# A 2-Categorical Formalism for Quantum Information

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# Introduction

In quantum protocols, information flow is topological.

This perspective was emphasized by Abramsky and Coecke, with the following account of quantum teleportation in terms of the topology of 1-dimensional manifolds:



In this talk we develop this significantly:

- Topological account extended from fragments to entire protocols, including relevant classical information and 'branching'.
- Higher 2-categorical syntax replaces a 1-categorical one.
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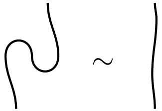

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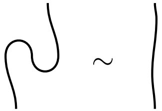

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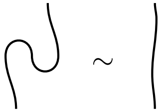

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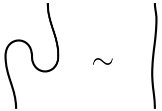

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# Formal Algorithms

A **formal protocol** consists of the following components:

- **Regions** representing classical information
- **Lines** representing quantum systems
- **Vertices** representing quantum dynamics

A **formal algorithm** is an equation between formal protocols with the same input and output types.



**Teleportation / E.S.**



**Complementarity**



**Dense coding**



**Interlaced teleportation**

Use **2-category theory**, an algebraic setting for 2d composition.

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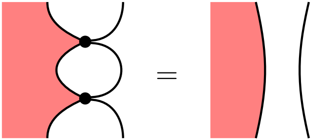

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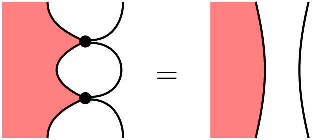

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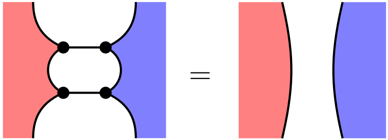

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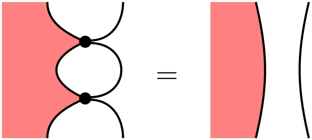

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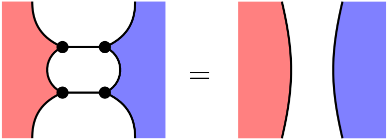

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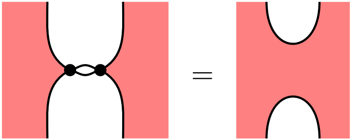

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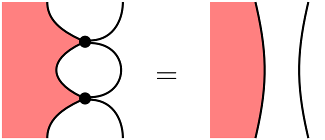

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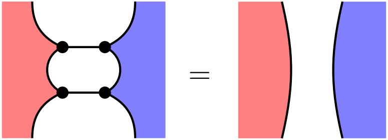

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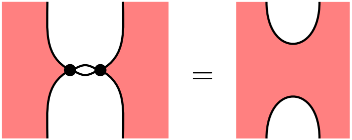

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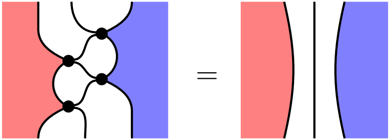

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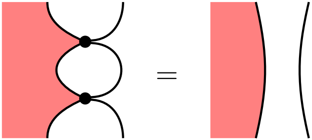

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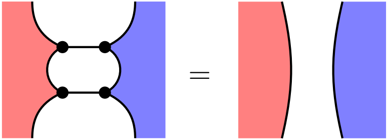

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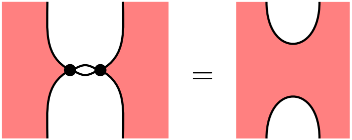

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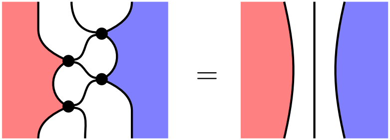

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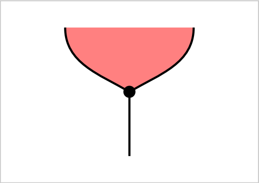
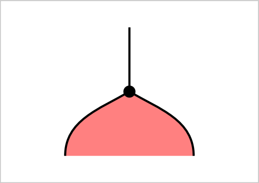
# Graphical Calculus

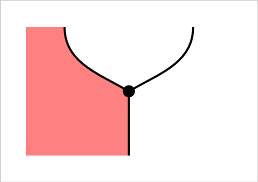


**Perform a measurement**

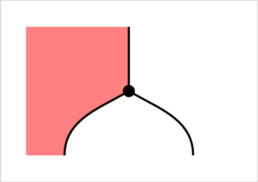


**Undo classical correlations**

# Graphical Calculus

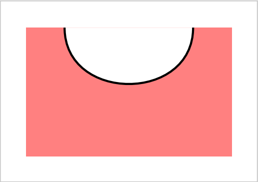


**Extract a copy of the classical data**

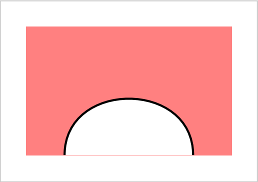


**Compare quantum data with classical data** (postselecting)

# Graphical Calculus



**Copy classical data**



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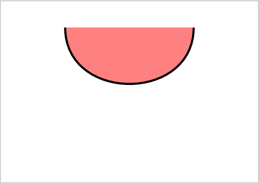
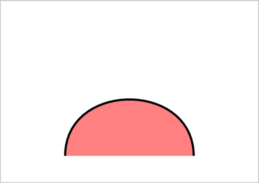
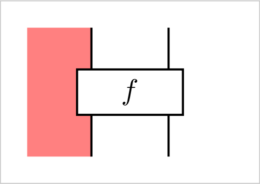


**Create uniform classical data**



**Forget classical data**

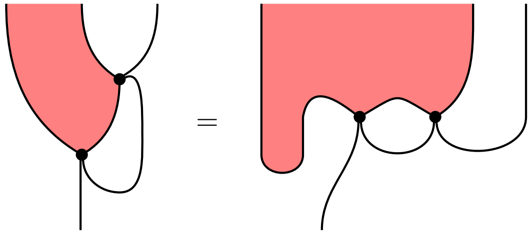
# Graphical Calculus




**Perform a controlled operation**

# Deformations

If the diagrams for any two protocols are **homotopic** relevant to the fixed boundary, then they will always give rise to the same quantum information flow, and we consider them to be equal.



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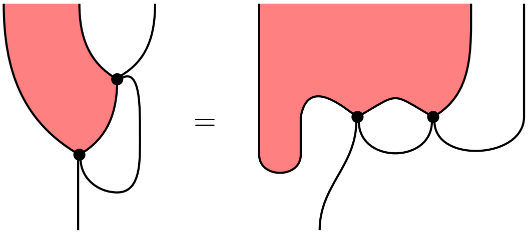

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# Implementation

To **implement** a formal algorithm is to find a solution for it in a particular target **2-category**.

Not all formal algorithms are implementable, and implementability will depend on the choice of target.

For implementations in standard quantum physics, the correct target is the 2-category **2Hilb** of **2–Hilbert spaces** due to Baez:

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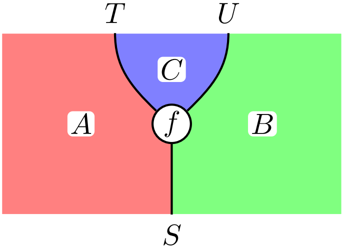

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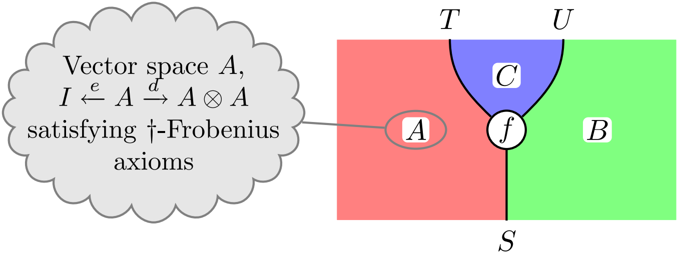

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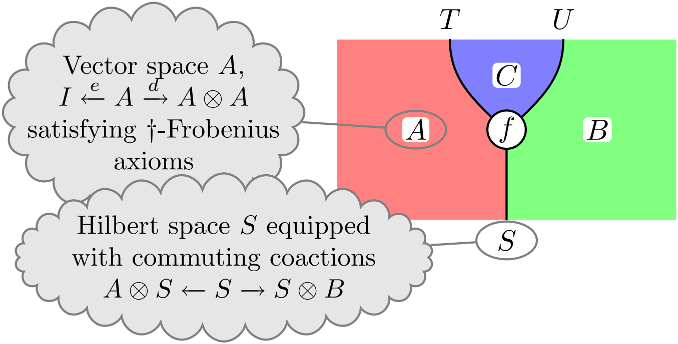

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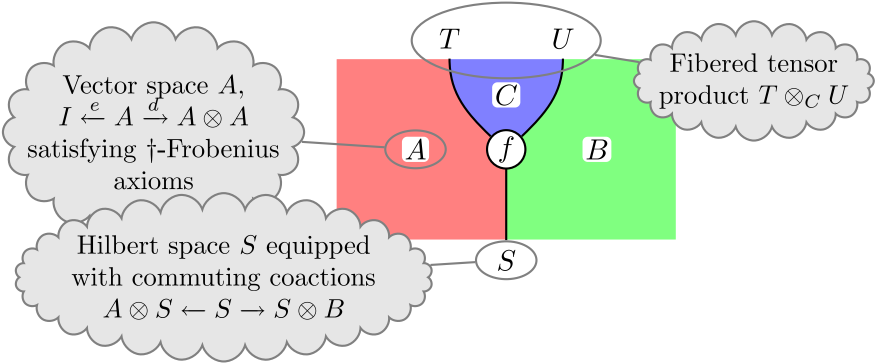

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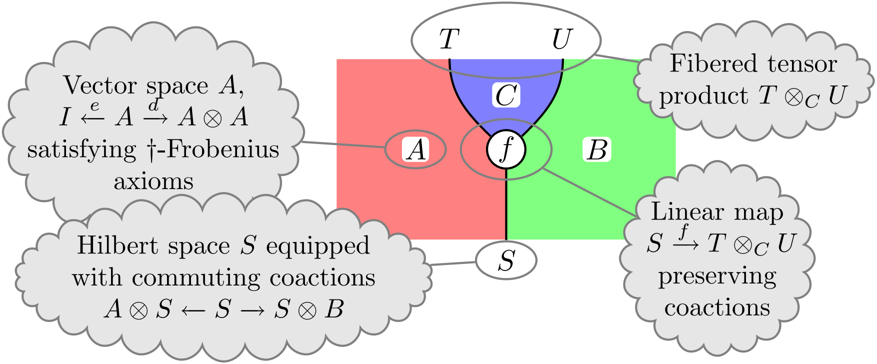

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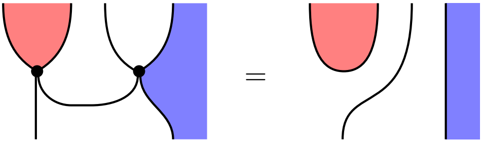
An 'impossible algorithm' is a formal algorithm which has no quantum implementation. Sometimes our quantum intuition tells us why. For example, it's reassuring that this algorithm is impossible:



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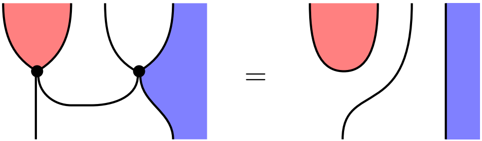


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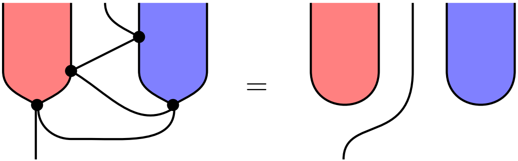


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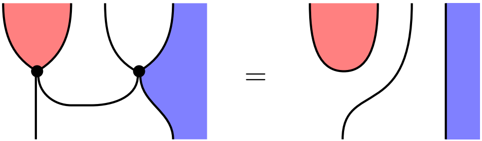


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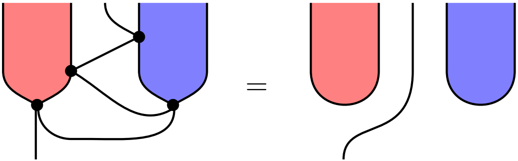


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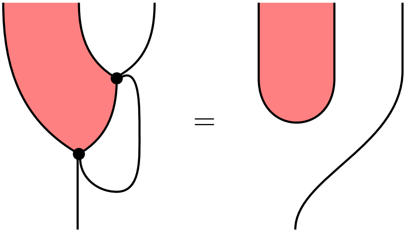

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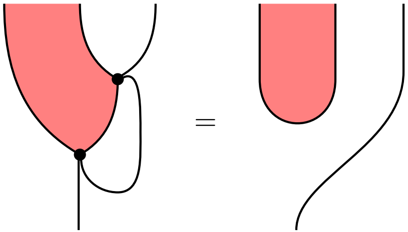

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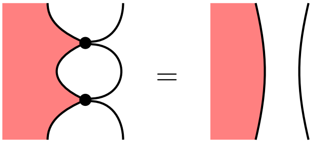

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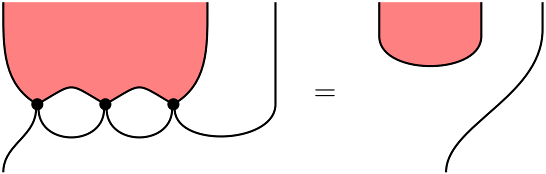


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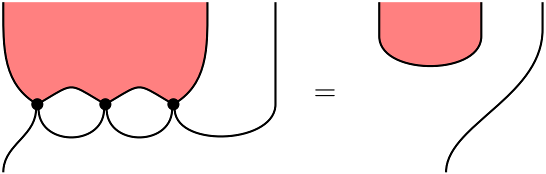

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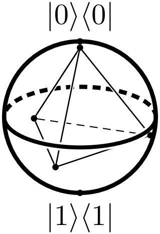

# What's Going On?

Begin with a pure state

$$\phi \in H.$$

This lives in **Hilb**, the category of Hilbert spaces and linear maps.

After a two-outcome measurement, obtain

$$(P_1\phi) \oplus (P_2\phi) \in H \oplus H.$$

Future unitary dynamics cannot mix these components.
The relevant category is now $\mathbf{Hilb} \oplus \mathbf{Hilb}$.

In general, will need to access $\mathbf{Hilb}^{\oplus n}$ to describe a given protocol.
These are called **2–Hilbert spaces**, and they form a 2-category.
Makes sense to use this to encode quantum algorithms.

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# Future Directions

- Implementability is a **topological invariant** of a formal algorithm. Can we compute it directly?
- For the **impossible algorithms**: why are they impossible? What does this tell us about QM? Would QM break down if they could be implemented?
- Can we find a broad class of **novel implementable protocols**? How can these be implemented physically?
- Can we treat **more general** sorts of algorithms (e.g. Deutsch-Josza) using this formalism?
- Investigate connections with closely-related areas:
  - Group representation theory
  - Linear logic
  - Combinatorics
  - Topological quantum computing

Thanks for listening!

# Future Directions

- Implementability is a **topological invariant** of a formal algorithm. Can we compute it directly?
- For the **impossible algorithms**: why are they impossible? What does this tell us about QM? Would QM break down if they could be implemented?
- Can we find a broad class of **novel implementable protocols**? How can these be implemented physically?
- Can we treat **more general** sorts of algorithms (e.g. Deutsch-Josza) using this formalism?
- Investigate connections with closely-related areas:
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