Dcpos of Commutative C*-subalgebras

Chris Heunen
Department of Computer Science
University of Oxford, United Kingdom
heunen@cs.ox.ac.uk

Bert Lindenhovius
Institute for Mathematics, Astrophysics, and Particle Physics
Radboud University Nijmegen, The Netherlands
aj.lindenhovius@math.ru.nl

This short note summarises the results in:

- A.J. Lindenhovius (2015), *Classifying finite-dimensional C*-algebras by posets of their commutative C*-subalgebras*, to appear in International Journal of Theoretical Physics. arXiv:1501.03030,
- C. Heunen and A.J. Lindenhovius (2015), *Domains of Commutative C*-subalgebras*, to appear in the proceedings of Logic in Computer Science, arXiv:1504.02730.

1 Introduction

In this contribution, we study *C*-algebras* from a new point of view. We recall that C*-algebras are defined as norm-closed selfadjoint subalgebras of the algebra $B(H)$ of bounded operators on some Hilbert space H. Since observables in quantum mechanics are usually described in terms of selfadjoint operators on a Hilbert space, C*-algebras can be used in order to describe quantum systems. Moreover, the first Gelfand-Naimark Theorem assures that each commutative unital C*-algebra A is isomorphic to the algebra $C(X)$ of continuous functions $X \to \mathbb{C}$ for some compact Hausdorff space X, called the *spectrum* of A. Since systems in classical mechanics are usually described by function spaces, this allows us to use C*-algebras for representing classical systems as well.

It follows that C*-algebras are an appropriate tool for describing the interplay between classical and quantum mechanics. For instance, the classical limit can be described in the framework of C*-algebras [22]. Moreover, C*-algebras can be used in order to translate *Bohr's doctrine of classical concepts* into a mathematical problem. Bohr's doctrine, roughly speaking, states that a measurement provides a "classical snapshot of quantum reality" and knowledge of all classical snapshots should provide a picture of quantum reality that is as complete as possible. If every commutative C*-subalgebra of a C*-algebra A corresponds practically to such a classical snapshot of the quantum system represented by A, Bohr's doctrine suggests the problem whether every C*-algebra is completely determined by its commutative C*-subalgebras.

Commutative C*-subalgebras also yield an inherent notion of *coarse graining*, or *approximation*. Observables are compatible when we can learn their joint value simultaneously, meaning that they commute as operators. Thus a measurement of the intermediate result of a partial computation is a commutative C*-subalgebra. Larger measurements, involving more observables, give us more information, leading us to use the partial order of inclusion: if $C \subseteq D$, then D contains more observables, and hence provides more information. Thus we can model approximation of quantum computations by classical ones. Performing a measurement halfway terminates a quantum computation and results in classical information, which is the only way to access quantum data. The later the measurement, the closer the approximation.

This sort of informational approximation is more commonly studied by *domain theory* [1]. As we are speaking of a continuous amount of observables, but in practice only have access to a discrete number of

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them, we are most interested in partial orders where every element can be approximated by empirically accessible ones. In domain-theoretic terms, such partial orders are called *continuous*, or a *domain*. If every element is approximable by finite ones, the domain is *algebraic*. There are also weaker versions called quasi-continuity and quasi-algebraicity. For nice enough partial orders, there is also a weaker version called meet-continuity. Another practically accessible notion of approximation in partial orders is that they be *atomistic*, meaning that the computation proceeds in indivisible steps. Finally, one can also endow domains with a topology, whose notion of limits then models approximation, such as the *Scott topology* and the *Lawson topology* [15]. In operator-algebraic terms, one might expect so-called *approximately finite-dimensional* C*-algebras, in which every observable can be approximated by observables with a finite number of outcomes [3]. We will investigate the various relationships between these notions of approximation.

2 Contributions

We start with a C*-algebra A with identity element 1_A and consider the set

$$\mathscr{C}(A) = \{C \subseteq A : C \text{ is a commutative C*-subalgebra of } A, 1_A \in C\},$$

which we order by inclusion. We emphasize that we only consider the order-theoretic properties of $\mathscr{C}(A)$, i.e., we forget about the C*-structure of elements of $\mathscr{C}(A)$. However, it turns out that actually nothing is lost, see Section 3. We prove that $\mathscr{C}(A)$ is always a *directed-complete partial order (dcpo)*, i.e., suprema of directed subsets exists, and that each *-homomorphism $f : A \to B$ between unital C*-algebras A and B induces an order morphism $\mathscr{C}(f) : \mathscr{C}(A) \to \mathscr{C}(B)$, $C \mapsto f[C]$, which is *Scott continuous*, i.e., a map that preserves directed suprema.

We proceed by translating C*-algebraic properties of A into order-theoretic properties of $\mathscr{C}(A)$. In particular, we show that A is finite dimensional if and only if $\mathscr{C}(A)$ is *Noetherian* if and only if $\mathscr{C}(A)$ is *Artinian*. Here a Noetherian poset is defined as a poset in which every ascending sequence eventually stabilizes; the Artinian condition is defined dually. Moreover, we show that the center $Z(A)$ of A corresponds to the least upper bound of the set of all maximal elements of $\mathscr{C}(A)$. If $A = \bigoplus_{i=1}^n A_i$ for C*-algebras A_i, we show that

$$\uparrow Z(A) = \{C \in \mathscr{C}(A) : Z(A) \leq C\} \cong \prod_{i=1}^n \mathscr{C}(A_i),$$

where $\mathscr{C} \cong \mathscr{D}$ denotes that two posets $\mathscr{C}$ and $\mathscr{D}$ are order isomorphic.

We proceed with showing that $\mathscr{C}(A)$ completely determines finite-dimensional C*-algebras A by using the Artin-Wedderburn Theorem, which states that A must be isomorphic as a C*-algebra to $\bigoplus_{i=1}^k M_{n_i}(\mathbb{C})$, where $k, n_1, \ldots, n_k \in \mathbb{N}$. Then the order isomorphism between $\uparrow Z(A)$ and $\prod_{i=1}^k \mathscr{C}(M_{n_i}(\mathbb{C}))$ induces an order isomorphism between

$$[Z(A), M] = \{C \in \mathscr{C}(A) : Z(A) \leq C \leq M\}$$

and $\prod_{i=1}^k \mathscr{C}(\mathbb{C}^{n_i})$ for each maximal abelian C*-subalgebra M of A. We show that this product factorization of $[Z(A), M]$ is unique, and indicate how this gives the possibility of retrieving the numbers $k, n_1, \ldots, n_k$ from $\mathscr{C}(A)$.

The next step is considering *AF-algebras*, i.e., C*-algebras that can be approximated by finite-dimensional C*-algebras. From an order-theoretical point of view, we are interested in several domain-theoretical properties of $\mathscr{C}(A)$ can be translated to C*-algebraic properties of A. Both issues result in the equivalence of the following notions:

- $\mathscr{C}(A)$ is continuous;
- $\mathscr{C}(A)$ is algebraic;
- $\mathscr{C}(A)$ is atomistic;
- $\mathscr{C}(A)$ is quasi-continuous;
- $\mathscr{C}(A)$ is quasi-algebraic;
- Every commutative C*-subalgebra of A containing the identity element of A is an AF-algebra;
- A is *scattered*.

The latter is an established but not very well-known notion; scattered C*-algebras form a subclass of AF-algebras, generalizing the notion of scattered topological spaces. Additionally, these notions imply meet-continuity of the domain, and we prove a partial converse. Our results thus make precise exactly 'how much' approximate finite-dimensionality on the algebraic side is required for these desirable notions of approximation on the domain-theoretic side, and epitomize the robustness of operator-algebraic semantics.

Conversely, when the above properties hold, $\mathscr{C}$(A) itself becomes the spectrum of a commutative C*-algebra in its own right if we equip it with the Lawson topology.

3 Related work

It was shown by Mendivil [24] and Hamhalter [17] in different ways that each commutative C*-algebra A is completely determined by $\mathscr{C}(A)$. A characterization of posets that are order isomorphic to $\mathscr{C}(A)$ for some commutative unital C*-algebra is given in [18]. These results also imply that nothing is lost if we forget about the C*-structure of an element C of $\mathscr{C}(A)$ even if A is not commutative, since the set $\{D \in \mathscr{C}(A) : D \leq C\}$ is order isomorphic to $\mathscr{C}(C)$, which determines the C*-structure of C. Moreover, in [17] it is shown as well that the Jordan structure of A can be reconstructed from $\mathscr{C}(A)$. Similar results for the special case of von Neumann algebras are proved in [13].

However, the question whether each C*-algebra A is completely determined by $\mathscr{C}(A)$ has (implicitly) been answered negatively by Connes [6], who constructed a C*-algebra A_c that is not isomorphic to its opposite algebra A_c^{op}. Here opposite algebra means the C*-algebra with the same underlying topological vector space, but with multiplication defined by $(a,b) \mapsto ba$, where $(a,b) \mapsto ab$ denotes the original multiplication. Since $\mathscr{C}(A)$ always equals $\mathscr{C}(A^{\mathrm{op}})$ for each C*-algebra A, the existence of this C*-algebra A_c shows that some extra structure is needed in order to reconstruct arbitrary C*-algebras A from $\mathscr{C}(A)$.

It is shown in [20] that *AW*-algebras*, a particular class of C*-algebras, which contains the class of von Neumann algebras, are completely determined by so-called *active lattices*, a structure which can be seen as an enrichment of $\mathscr{C}(A)$.

Apart from its mathematical relevance, the problem whether a C*-algebra is completely determined by its commutative C*-subalgebras is of great importance for the program of describing quantum mechanics in terms of topos theory (see e.g., [4, 14, 19, 30]). In this program, the central objects of research are the topoi $\mathrm{Sets}^{\mathscr{C}(A)}$ and $\mathrm{Sets}^{\mathscr{C}(A)^{\mathrm{op}}}$, for which the study of the properties of $\mathscr{C}(A)$ is essential.

Commutative C*-algebras provide relatively standard semantics for labelled Markov processes, albeit not often phrased that way, and bisimulations can be expressed algebraically [25, 29, 21, 26, 23]. But also noncommutative approximately finite-dimensional C*-algebras have been used as operational semantics of probabilistic languages [10, 11]. Furthermore, C*-algebras find applications in computer science in minimization of automata [2], and via graph theory: any directed graph gives rise to a C*-algebra which contains almost all information about the graph [27]. Additionally, C*-algebras give semantics for linear logic [9] and geometry of interaction [16]. A domain-theoretic study is new, however. Domains have played a role in labelled Markov processes [7, 8], but not in terms of C*-algebras. As far as we know, the only work similar to the current one is in quantum computing: modeling weakest preconditions [28] and giving semantics for quantum programming languages [5] in an enriched category of certain C*-algebras, but with an alternative order.

Finally, the special case of von Neumann algebras has been studied domain-theoretically [12], but forms a somewhat degenerate setting: the domain-theoretic notions listed above are not equivalent there, and come down to finite-dimensionality, which is relatively uninteresting from the perspective of information approximation.

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