac{1}{N+1} \qquad \text{for } i \neq j$$

- A 2-design with the minimal number of elements.
- A special kind of (doable) measurement.
- Potentially a "standard quantum measurement" (cf. quantum Bayesianism)

Zauner's Conjecture

SICs exist in every dimension

- Exact solutions in dimensions $2-16$, 19, 24, 35 and 48
- Numerical solutions in dimensions $2-67$

They can be chosen so that they form an orbit of $H(N)$ i.e. the SIC has the form $D_{ij}|\psi\rangle$ *and* there is a special order 3 element of the Clifford group such that

$$U_z|\psi\rangle = |\psi\rangle$$

- All available evidence supports this conjecture

Images of SICs in the probability simplex

$$|\psi\rangle = (\psi_{00}, \psi_{01}, \psi_{10}, \ldots)^T = (\sqrt{p_{00}}, \sqrt{p_{01} e^{i\mu_{01}}}, \sqrt{p_{10} e^{i\mu_{10}}}, \ldots)^T$$

$\downarrow$ image w.r.t the basis

prob vector $= (p_{00}, p_{01}, p_{10}, \ldots)^T$

Consider the equations

$$\langle\psi|X^{nu}Z^{nv}|\psi\rangle = \sum_{r,s} p_{rs} q^{ru+sv}$$

$$|\langle\psi|X^{nu}Z^{nv}|\psi\rangle|^2 = \sum_{r,s}\sum_{r',s'} p_{rs}p_{r's'} q^{(r-r')u+(s-s')v}$$

$$\frac{1}{N+1} = \sum_{r,s}\sum_{r',s'} p_{rs}p_{r's'} q^{(r-r')u+(s-s')v}$$

Take the Fourier transform

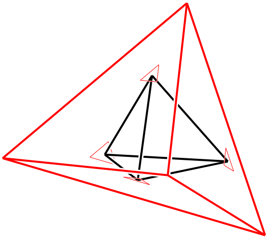
$$\sum_{r,s} p_{rs}^2 = \frac{2}{N+1}$$

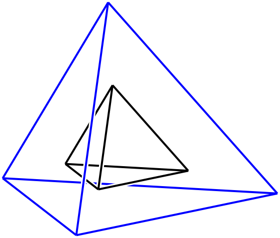
$$\sum_{r,s} p_{rs}p_{r+x,s+y} = \frac{1}{N+1} \quad \text{for } (x,y) \neq (0,0)$$

SICs are nicely aligned in the new basis

Geometric interpretation: When the SIC is projected to the basis simplex, we see a regular simplex centered at the origin with N vertices.

The new basis is nicely orientated...





Solutions to the SIC problem

- Dimension $N = 2^2$ is now trivial
- Dimension $N = 3^2$ can be solved on a blackboard
- Dimension $N = 4^2$ can be solved with a computer

Solving the SIC problem in dimension $N = 2^2$

First write the SIC fiducial as

$$|\psi\rangle = \begin{pmatrix} \psi_{00} \\ \psi_{01} \\ \psi_{10} \\ \psi_{11} \end{pmatrix}$$

In the monomial basis, Zauner's unitary is

$$U_z = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence, Zauner's conjecture, $U_z|\psi\rangle = |\psi\rangle$, gives us

$$|\psi\rangle = \begin{pmatrix} a \\ a \\ a \\ be^{i\theta} \end{pmatrix}$$

Solving the SIC problem in dimension $N = 2^2$

The SIC fiducial is

$$|\psi\rangle = \begin{pmatrix} a \\ a \\ a \\ be^{i\theta} \end{pmatrix}$$

Then we have two conditions,

$$\text{Norm} = 1 \Rightarrow \qquad 3a^2 + b^2 = 1 \qquad (1)$$

$$\sum p_{rs}^2 = \frac{2}{N+1} \Rightarrow \qquad 3a^4 + b^4 = \frac{2}{5} \qquad (2)$$

Solving these equations gives

$$a = \sqrt{\frac{5-\sqrt{5}}{20}} \qquad b = \sqrt{\frac{5+3\sqrt{5}}{20}}$$

Solving the SIC problem in dimension $N = 2^2$

Plugging these values into the equation

$$|\langle\psi|X|\psi\rangle|^2 = \frac{1}{5}$$

Gives

$$\frac{3-\sqrt{5}}{20} + \frac{1+\sqrt{5}}{10}\cos^2\theta = \frac{1}{5}$$

Hence

$$\theta = \frac{(2\lambda+1)\pi}{4} \quad \lambda = 0, 1, 2, 3$$

Conclusion: The Zauner eigenspace contains the fiducials

$$|\psi_\lambda\rangle = \sqrt{\frac{5-\sqrt{5}}{20}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \sqrt{2+\sqrt{5}}\, e^{\pi i/4}\, i^\lambda \end{pmatrix}$$

Solving the SIC problem in dimension $N = 3^2$

In the new basis, we can solve the SIC problem on the blackboard...

Solving the SIC problem in dimension $N = 3^2$

Zauner's conjecture implies that

$$|\psi\rangle = \quad -z_1\omega^7|1,1\rangle - z_2\omega|2,2\rangle + z_3(\omega^6|0,2\rangle + |1,0\rangle + \omega^8|2,1\rangle) + z_4(\omega^6|0,1\rangle + |2,0\rangle + \omega^5|1,2\rangle)\ .$$

$$z_1 = \sqrt{p_1}e^{i\mu_0} \qquad z_2 = \sqrt{p_2}e^{-i\mu_0} \qquad z_3 = \sqrt{p_3}e^{i\mu_3} \qquad z_4 = \sqrt{p_4}e^{i\mu_4}$$

Solving the SIC problem in dimension $N = 3^2$

The absolute values:

$$p_1 = a_1 + b_1 \ , \quad p_2 = a_1 - b_1 \ , \quad p_3 = a_3 + b_3 \ , \quad p_4 = a_3 - b_3$$

$$\begin{aligned}
a_1 &= \frac{1}{40}\left(5 - s_0 5\sqrt{3} + s_0 3\sqrt{5} + \sqrt{15}\right) \\
b_1 &= \frac{s_2}{60}\sqrt{15\left(\sqrt{15} + s_0\sqrt{3}\right)} \\
a_3 &= \frac{1}{120}\left(15 + s_0 5\sqrt{3} - s_0 3\sqrt{5} - \sqrt{15}\right) \\
b_3 &= \frac{s_1}{60}\sqrt{5\left(-18 - s_0 7\sqrt{3} + s_0 6\sqrt{5} + 5\sqrt{15}\right)}
\end{aligned}$$

where $s_0 = s_1 = s_2 = \pm 1$

Solving the SIC problem in dimension $N = 3^2$

The phases:

$$
\begin{aligned}
e^{i\mu_0} &= \sqrt{\frac{1}{2} + c_0} - is_1\sqrt{\frac{1}{2} - c_0} \\
e^{i\mu_3} &= q^{m_3}\left(-\sqrt{\frac{1}{2} - c_1 + c_2} + is_1 s_2\sqrt{\frac{1}{2} + c_1 - c_2}\right)^{\frac{1}{3}} \\
e^{i\mu_4} &= q^{m_4}\left(-\sqrt{\frac{1}{2} - c_1 - c_2} + is_1 s_2\sqrt{\frac{1}{2} + c_1 + c_2}\right)^{\frac{1}{3}}
\end{aligned}
$$

$$
\begin{aligned}
c_0 &= \frac{1}{8}\sqrt{2(6 + s_0\sqrt{3} - \sqrt{15})} \\
c_1 &= \frac{s_0}{8}\sqrt{9 - s_0 4\sqrt{3} + s_0 3\sqrt{5} - 2\sqrt{15}} \\
c_2 &= \frac{s_1 s_0}{24}\sqrt{15(-19 + s_0 12\sqrt{3} - s_0 9\sqrt{5} + 6\sqrt{15})}
\end{aligned}
$$

Solving the SIC problem in dimension $N = 4^2$

The new basis allows us to solve the SIC problem in dimension 16...

The solutions are given in a number field

$$\mathbb{K} = \mathbb{Q}(\sqrt{2}, \sqrt{13}, \sqrt{17}, r_2, r_3, t_1, t_2, t_3, t_4, \sqrt{-1}),$$

of degree 1024, where

$$r_2 = \sqrt{\sqrt{221} - 11} \qquad r_3 = \sqrt{15 + \sqrt{17}}$$

$$t_1 = \sqrt{15 + (4 - \sqrt{17})r_3 - 3\sqrt{17}}$$

$$t_2^2 = ((3 - 5\sqrt{17})\sqrt{13} + (39\sqrt{17} - 65))r_3 + ((16\sqrt{17} - 72)\sqrt{13} + 936))t_1 - 208\sqrt{13} + 2288$$

$$t_3 = \sqrt{2 - \sqrt{2}} \qquad t_4 = \sqrt{2 + t_3}$$

Conclusion

- A basis were every element of the Clifford group is a monomial matrix
- The SICs are nicely orientated in the new basis
- The solutions to the SIC problem in dimensions 4, 9 and 16 are given entirely in terms of radicals, as expected (but not understood!)
- The result can be extended to non-square dimensions kn^2
- Are there other applications in quantum information?

$N = n^2$: *QIC vol 12, 0404 (2012), arXiv:1102.1268*
$N = kn^2$: *in preparation*