Quotient–Comprehension Chains

Kenta Cho, Bart Jacobs, Bas Westerbaan and Bram Westerbaan

Institute for Computing and Information Sciences (iCIS),
Radboud University Nijmegen, The Netherlands.

{k.cho,bart,bwesterb,awesterb}@cs.ru.nl

Quotients and comprehension are fundamental mathematical constructions that can be described via adjunctions in categorical logic. This paper reveals that quotients and comprehension are related to measurement, not only in quantum logic, but also in probabilistic and classical logic. This relation is presented by a long series of examples, some of them easy, and some also highly non-trivial (esp. for von Neumann algebras). We have not yet identified a unifying theory. Nevertheless, the paper contributes towards such a theory by introducing the new quotient-and-comprehension perspective on measurement instruments, and by describing the examples on which such a theory should be built.

1 Introduction

Measurement is a basic operation in quantum theory: the act of observing a quantum system. It is characteristic of the quantum world that such an observation disturbs the system under measurement: it has a side-effect. In [12] a categorical description of measurement is given that takes such side-effects into account. We sketch the essentials, omitting many details. For each predicate p on a type/object A in this theory, there is an 'instrument' map

$$A \xrightarrow{\mathrm{instr}_p} A+A \tag{1}$$

that performs the act of measuring p. We write $A+A$ for the coproduct/sum of A with itself, which comes equipped with left and right insertion/coprojection maps $\kappa_1, \kappa_2 \colon A \to A+A$. Intuitively, the map instr_p gives an outcome in the left summand of $A+A$ if p holds, and in the right component otherwise. The side-effect associated with the instrument is the map $\nabla \circ \mathrm{instr}_p \colon A \to A$, where $\nabla = [\mathrm{id},\mathrm{id}] \colon A+A \to A$ is the codiagonal. If $\nabla \circ \mathrm{instr}_p$ is the identity map $A \to A$, one calls p *side-effect free*. Measurement in a probabilistic setting is side-effect free, but proper quantum measurement is not.

The set-theoretic case may help to understand this instrument map. For each predicate $p \subseteq A$ one has $\mathrm{instr}_p(a) = \kappa_1(a)$ if $a \in P$ and $\mathrm{instr}_p(a) = \kappa_2(a)$ if $a \notin P$. In [12] it is shown that such instrument maps also exist in a probabilistic and in a quantum setting. In the latter case one works in the opposite of the category of C^*-algebras, with completely positive unital maps. The instrument (1) then has type $A \times A \to A$, and is defined as $\mathrm{instr}_p(a,b) = \sqrt{p} \cdot a \cdot \sqrt{p} + \sqrt{1-p} \cdot b \cdot \sqrt{1-p}$. This is the (generalised) Lüders rule, see for instance, in [2, Eq.(1.3)].[1]

[1] Three notions of measurement (instrument) commonly appear in the literature. *Sharp* or *projective measurement* corresponds to instr_p where p is a projection [17, §2.2.5], and appears in von Neumann's projection postulate. *POVM measurement* corresponds to arbitrary instr_p, although the post-measurement states are usually left out [17, §2.2.6]. *Generalized measurements* capture the different ways the same POVM can be measured [17, §2.2.3]; in the finite dimensional case, every generalized measurement corresponds to a composition $(\varphi + \psi) \circ \mathrm{instr}_p$, where φ and ψ are automorphisms.

To appear in EPTCS.

The paper [12] lists several requirements for instrument maps (1). The question remained: do these requirement uniquely determine the instrument maps? Put differently: is the presence of these maps a *property* of a category, or *structure*? The current paper does not solve this fundamental problem. But it does uncover the relevance of the logical notions of quotient and comprehension for measurement.

After the formulation of the theory of instruments (1), it became clear (see [4]) that one can also work with *partial* maps $A \to A+1$ and $A \to 1+A$. The two of them can be combined into a single instrument map $A \to A+A$ via a suitable pullback. More importantly, it was noted that in all of the examples the relevant partial map, called 'assert' and written as $\mathrm{asrt}_p \colon A \to A$ in the category of partial maps, is a composite of a quotient map ξ and a comprehension map π, as in:

$$
\begin{array}{ccc}
A & \xrightarrow{\quad \mathrm{asrt}_p \quad} & A \\
{\scriptstyle \xi} \searrow & & \nearrow {\scriptstyle \pi} \\
& A/p^{\perp} \overset{(*)}{=} \{A \mid p\} &
\end{array}
\tag{2}
$$

where $p^{\perp}$ is the negation of p. Such a connection between the fundamental concepts of quotient, comprehension and measurement is fascinating! Quotients and comprehension have a clean description in categorical logic as adjoints (see below for details). Does that lead to instruments as a property? This question remains unsolved, but now takes another form: diagram (2) involves an equality, marked with $(*)$, that seems highly un-categorical: adjoints are determined up-to-isomorphism, so having an equality between them is strange. Still this is what we see in all examples, via obvious choices of quotient and comprehension functors. It is not clear if an equality (or isomorphism) between a quotient $A/p^{\perp}$ and a comprehension $\{A \mid p\}$ is property or structure. This is a topic of active research, that requires investigation of many examples. (We have slightly simplified the picture (2) since there is another operation $\lceil p \rceil$ involved, but that is not essential at this stage; it will be adjusted below.)

This paper is about the following. Once we started looking for quotients and comprehension in the relevant mathematical models we found them everywhere, often in somewhat disguised form. Uncovering familiar constructions, like (co)support for von Neumann algebras, as quotient and comprehension is mathematically relevant on its own. It changes one's perspective. Thus, the paper only contains examples. Many different examples, each showing that certain constructions are instances of quotient and comprehension. The examples include vector and Hilbert spaces, sets and topological spaces, various Kleisli categories of monads used for probability theory, commutative rings, MV-modules and C^*-algebras, and finally (non-commutative) von Neumann algebras. The examples point to decomposition of (commutative) mathematical structures as products of quotients and comprehension, like in ring theory, and used for the sheaf theory of commutative rings.

In summary, we think that quotients and comprehension provide a new fruitful perspective on the nature of quantum measurement. This is illustrated here in many examples. We are fully aware that the general, final explanation is lacking at this stage. But such a general theory must be based on a thorough understanding of the examples. That is the focus of the current paper.

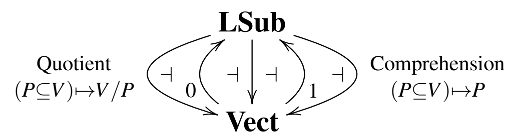
This (missing) underlying general theory will bear some resemblance to recent work in (non-Abelian) homological algebra, see in particular [21] (where similar adjunction chains are studied), but also [13, 7]. Part of the motivation is axiomatising the category of (non-Abelian) groups, following [15]. As a result, stronger properties are used than occur in the current setting (for instance the first isomorphism theorem and left adjoints to substitution, corresponding to bifibrations), which excludes not only our motivating example, the category of von Neumann algebras, but also $\mathcal{K}\ell(\mathcal{D})$ and **Sets** to name but two.

2 Comprehension and Quotients for Vector Spaces

This section briefly reviews comprehension and quotients for vector spaces. These constructions are fairly familiar. Their categorical description via a chain of adjunctions, as in (3) below, is probably less familiar. This re-description may help to understand similar such chains in the rest of this paper.

We write **Vect** for the category of vector spaces over some fixed field with linear maps between them. Linear subspaces are organised in a category **LSub**. Its objects are pairs (V,P), where V is a vector space and $P \subseteq V$ is a linear subspace. A morphism $(P \subseteq V) \to (Q \subseteq W)$ in **LSub** is a linear map $f\colon V \to W$ that restricts to $P \to Q$, *i.e.*, that satisfies $P \subseteq f^{-1}(Q)$. There is then an obvious forgetful functor **LSub** $\to$ **Vect**. It is a poset fibration [10], but that does not play a role here. We view **LSub** as a category of linear predicates, over the category **Vect** of linear types.

Interestingly, there is a chain of adjunctions like in (3). The up going functors $0,1\colon$ **Vect** $\to$ **LSub** are for falsum and truth respectively. They send a vector space V to the least $0(V) = (\{0\} \subseteq V)$ and greatest $1(V) = (V \subseteq V)$ subspace. There is a comprehension functor $(V,P) \mapsto P$ that is right adjoint to truth, and a quotient functor $(V,P) \mapsto V/P$ that is left adjoint to falsum. The outer adjunctions involve (natural) bijective correspondences:



(3)

$$\frac{1V = (V \subseteq V) \xrightarrow{\;f\;} (Q \subseteq W)}{V \xrightarrow[\;g\;]{} Q} \qquad\qquad \frac{(P \subseteq V) \xrightarrow{\;f\;} (\{0\} \subseteq W) = 0W}{V/P \xrightarrow[\;g\;]{} W}$$

The second correspondence says that if $P \subseteq f^{-1}(\{0\}) = \ker(f)$, then f corresponds to a map $V/P \to W$. The quotient uses the equivalence relation $v \sim_P v'$ iff $v - v' \in P$.

The category **LSub** is obtained via what is called the 'Grothendieck construction'. Since we will use it many times in the sequel, we make it explicit. For convenience we restrict it to posets. We write **PoSets** for the category of posets with monotone functions between them.

Definition 1 *Let* $\boldsymbol{B}$ *be a category, with a functor* $F\colon \boldsymbol{B} \to \boldsymbol{PoSets}^{op}$. *We write* $\int F$ *for the category with pairs* (X,P) *as objects, where* $X \in \boldsymbol{B}$ *and* $P \in F(X)$. *A morphism* $f\colon (X,P) \to (Y,Q)$ *is a map* $f\colon X \to Y$ *in* $\boldsymbol{B}$ *with* $P \le F(f)(Q)$. *There is an obvious forgetful functor* $\int F \to \boldsymbol{B}$, *given by* $(X,P) \mapsto X$ *and* $f \mapsto f$.

The category **LSub** of linear subspaces is obtained via this Grothendieck construction from the functor $F\colon$ **Vect** $\to$ $\mathbf{PoSets}^{\mathrm{op}}$, where $F(V)$ is the poset of linear subspaces of V, ordered by inclusion; on a linear map $f\colon V \to W$ we get $F(f)\colon F(W) \to F(V)$ by inverse image: $F(f)(Q) = f^{-1}(Q)$.

The following general observation about the Grothendieck construction is useful.

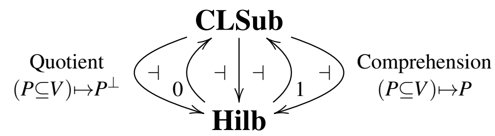
Lemma 2 *Assume for a functor* $F\colon \boldsymbol{B} \to \boldsymbol{PoSets}^{op}$,

- *each 'fibre'* $F(X)$ *has a least element* 0_X*;*
- *each* $F(X)$ *also has a greatest element* 1_X, *and each* $F(f)\colon F(Y) \to F(X)$ *satisfies* $F(f)(1_Y) = 1_X$.

Then there are functors $0,1\colon \boldsymbol{B} \to \int F$, *namely* $0(X) = (X, 0_X)$ *and* $1(X) = (X, 1_X)$, *which are left and right adjoints to the forgetful functor* $\int F \to \boldsymbol{B}$. ■

We briefly sketch the situation for Hilbert spaces, where quotients are given by (ortho)complements. So let **Hilb** $\hookrightarrow$ **Vect** be the category of Hilbert spaces, with bounded linear maps between them. Mapping a Hilbert space V to the poset of closed linear subspaces yields a functor **Hilb** $\to$ $\mathbf{PoSets}^{\mathrm{op}}$. We write **CLSub** for the resulting Grothendieck completion, with forgetful functor **CLSub** $\to$ **Hilb**. Since both $\{0\} \subseteq V$ and $V \subseteq V$ are closed, this functor has both a left and right adjoint, by Lemma 2.

We get a situation like in (3), see (4). For the quotient adjunction, note that if $f\colon V \to W$ in **Hilb** satisfies $P \subseteq \ker(f) = f^{-1}(\{0\})$, for a closed $P \subseteq V$, then f is determined by its restriction $P^{\perp} \to W$, using that $V \cong P \oplus P^{\perp}$. The latter decomposition of the space V exists because each vector $v \in V$ can be written in a unique way as $v = v_1 + v_2$ with $v_1 \in P$ and $v_2 \in P^{\perp}$. This is a basic result in the theory of Hilbert spaces.



(4)

3 Set-Theoretic Examples

Standardly it is a *relation* $R \subseteq X \times X$ on a set X that gives rise to a quotient X/R, and not a *predicate*, like for vector spaces in the previous section. Such a quotient $R \mapsto X/R$ is described as a left adjoint to the equality functor, see [10] for details. It turns out that a quotient of a predicate also exists in set-theoretic and other contexts if we switch to partial functions. Categorically this will be done via the lift monad (sometimes called *maybe monad*). We isolate the general construction first.

Definition 3 *Let **B** be a category with binary coproducts + and a final object* 1*. The functor* $X \mapsto X+1$ *is then a monad on **B**, called the lift monad. We write* $\mathbf{B}_{+1}$ *for the Kleisli category of this monad.*

The category $\mathbf{B}_{+1}$ thus has the same objects as **B**, and maps $X \to Y$ in $\mathbf{B}_{+1}$ are maps $X \to Y+1$ in **B**. We denote the composition in $\mathbf{B}_{+1}$ by $g \odot f = [g, \kappa_2] \circ f$. For the category **Sets** of sets and functions, the final object is a singleton $1 = \{*\}$ and coproducts are given by disjoint union. So in **Sets** the maps $f\colon X \longrightarrow Y+1 \equiv Y \cup \{*\}$ correspond exactly to partial maps from X to Y. Hence $\mathbf{Sets}_{+1}$ is the category of sets and *partial* functions.

We define a functor $\Box\colon \mathbf{Sets}_{+1} \to \mathbf{PoSets}^{\mathrm{op}}$ by $\Box(X) = \mathscr{P}(X)$, the poset of subsets of X, ordered by inclusion. For a function $f\colon X \longrightarrow Y+1 \equiv Y \cup \{*\}$ we define $\Box(f)\colon \mathscr{P}(Y) \to \mathscr{P}(X)$ as:

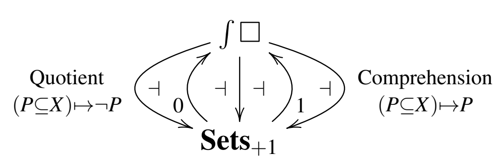
$$\Box(f)(Q) \;=\; f^{-1}(Q \cup \{*\}) \;=\; \{x \in X \mid \forall y \in Y.\, f(x) = y \implies y \in Q\}.$$

A morphism $f\colon (X,P) \longrightarrow (Y,Q)$ in $\int \Box$ is a map $f\colon X \to Y+1$ such that $f(P) \subseteq Q \cup \{*\}$.

Each poset $\Box(X) = \mathscr{P}(X)$ has a greatest element $1 = X \subseteq X$ and a least element $0 = \emptyset \subseteq X$. Moreover, $\Box(f)(1) = 1$. Hence the conditions of Lemma 2 are satisfied, so that the forgetful functor $\int \Box \to \mathbf{Sets}_{+1}$ has both a left and a right adjoint. But there is more.

Proposition 4 *In the set-theoretic case we have a chain of adjunctions as shown in* (5) *below.*

We note that there is a clear similarity with the earlier vector space and Hilbert space examples: in a quotient V/P, for a linear subspace $P \subseteq V$, all elements from P are identified. Similarly, in the above set-theoretic case, a subset $P \subseteq X$ yields as quotient the complement $\neg P = \{x \mid x \notin P\}$. It is the part of X that remains when all elements from P are removed (or, identified with the base point, $*$, in a setting with partial functions). Thus, the quotient of $P \subseteq X$ is the comprehension of $\neg P$.



(5)

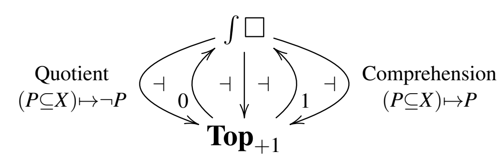
The unit of the adjunction between 0 and quotient in (5) for an object (X,P) in $\int \Box$ is obtained via the decomposition $X = P + \neg P$. The unit map $\xi_P\colon X \to \neg P + 1$ sends $x \in \neg P$ to itself and $x \in P$ to $* \in 1$. Unfolded the universal property of ξ_P reads: for every map $f\colon X \to Y+1$ such that $f(P) \subseteq \{*\}$ there is unique map $\overline{f}\colon \neg P \to Y+1$ such that $\overline{f} \odot \xi_P = f$. The map $\overline{f}$ will simply be the restriction of f to $\neg P$.

The counit of the adjunction between 1 and comprehension for an object (Y,Q) in $\int\Box$ is the inclusion $\pi_Q\colon Q \to Y$. It has the following universal property. For every $f\colon X \to Y+1$ with $f(X) \subseteq Q \cup \{*\}$ there is a unique map $\overline{f}\colon X \to Q+1$ such that $f = \pi_Q \circ \overline{f}$. The map $\overline{f}$ will simply be the restriction of f to a partial map from X to Q.

In this situation consider the following (composite) maps, the first two in $\mathbf{Sets}_{+1}$, the last one in **Sets**.

$$
\begin{array}{ccc}
P \xrightarrow{\pi_P} X \xrightarrow{\xi_{\neg P}} P & X \xrightarrow{\xi_{\neg P}} P \xrightarrow{\pi_P} X & X \xrightarrow{\mathrm{instr}_P} X+X \\
x \longmapsto x \longmapsto x & x \longmapsto \begin{cases} x & \text{if } x \in P \\ * & \text{if } x \notin P \end{cases} \longmapsto \begin{cases} x & \text{if } x \in P \\ * & \text{if } x \notin P \end{cases} & x \longmapsto \begin{cases} \kappa_1 x & \text{if } x \in P \\ \kappa_2 x & \text{if } x \notin P \end{cases}
\end{array}
\qquad (6)
$$

The first map is the identity; the second one is the 'assert' map asrt_P from the introduction; and the third one is obtained by combining asrt_P and $\mathrm{asrt}_{\neg P}$ via a suitable pullback. It is the instrument map for measurement, associated with the predicate $P \subseteq X$.



(7)

There are some relatively straightforward variations of the chain of adjunctions in (5). If one replaces the poset $\mathscr{P}(X)$ of subsets of a set X by the poset $\mathrm{Clopen}(X)$ of clopens of a topological space X one gets a functor $\Box\colon \mathbf{Top} \to \mathbf{PoSets}^{\mathrm{op}}$. For a continuous function $f\colon X \to Y+1$ (which corresponds to a continuous partial function $f\colon X \to Y$ with clopen domain) and clopen $Q \subseteq Y$ we define $\Box(f)(Q) = f^{-1}(Q \cup \{*\})$ as before. (Note that $f^{-1}(Q \cup \{*\})$ is clopen.) Again one gets a quotient–comprehension chain (7). For a clopen $P \subseteq X$ the quotient $\neg P$ and comprehension P are the same as in the case of sets (5) but now come with a natural topology induced by X. To see that this works one checks that all maps involved are continuous.

One obtains a similar chain for the category **Meas** of measurable spaces and measurable maps if one replaces the poset $\mathscr{P}(X)$ of subsets of a set X by the poset $\mathrm{Meas}(X)$ of measurable subsets of a measurable space X.

Let us think some more about the chain for topological spaces. Since a closed subset of a compact Hausdorff space is again compact, we may restrict the chain (7) to the category **CH** of compact Hausdorff spaces and the continuous maps between them. Since **CH** is dual to a whole slew of 'algebraic' categories (as opposed to 'spacial' such as **Top**) we get quotient–comprehension chains for (the opposite of) all those categories as well. For example, we get a quotient–comprehension chain for the opposite category of commutative unital C^*-algebras with unital $*$-homomorphisms via Gelfand's duality (see e.g. [6]), and for the opposite category of unital Archimedean Riesz spaces with Riesz homomorphisms via Yosida's duality [23]. Interestingly, there are quotient–comprehension chains for 'algebraic' categories which do not seem to have a 'spacial' counterpart such as the category of commutative rings and homomorphisms, such as the category $\mathbf{CRng}^{\mathrm{op}}$ of commutative rings and homomorphisms, as we will see in Section 5.

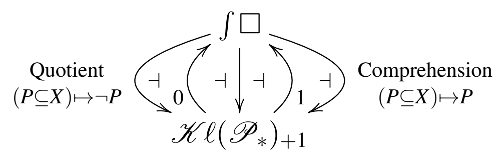
The categories **Sets**, **Top**, **Meas**, **CH** and $\mathbf{CRng}^{\mathrm{op}}$ are all *extensive* [3]. In fact, any extensive category $\mathscr{E}$ with final object has a quotient–adjunction chain of which (5) and (7) are instances. In particular, any topos will have a quotient–adjunction chain. In this general setting, the poset of subsets of a set X is replaced by the poset of *complemented* subobjects of an object X of $\mathscr{E}$. Details will appear elsewhere.

For our next example we write $\mathscr{P}_*$ for the *nonempty* powerset monad on **Sets**, $\mathscr{K}\ell(\mathscr{P}_*)$ for its Kleisli category, and $\mathscr{K}\ell(\mathscr{P}_*)_{+1}$ for the Kleisli category of the lift monad on $\mathscr{K}\ell(\mathscr{P}_*)$. Thus, maps $X \to Y$ in $\mathscr{K}\ell(\mathscr{P}_*)_{+1}$ are functions $X \to \mathscr{P}_*(Y+1)$. They capture non-deterministic computation, with multiple successor states and possibly also non-termination.

There is again a predicate functor $\Box\colon \mathcal{K}\ell(\mathcal{P}_*)_{+1} \to \mathbf{PoSets}^{\mathrm{op}}$ with $\Box(X) = \mathcal{P}(X)$ for a set X. For a map $f\colon X \to \mathcal{P}_*(Y+1)$ we define: $\quad \Box(f)(Q) = \{x \in X \mid \forall y \in Y.\, y \in f(x) \Rightarrow Q(y)\}$.

Proposition 5 *Also for non-deterministic computation via the non-empty powerset monad $\mathcal{P}_*$ we have a chain of adjunctions as shown in* (8) *below.*

Proof The truth functor $1(X) = (X \subseteq X)$ and falsum functor $0(X) = (\emptyset \subseteq X)$ are obtained via Lemma 2.



(8)

The comprehension adjunction is easy: for a map $f\colon 1X \to (Y,Q)$ in $\int\Box$, so $f\colon X \to \mathcal{P}_*(Y+1)$, we have $1X \subseteq \Box(f)(Q)$. This means that for each $x \in X$ and $y \in Y$ we have: $y \in f(x) \Rightarrow Q(y)$. Thus we can factor f as $\overline{f}\colon X \to \mathcal{P}_*(Q+1)$, giving us a map $\overline{f}\colon X \to Q$ in $\mathcal{K}\ell(\mathcal{P}_*)_{+1}$.

$$\frac{(P \subseteq X) \xrightarrow{\;f\;} 0Y \qquad \text{in } \int\Box}{\neg P \xrightarrow[g]{} Y \qquad \text{in } \mathcal{K}\ell(\mathcal{P}_*)_{+1}} \tag{9}$$

The quotient adjunction involves correspondences shown in (9). We spell out the transpose operations of this adjunction below.

Given a map $f\colon (P \subseteq X) \to (\emptyset \subseteq Y)$ in $\int\Box$, we have $P \subseteq \Box(f)(\emptyset) = \{x \mid * \in f(x)\}$. We can define $\overline{f}\colon \neg P \to \mathcal{P}_*(Y+1)$ simply as $\overline{f}(x) = f(x)$.

For $g\colon \neg P \to \mathcal{P}_*(Y+1)$ we get $\overline{g}\colon X \to \mathcal{P}_*(Y+1)$ by putting $\overline{g}(x) = g(x)$ for $x \in P$ and $\overline{g}(x) = \{*\}$ for $x \in \neg P$. This $\overline{g}$ is a map $(P \subseteq X) \to (\emptyset \subseteq Y)$ in $\int\Box$ since $\Box(\overline{g})(\emptyset) = \{x \mid \overline{g}(x) = \{*\}\} \supseteq P$.

Then for $x \in X$, we have $\overline{\overline{f}}(x) = f(x)$, and also $\overline{\overline{g}}(x) = g(x)$ for $x \in \neg P$. ∎

4 Probabilistic Examples

In this section we show how the quotient–comprehension chains of adjunctions also exist in probabilistic computation, via the (finite, discrete probability) distribution monad $\mathcal{D}$ on **Sets**, and via the Giry monad $\mathcal{G}$ on **Meas**. The monad $\mathcal{D}$ sends a set X to the set of distributions:

$$\begin{aligned} \mathcal{D}(X) \;&=\; \{r_1|x_1\rangle + \cdots r_n|x_n\rangle \mid r_i \in [0,1], x_i \in X, \textstyle\sum_i r_i = 1\} \\ &\cong\; \{\varphi\colon X \to [0,1] \mid \mathit{supp}(\varphi) \text{ is finite, and } \textstyle\sum_x \varphi(x) = 1\}, \end{aligned}$$

where $\mathit{supp}(\varphi) = \{x \mid \varphi(x) \neq 0\}$. The 'ket' notation $|x\rangle$ is just syntactic sugar, used to distinguish an element $x \in X$ from its occurrence in a formal convex sum in $\mathcal{D}(X)$. In the sequel we shall freely switch between the above two descriptions of distributions. The unit of the monad is $\eta(x) = 1|x\rangle$, and the multiplication is $\mu(\Phi)(x) = \sum_\varphi \Phi(\varphi) \cdot \varphi(x)$.

We are primarily interested in the Kleisli category $\mathcal{K}\ell(\mathcal{D})$ of the distribution monad. This category has coproducts, like in **Sets**, and the singleton set $1 = \{*\}$ as final object, because $\mathcal{D}(1) \cong 1$. Hence we can consider the Kleisli category $\mathcal{K}\ell(\mathcal{D})_{+1}$ of the lift monad $(-)+1$ on $\mathcal{K}\ell(\mathcal{D})$. Its objects are sets, and its maps $X \to Y$ are functions $X \to \mathcal{D}(Y+1)$. Elements of $\mathcal{D}(Y+1)$ are called subdistributions on Y.

As before we define a 'predicate' functor $\Box\colon \mathcal{K}\ell(\mathcal{D})_{+1} \to \mathbf{PoSets}^{\mathrm{op}}$. For a set X, take $\Box(X) = [0,1]^X$, the set of 'fuzzy' predicates $X \to [0,1]$ on X. They form a poset, by using pointwise the order on $[0,1]$. This poset $[0,1]^X$ contains a top (1) and bottom (0) element, namely the constant functions $x \mapsto 1$ and $x \mapsto 0$ respectively. For a predicate $p \in [0,1]^X$ we write $p^\perp \in [0,1]^X$ for the orthocomplement, given by $p^\perp(x) = 1 - p(x)$. Notice that $p^{\perp\perp} = p$, $1^\perp = 0$ and $0^\perp = 1$. Together with its partial sum operation, the set of fuzzy predicates $[0,1]^X$ forms a what is called an effect module, that is, an effect algebra with a $[0,1]$-action (see [12] for details).

A predicate $p \in [0,1]^X$ is called *sharp* if $p^2 = p$. This means that $p(x) \in \{0,1\}$, so that p is a Boolean predicate in $\{0,1\}^X$. Equivalently, p is sharp if $p \wedge p^\perp = 0$. For each predicate $p \in [0,1]^X$ there is a least sharp predicate $\lceil p \rceil$ with $p \leq \lceil p \rceil$, and a greatest sharp predicate $\lfloor p \rfloor \leq p$, namely:

$$\lceil p \rceil(x) = \begin{cases} 0 & \text{if } p(x) = 0 \\ 1 & \text{otherwise.} \end{cases} \qquad\qquad \lfloor p \rfloor(x) = \begin{cases} 1 & \text{if } p(x) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

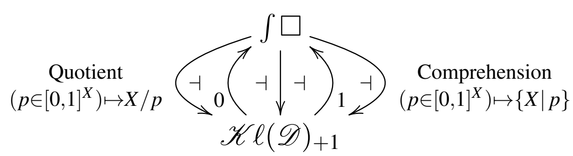
It is easy to see that these least sharp and greatest sharp predicates are each others De Morgan duals, that is, $\lfloor p^\perp \rfloor = \lceil p \rceil^\perp$. If p is a sharp, then $\lfloor p \rfloor = p = \lceil p \rceil$.

For a function $f \colon X \to \mathcal{D}(Y+1)$ we define $\Box(f) \colon [0,1]^Y \to [0,1]^X$ as:

$$\Box(f)(q)(x) = \textstyle\sum_{y \in Y} f(x)(y) \cdot q(y) + f(x)(*).$$

Since $f(x) \in \mathcal{D}(Y+1)$ is a distribution, we have $\sum_{y \in Y} f(x)(y) + f(x)(*) = 1$, so that $\Box(f)(1) = 1$. Hence Lemma 2 applies, so that we have a functor $\int \Box \to \mathcal{K}\ell(\mathcal{D})_{+1}$ with falsum 0 as left adjoint, and truth 1 as right adjoint. Recall that a map $(X,p) \to (Y,q)$ in $\int \Box$ is a function $f \colon X \to \mathcal{D}(Y+1)$ with $p(x) \leq \Box(f)(q)(x)$ for all $x \in X$.

Proposition 6 *The distribution monad $\mathcal{D}$ on **Sets**, used to model probabilistic computation, gives rise to the chain of adjunctions* (10) *to the right where* $\{X|p\} = \{x \in X \mid p(x) = 1\}$, *and* $X/p = \{X|\lceil p^\perp \rceil\} = \{x \mid p(x) \neq 1\}$.



(10)

Proof For a map $f \colon 1Y \to (X,p)$ in $\int \Box$ we have $f \colon Y \to \mathcal{D}(X+1)$ satisfying $1 \leq \Box(f)(p)$. This means $1 = \big(\sum_x f(y)(x) \cdot p(x)\big) + f(y)(*)$, for each $y \in Y$. Since $\sum_x f(y)(x) + f(y)(*) = 1$, this can only happen if $f(y)(x) \neq 0 \Rightarrow p(x) = 1$. But then we can factor f as $\overline{f} \colon Y \to \{X|p\}$ in $\mathcal{K}\ell(\mathcal{D})_{+1}$, where $\overline{f}(y) = \sum_{x, f(y)(x) \neq 0} f(y)(x)|x\rangle + f(y)(*)|*\rangle$.

In the other direction, given a function $g \colon Y \to \mathcal{D}(\{X|p\}+1)$ we define the map $\overline{g} \colon Y \to \mathcal{D}(X+1)$ as $\overline{g}(y) = \sum_{x,p(x)=1} g(y)(x)|x\rangle + g(y)(*)|*\rangle$. Then, for each $y \in Y$,

$$\Box(\overline{g})(p)(y) = \textstyle\sum_{x,p(x)=1} \overline{g}(y)(x) \cdot p(x) + \overline{g}(y)(*) = \sum_{x,p(x)=1} g(x)(y) + g(y)(*) = 1.$$

The quotient adjunction involves the correspondence (11), which works as follows. Given $f \colon (X,p) \to 0Y$ in $\int \Box$, then $f \colon X \to \mathcal{D}(Y+1)$ satisfies $p \leq \Box(f)(0)$. This means that $p(x) \leq \sum_y f(x)(y) \cdot 0(y) + f(x)(*) = f(x)(*)$, for each $x \in X$. We then define $\overline{f} \colon X/p \to \mathcal{D}(Y+1)$ as $\overline{f}(x) = \sum_y \frac{f(x)(y)}{p^\perp(x)}|y\rangle + \frac{f(x)(*) - p(x)}{p^\perp(x)}|*\rangle$. This is well-defined, since $p^\perp(x) \neq 0$ for $x \in X/p$.

$$\frac{(X,p) \xrightarrow{\;f\;} 0Y}{X/p \xrightarrow[\;g\;]{} Y} \tag{11}$$

In the other direction, given $g \colon X/p \to \mathcal{D}(Y+1)$ we define $\overline{g} \colon X \to \mathcal{D}(Y+1)$ as:

$$\overline{g}(x) = \textstyle\sum_y p^\perp(x) \cdot g(x)(y)|y\rangle + \big(p(x) + p^\perp(x) \cdot g(x)(*)\big)|*\rangle.$$

Notice that this extension of g outside the subset $\{X|\lceil p^\perp \rceil\} \hookrightarrow X$ is well-defined, since if $x \notin \{X|\lceil p^\perp \rceil\}$, then $p(x) = 1$, so $p^\perp(x) = 0$, which justifies writing $p^\perp(x) \cdot g(x)(y)$. In that case, when $p(x) = 1$, we get $\overline{g}(x) = 1|*\rangle$. This $\overline{g}$ is a morphism $(X,p) \to 0Y$ in $\int \Box$, since $p \leq \Box(\overline{g})(0)$, that is $p(x) \leq \overline{g}(x)(*)$. This follows since $p^\perp(x) \geq 0$ and $g(x)(*) \geq 0$ in $\overline{g}(x)(*) = p(x) + p^\perp(x) \cdot g(x)(*) \geq p(x)$. ■

The counit map $\pi_p\colon \{X|p\} \to \mathcal{D}(X+1)$ and the unit $\xi_p\colon X \to \mathcal{D}(X/p+1)$ are given by $\pi_p(x) = 1|x\rangle$ and $\xi_p(x) = p^{\perp}(x)|x\rangle + p(x)|*\rangle$. We can consider their combination, like in diagram (6), in $\mathcal{K}\ell(\mathcal{D})_{+1}$.

$$\begin{array}{ccc}
\{X|\lceil p\rceil\} \xrightarrow{\pi_{\lceil p\rceil}} X \xrightarrow{\xi_{p^\perp}} X/p^{\perp} = \{X|\lceil p\rceil\} & \qquad & X \xrightarrow{\xi_{p^\perp}} X/p^{\perp} = \{X|\lceil p\rceil\} \xrightarrow{\pi_{\lceil p\rceil}} X \\
x \longmapsto 1|x\rangle \mapsto p(x)|x\rangle + p^{\perp}(x)|*\rangle & & x \mapsto p(x)|x\rangle + p^{\perp}(x)|*\rangle \mapsto p(x)|x\rangle + p^{\perp}(x)|*\rangle
\end{array}$$

The map on the left is the identity if the predicate p is sharp. The map on the right is the 'assert' map asrt_p, which yields, together with $\mathrm{asrt}_{p^\perp}$ the instrument $\mathrm{instr}_p\colon X \to X+X$ in $\mathcal{K}\ell(\mathcal{D})$ given by $\mathrm{instr}_p(x) = p(x)|\kappa_1 x\rangle + (1-p(x))|\kappa_2 x\rangle$, precisely as in [12].

We can generalise the situation from (finite) discrete probabilistic computation via the monad $\mathcal{D}$, to continuous probabilistic computation via the Giry monad $\mathcal{G}$ on the category **Meas** of measurable spaces and measurable functions. The category $\mathcal{K}\ell(\mathcal{G})_{+1}$ of partial maps in the associated Kleisli category is isomorphic to the Kleisli category $\mathcal{K}\ell(\mathcal{G}_{\leq 1})$ of the 'subprobability' Giry monad. We prefer to work with the latter. Thus, for a measurable space (X, Σ_X), which is referred to simply by X, we set:

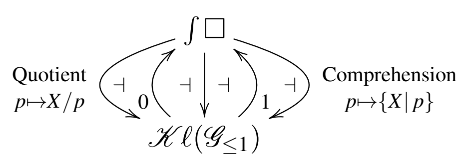
$$\mathcal{G}_{\leq 1}(X) \;=\; \{\phi\colon \Sigma_X \to [0,1] \mid \phi \text{ is a subprobability measure}\},$$

where a *subprobability measure* is a countably additive map $\phi\colon \Sigma_X \to [0,1]$ with $\phi(\emptyset) = 0$, but not necessarily $\phi(X) = 1$. As predicates $\mathit{Pred}(X)$ on $X \in \mathbf{Meas}$ we use measurable functions $X \to [0,1]$. They form an effect module, see [11] for details.

Now we define a predicate functor $\Box\colon \mathcal{K}\ell(\mathcal{G}_{\leq 1}) \to \mathbf{PoSets}^{\mathrm{op}}$. For a measurable space X we define $\Box(X) = \mathit{Pred}(X)$. For a Kleisli map $f\colon X \to \mathcal{G}_{\leq 1}(Y)$, define $\Box(f)\colon \mathit{Pred}(Y) \to \mathit{Pred}(X)$ by integration: $\Box(f)(q)(x) = \int q \,\mathrm{d}f(x) + (1 - f(x)(Y))$.

Proposition 7 *For the 'subprobability' Giry monad $\mathcal{G}_{\leq 1}$ on **Meas** there is the chain of adjunctions* (12) *where $\{X|p\} = \{x \in X \mid p(x) = 1\}$, and $X/p = \{x \mid p(x) \neq 1\}$.*

Proof The verification's proceed much like for Proposition 6, with summation $\sum$ for discrete distributions replaced by integration $\int$ for continuous distributions. Details are left to the interested reader. ■



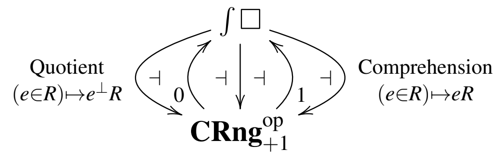
(12)

5 Commutative Ring Examples

In Section 3 it was mentioned that extensive categories have quotient–comprehension chains. This applies in particular to the extensive category $\mathbf{CRng}^{\mathrm{op}}$, where **CRng** is the category of commutative rings. Nevertheless, we describe the ring-theoretic construction here in some detail, because (1) it forms a good preparation for the more complicated example of von Neumann algebras in the next section, (2) it points to a relation with decomposition in the sheaf theory of rings.

An element $e \in R$ in a ring is called idempotent if $e^2 = e$. The set $\mathit{Pred}(R)$ of idempotents in R is an effect algebra in general, and a Boolean algebra if R is commutative. We concentrate on the latter case; then $e \leq d$ iff $ed = e$, with $e \wedge d = ed$ and $e^{\perp} = 1 - e$. The ring of integers $\mathbb{Z}$ is initial in **CRng**, and thus final in $\mathbf{CRng}^{\mathrm{op}}$. The Kleisli category $\mathbf{CRng}^{\mathrm{op}}_{+1}$ of the lift monad has ring homomorphisms $R \times \mathbb{Z} \to S$ as maps $S \to R$. They correspond to *subunital* maps $R \to S$ that preserves sums $0, +$ and multiplication, but not necessarily the unit. We define a functor $\Box\colon \mathbf{CRng}_{+1}^{\mathrm{op}} \to \mathbf{PoSets}^{\mathrm{op}}$ by $\Box(R) = \mathit{Pred}(R)$, the set of idempotents. For a subunital map $f\colon R \to S$ with define $\Box(f)\colon \mathit{Pred}(R) \to \mathit{Pred}(S)$ by $\Box(f)(e) = f(e) + f(1)^{\perp}$. We see that $\Box(f)(1) = 1$, so Lemma 2 applies.

Also in this case we have quotient and comprehension, see (13). Comprehension $\{R|e\}$ for an idempotent $e \in R$ is given by the principal ideal eR, or equivalently the ring of fractions $R[e^{-1}]$. The associated projection map $\pi_e\colon R \to eR$ is given by $\pi_e(x) = ex$. For a subunital map $f\colon R \to S$ with $1 \leq \Box(f)(e) = f(e) + f(1)^{\perp}$ we get $f(e) = f(1)$. The restriction $\overline{f}\colon eR \to S$ of f then satisfies $\overline{f} \circ \pi_e = f$, since $\overline{f}(\pi_e(x)) = f(ex) = f(e)f(x) = f(1)f(x) = f(1x) = f(x)$.



(13)

We also show that quotients are given by $R/e = e^{\perp}R$, with inclusion $\xi_e\colon e^{\perp}R \to R$ as subunital quotient map. Let $f\colon S \to R$ be a subunital map with $e \leq \Box(f)(0) = f(1)^{\perp}$. Hence $f(1) \leq e^{\perp}$ and thus $e^{\perp}f(1) = f(1)$. We define $\overline{f}\colon S \to e^{\perp}R$ as $\overline{f}(x) = f(x)$. Then:

$$\big(\xi_e \circ \overline{f}\big)(x) \;=\; \xi_e(f(x)) \;=\; e^{\perp}f(1\cdot x) \;=\; e^{\perp}f(1)f(x) \;=\; f(1)f(x) \;=\; f(x).$$

Finally we notice that each idempotent $e \in R$ gives a decomposition $R \cong eR \times e^{\perp}R = \{R|e\} \times Q/e$. This decomposition is essential in the sheaf theory of commutative rings, see [14, Chap. IV] or [1, Part III] for details. The instrument takes the form $\mathrm{instr}_e\colon R \times R \to R$, and implicitly uses this decomposition in: $\mathrm{instr}_e(x,y) = ex + e^{\perp}y$.

A similar example can be constructed for MV-modules, that is for MV-algebras with a suitable $[0,1]$-scalar multiplication. They are effect modules with a join $\vee$ (and then also meet $\wedge$) interacting appropriately with the other structure. MV-modules are also called Riesz MV-algebras, see [18]. The predicates on an MV-module A are the 'sharp' elements $p \in A$ satisfying $p^{\perp} \wedge p = 0$; they form a Boolean algebra. Comprehension $\{A|p\}$ is $\downarrow p$ and quotient A/p is $\downarrow p^{\perp}$. Again, there is a decomposition $A \cong \downarrow p \times \downarrow p^{\perp} = \{A|p\} \times A/p$, like for rings, see also [5, 6.4]. Details will be elaborated elsewhere.

The opposite of the category of commutative C^*-algebra with *-homomorphisms fits in this same pattern. We have already seen in Section 3 that it has a quotient–comprehension chain, because of the equivalence with the (extensive) category **CH** of compact Hausdorff spaces.

6 A Quantum Example

Von Neumann algebras also yield a quotient–comprehension chain, see (14) below. Strikingly, in this setting of quantum computation, one quotient–comprehension chain gives us the sequential product $a * b = \sqrt{a}b\sqrt{a}$ which is used to describe (sequential) measurement on quantum systems [8]. Since a rigorous treatment of the results in this section requires solid understanding of functional analysis we have collected the proofs and details in a separate manuscript [22] and we permit ourselves here an easygoing narrative.

We model a quantum system by a *von Neumann algebra* $\mathscr{A}$ (see [16, 20]). A finite dimensional von Neumann algebra is just a ring of matrices (closed under complex conjugation). The reader is encouraged to keep this example in mind! Roughly speaking an element a of the von Neumann algebra $\mathscr{A}$ (called an operator) represents both an observable, and the act of measuring it. *Qubits* are modelled as 2×2 complex matrices over $\mathbb{C}$.

Operators of the form a^*a are called *positive*. Their significance lies in the fact that the linear maps $\varphi\colon \mathscr{A} \to \mathbb{C}$ with $\varphi(1) = 1$ which map positive operators to positive numbers represent the *states* of the system; the number $\varphi(a)$ for an operator $a \in \mathscr{A}$ is the expectation value when measuring observable a in state φ. We are only interested in states that are *normal*, i.e. preserve directed suprema of positive operators. (Normality is only a concern for infinite dimensional von Neumann algebras: a state on a ring of matrices is always normal.) A computation which takes its input from a quantum system $\mathscr{A}$ and ends up in $\mathscr{B}$ is represented by a linear map $f\colon \mathscr{B} \to \mathscr{A}$ which is *positive* (maps positive operators to positive operators), *normal* (preserves directed suprema of positive operators) and *unital* ($f(1) = 1$); we say that f is a *PNU-map*. If the type of the map f surprises you, note that f allows us to transforms a normal state $\varphi\colon \mathscr{A} \to \mathbb{C}$ on $\mathscr{A}$ to a normal state $\varphi \circ f$ on $\mathscr{B}$. The von Neumann algebra $\mathbb{C}$ has only one state, and so contains no data. The state $\varphi\colon \mathscr{A} \to \mathbb{C}$ thus represents the computation without input that initialises $\mathscr{A}$ in state φ.

The (parallel) composition of two quantum systems $\mathscr{A}$ and $\mathscr{B}$ is represented by the tensor product $\mathscr{A} \otimes \mathscr{B}$ of which the details are delicate. One subtlety is that given computations (=PNU-maps) $f_1\colon \mathscr{A}_1 \to \mathscr{B}_1$ and $f_2\colon \mathscr{A}_2 \to \mathscr{B}_2$ their combination $f_1 \otimes f_2\colon : \mathscr{A}_1 \otimes \mathscr{A}_2 \to \mathscr{B}_1 \otimes \mathscr{B}_2$ need not be positive (i.e. map positive operators to positive operators), even when $f_2 \equiv \mathrm{id}_{\mathscr{A}}\colon \mathscr{A} \to \mathscr{A}$.

A PNU-map f for which $f \otimes \mathrm{id}_{\mathscr{A}}$ is positive for every $\mathscr{A}$ is called *completely positive* [19]. Such maps, *cPNU-maps* for short, are for our purposes the properly behaved quantum computations. Let $\mathbf{W}^*$ denote the category of cPNU-maps between von Neumann algebras.

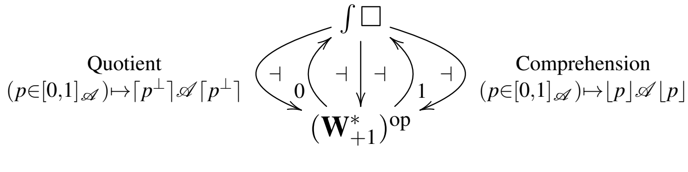
One final detail: as in the classical and probabilistic examples, we need to consider partial maps to obtain a quotient–comprehension chain. A partial quantum computation from $\mathscr{A}$ to $\mathscr{B}$ is simply a completely positive normal linear map $f\colon \mathscr{B} \to \mathscr{A}$ which is *subunital*, i.e., $f(1) \le 1$. These maps between von Neumann algebras which we will call *cPNsU-maps* form a category $\mathbf{W}^*_{+1}$. Interestingly, any ('partial') cPNsU-map $f\colon \mathscr{B} \to \mathscr{A}$ gives us a ('total') cPNU-map $g\colon \mathscr{B} \times \mathbb{C} \to \mathscr{A}$ via the equality $g(b,\lambda) = f(b) + \lambda \cdot 1$. This gives a bijection between cPNsU-maps $\mathscr{B} \to \mathscr{A}$ and cPNU-maps $\mathscr{B} \times \mathbb{C} \to \mathscr{A}$. In fact, $\mathbf{W}^*_{+1}$ is isomorphic to the Kleisli category of the comonad $(-) \times \mathbb{C}$ on the category $\mathbf{W}^*$. Put differently, $(\mathbf{W}^*_{+1})^{\mathrm{op}}$ is isomorphic to the Kleisli category of the lift monad $(-)+1$ on the opposite category $(\mathbf{W}^*)^{\mathrm{op}}$, as is consistent with Definition 3.

The predicates on a quantum system (represented by a von Neumann algebra $\mathscr{A}$) are the operators p in $\mathscr{A}$ with $0 \le p \le 1$ called *effects*. The set of effects, $[0,1]_{\mathscr{A}}$, is ordered by: $p \le q$ if $q - p$ is positive. Note that 1 is the greatest and 0 is the least element of $[0,1]_{\mathscr{A}}$. Given $p \in [0,1]_{\mathscr{A}}$ we write $p^{\perp} := 1 - p$. The effects p for which $p \wedge p^{\perp} = 0$ are called *projections*. It is notable that the projections (in a von Neumann algebra) form a complete lattice while $[0,1]_{\mathscr{A}}$ might not even be a lattice. The least projection above an effect p is denoted by $\lceil p \rceil$; the greatest projection below p is denoted by $\lfloor p \rfloor$.

We can now get down to business. Let $\Box\colon (\mathbf{W}^*_{+1})^{\mathrm{op}} \to \mathbf{PoSets}^{\mathrm{op}}$ be given by $\Box(\mathscr{A}) = [0,1]_{\mathscr{A}}$ for every von Neumann algebra $\mathscr{A}$ and $\Box(f)(p) = f(p^{\perp})^{\perp}$ for every $f\colon \mathscr{A} \to \mathscr{B}$ and $p \in \mathscr{B}$. The definition of $\Box$ is designed to give us $\Box(f)(1) = 1$ so that by Lemma 2 the forgetful functor $\int \Box \to (\mathbf{W}^*_{+1})^{\mathrm{op}}$ has a left adjoint 0 and right adjoint 1.

Proposition 8 *We have two more adjunctions giving the chain* (14) *below, see [22].*

The comprehension functor sends an effect $p \in [0,1]_{\mathscr{A}}$ to the von Neumann algebra $\lfloor p \rfloor \mathscr{A} \lfloor p \rfloor$ which has unit $\lfloor p \rfloor$. The counit of the adjunction between 1 and comprehension on p is the cPNsU-map $\pi_p\colon \mathscr{A} \to \lfloor p \rfloor \mathscr{A} \lfloor p \rfloor$ which sends a to $\lfloor p \rfloor a \lfloor p \rfloor$.



(14)

The quotient functor sends an effect p of a von Neumann algebra $\mathscr{A}$ to the set of elements of $\mathscr{A}$ of the form $\lceil p^\perp \rceil a \lceil p^\perp \rceil$ which is denoted by $\lceil p^\perp \rceil \mathscr{A} \lceil p^\perp \rceil$. We should note that $\lceil p^\perp \rceil \mathscr{A} \lceil p^\perp \rceil$ is a linear subspace of $\mathscr{A}$ which is closed under multiplication, involution $(-)^*$ and is closed in the weak operator topology, so that $\lceil p^\perp \rceil \mathscr{A} \lceil p^\perp \rceil$ is itself (isomorphic to) a von Neumann algebra. The unit of $\lceil p^\perp \rceil \mathscr{A} \lceil p^\perp \rceil$ is $\lceil p^\perp \rceil$ which might be different from the unit of $\mathscr{A}$. The unit of the adjunction between quotient and 0 on the effect $p \in \mathscr{A}$ is the cPNsU-map $\xi_p \colon \lceil p^\perp \rceil \mathscr{A} \lceil p^\perp \rceil \to \mathscr{A}$ which sends a to $\sqrt{p^\perp} a \sqrt{p^\perp}$.

As in the probabilistic example, we can form the following composites in $\mathbf{W}^*_{+1}$.

$$\begin{array}{ccccc} \lceil p \rceil \mathscr{A} \lceil p \rceil & \xleftarrow{\pi_{\lceil p \rceil}} & \mathscr{A} & \xleftarrow{\xi_{p^\perp}} & \lceil p \rceil \mathscr{A} \lceil p \rceil \\ \sqrt{p} a \sqrt{p} & \longleftarrow\!\shortmid & \sqrt{p} a \sqrt{p} & \longleftarrow\!\shortmid & a \end{array} \qquad \begin{array}{ccccc} \mathscr{A} & \xleftarrow{\xi_{p^\perp}} & \lceil p \rceil \mathscr{A} \lceil p \rceil & \xleftarrow{\pi_{\lceil p \rceil}} & \mathscr{A} \\ \sqrt{p} a \sqrt{p} & \longleftarrow\!\shortmid & \lceil p \rceil a \lceil p \rceil & \longleftarrow\!\shortmid & a \end{array} \tag{15}$$

The map on the left is the identity if the predicate p is sharp. The map on the right is the 'assert' map asrt_p, which yields, together with $\mathrm{asrt}_{p^\perp}$ the instrument $\mathrm{instr}_p \colon \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ in $\mathbf{W}^*_{+1}$ given by

$$\mathrm{instr}_p(a,b) \;=\; \sqrt{p}\, a \sqrt{p} + \sqrt{1-p}\, b \sqrt{1-p}$$

precisely as in [12]. Hence we see how the instrument map for measurement for von Neumann algebras is obtained via the logical constructions of quotient and comprehension.

Proofsketch of Proposition 8 *(Comprehension)* We must show that given a von Neumann algebra $\mathscr{A}$, an effect $p \in \mathscr{A}$, and a map $f \colon \mathscr{A} \to \mathscr{B}$ in $\mathbf{W}^*_{+1}$ with $f(p) = f(1)$ there is a unique map $g \colon \lfloor p \rfloor \mathscr{A} \lfloor p \rfloor \to \mathscr{B}$ in $\mathbf{W}^*_{+1}$ with $g(\lfloor p \rfloor b \lfloor p \rfloor) = f(b)$. Put $g(b) = f(b)$; the difficulty it to show that $f(\lfloor p \rfloor b \lfloor p \rfloor) = f(b)$. By a variant of Cauchy–Schwarz inequality for the completely positive map f (see [19], exercise 3.4)

$$\|f(c^* d)\|^2 \;\leq\; \|f(c^* c)\| \cdot \|f(d^* d)\| \qquad (c, d \in \mathscr{A})$$

we can reduce this problem to proving that $f(\lfloor p \rfloor) = f(1)$, that is, $f(\lceil p^\perp \rceil) = 0$. Since $\lceil p^\perp \rceil$ is the supremum of $p^\perp \leq (p^\perp)^{1/2} \leq (p^\perp)^{1/4} \leq \cdots$ and f is normal, $f(\lceil p^\perp \rceil)$ is the supremum of the operators $f(p^\perp) \leq f((p^\perp)^{1/2}) \leq f((p^\perp)^{1/4}) \leq \cdots$, which all turn out to be zero by Cauchy–Schwarz since $f(p) = f(1)$. Thus $f(\lceil p^\perp \rceil) = 0$, and we are done. Again, for more details, see [22].

(Quotient) We must show that given a von Neumann algebra $\mathscr{A}$, an effect $p \in \mathscr{A}$, and a map $f \colon \mathscr{B} \to \mathscr{A}$ in $\mathbf{W}^*_{+1}$ with $f(1) \leq p^\perp$, there is a unique $g \colon \mathscr{B} \to \lceil p^\perp \rceil \mathscr{A} \lceil p^\perp \rceil$ in $\mathbf{W}^*_{+1}$ such that $\sqrt{p^\perp} g(b) \sqrt{p^\perp} = f(b)$. If $\sqrt{p^\perp}$ is invertible, then we may define $g(b) = (\sqrt{p^\perp})^{-1} f(b) (\sqrt{p^\perp})^{-1}$, and this works. The proof is also straightforward if $\sqrt{p^\perp}$ is pseudoinvertible (=has norm-closed range). The trouble is that in general $\sqrt{p^\perp}$ is not (pseudo)invertible. However, using the spectral theorem [9] we can find a sequence s_n (which converges ultraweakly to the (pseudo)inverse if it exists and) for which $g(b) = \mathrm{uwlim}_n\, s_n f(b) s_n$ exists and satisfies the requirements. For further details, see [22]. ■

7 Conclusions

This paper uncovers a fundamental chain of adjunctions for quotient and comprehension in many example categories of mathematical structures, in particular von Neumann algebras. This in itself is a discovery. Truly fascinating to us is the role that these adjunctions play in the description of measurement instruments in these examples. To our regret we are unable at this stage to offer a unifying categorical formalisation, since in each of the examples there is an equality connecting adjoints which are determined only up-to-isomorphism. To be continued!

References

[1] F. Borceux (1994): *Handbook of Categorical Algebra*. Encyclopedia of Mathematics 50, 51 and 52, Cambridge Univ. Press.

[2] P. Busch & J. Singh (1998): *Lüders theorem for unsharp quantum measurements*. Phys. Letters A 249, pp. 10–12.

[3] A. Carboni, S. Lack & R. Walters (1993): *Introduction to extensive and distributive categories*. Journal of Pure and Applied Algebra 84(2), pp. 145–158.

[4] K. Cho (2015): *Total and Partial Computation in Categorical Quantum Foundations*. In: QPL 2015. To appear.

[5] R. Cignoli, I. D'Ottaviano & D. Mundici (2000): *Algebraic foundations of many-valued reasoning*. Trends in Logic 7, Springer.

[6] R. Furber & B. Jacobs (2013): *From Kleisli categories to commutative C^*-algebras: Probabilistic Gelfand duality*. In: Algebra and Coalgebra in Computer Science, Springer, pp. 141–157.

[7] M. Grandis (2012): *Homological Algebra: The Interplay of Homology with Distributive Lattices and Orthodox Semigroups*. World Scientific, Singapore.

[8] Stan Gudder & Gabriel Nagy (2001): *Sequential quantum measurements*. Journal of Mathematical Physics 42, pp. 5212–5222.

[9] P. Halmos (1957): *Introduction to Hilbert space and the theory of spectral multiplicity*. Chelsea New York.

[10] B. Jacobs (1999): *Categorical logic and type theory*. North Holland, Amsterdam.

[11] B. Jacobs (2013): *Measurable spaces and their effect logic*. In: Logic in Computer Science, IEEE, Computer Science Press.

[12] B. Jacobs (2014): *New directions in categorical logic, for classical, probabilistic and quantum Logic*. See arxiv.org/abs/1205.3940v3.

[13] Z. Janelidze (1994): *On the Form of Subobjects in Semi-Abelian and Regular Protomodular Categories*. Appl. Categorical Struct. 22(5-6), pp. 755–766.

[14] P. Johnstone (1982): *Stone spaces*. Cambridge Studies in Advanced Mathematics 3, Cambridge Univ. Press.

[15] S. Mac Lane (1950): *An algebra of additive relations*. Bull. Amer. Math. Soc. 56, pp. 485–516.

[16] F. Murray & J. Neumann (1936): *On rings of operators*. Annals of Mathematics, pp. 116–229.

[17] Michael A Nielsen & Isaac L Chuang (2010): *Quantum computation and quantum information*. Cambridge university press.

[18] A. Di Nola & I. Leuştean (2014): *Łukasiewicz logic and Riesz spaces*. Soft Computing 18(12), pp. 2349–2363.

[19] V. Paulsen (2002): *Completely bounded maps and operator algebras*. 78, Cambridge University Press.

[20] S. Sakai (1971): *C*-algebras and W*-algebras*. 60, Springer Science & Business Media.

[21] T. Weighill (2014): *Bifibrational duality in non-abelian algebra and the theory of databases*. MSc Thesis.

[22] B. Westerbaan & A. Westerbaan (2015): *A universal property of sequential measurement*. Preprint is available at `http://westerbaan.name/~bas/math/univ-prop-seq-prod.pdf`.

[23] K. Yosida (1941): *On vector lattice with a unit*. Proceedings of the Imperial Academy 17(5), pp. 121–124.