Effect Algebras, Presheaves, Non-locality and Contextuality

Sam Staton
University of Oxford
sam.staton@cs.ox.ac.uk

Sander Uijlen
Radboud University, Nijmegen
suijlen1337@gmail.com

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Our paper is about generalized theories of probability that allow us to analyze the non-locality and contextuality paradoxes from quantum theory. Informally, these paradoxes have to do with the idea that it might not be possible to explain the outcomes of measurements in a classical way. We use now-standard techniques for local reasoning in computer science. Partial monoids play a crucial role in 'separation logic' which is a basic framework of locality especially relevant to memory locality and presheaves on natural numbers have already been used to study local memory and contexts in abstract syntax.

The paper is in two parts. In the first we establish new relationships between two generalized theories of probability. In the second we analyze the paradoxes of contextuality using our theories of probability, and we use this to recover earlier formulations of them in different frameworks.

1 Generalized probability measures

We start from the idea that a probability distribution on a finite measurable space (X,Ω), where $\Omega \subseteq \mathscr{P}(X)$ is a sub-Boolean algebra of the powerset of X, is a function $p:\Omega\to[0,1]$ such that $p(X)=1$ and if $A_1\dots A_n$ are disjoint sets in Ω, then $\sum_{i=1}^n p(A_i) = p(\bigcup_{i=1}^n A_i)$. In analyzing this we notice there is no actual reference to the surrounding space $\mathscr{P}(X)$. All we use is the disjoint structure of Ω. This leads us to consider two general notions of probability measure.

Partial monoids. Our first generalization involves partial monoids. More generally, we use a *pointed partial commutative monoid* (PPCM), which is a structure $(E,\ovee,0,1)$ where E is a set and $1\in E$ a distinguished point. The partial operation $\ovee : E\times E\to E$ is commutative, associative and has a unit 0. A morphism of PPCMs preserves 0,1 and the sum whenever it exists. Now $(\Omega,\uplus,\emptyset,X)$ and the interval $([0,1],+,0,1)$ are PPCMs, and a probability distribution is now the same thing as a PPCM homomorphism, $(\Omega,\uplus,\emptyset,X)\to([0,1],+,0,1)$. Thus PPCMs are a candidate for a generalized probability theory.

Functors. Our second generalization goes as follows. Every finite Boolean algebra Ω is isomorphic to one of the form $\mathscr{P}(N)$ for a finite set N, called the atoms of Ω. Now, a probability distribution $p:\Omega\to[0,1]$ is equivalently given by a function $q:N\to[0,1]$ such that $\sum_{a\in N} q(a)=1$. Let

$$D(N) = \{q : N\to[0,1] \mid \textstyle\sum_{a\in N} q(a) = 1\} \tag{1}$$

be the set of all distributions on a finite set N. It is well-known that D extends to a functor $D:\mathbf{Set_f}\to\mathbf{Set}$. The Yoneda lemma gives a bijection between distributions in $D(N)$ and natural transformations $\mathbf{Set_f}(N,-)\to D$. Thus we are led to say that a generalized finite measurable space is a functor $F:\mathbf{Set_f}\to\mathbf{Set}$ (aka presheaf), and a probability distribution on F is a natural transformation $F\to D$.

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Relationship. There is an adjunction between the two kinds of generalized measurable spaces: PPCMs, and presheaves $\mathbf{Set_f} \to \mathbf{Set}$. Given a PPCM E we obtain a functor $T(E) : \mathbf{Set_f} \to \mathbf{Set}$, where, whenever N is an n-element set, $T(E)(N)$ is the set of all n-tests on E. That is, the space of n-tuples $(e_1,\ldots,e_n)$ such that $e_1 \ovee \ldots \ovee e_n = 1$. 'Effect algebras' are a special class of PPCMs which additionally have an orthocomplement. The adjunction restricts to a reflection from effect algebras into presheaves $\mathbf{Set_f} \to \mathbf{Set}$, which gives us a slogan that 'effect algebras are well-behaved generalized finite measurable spaces'.

2 Relating non-locality and contextuality arguments

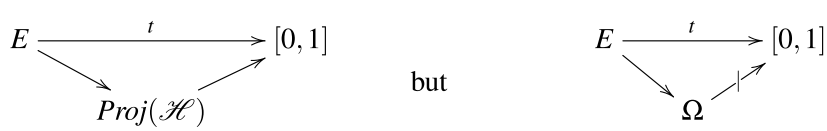
In the second part of the paper we investigate three paradoxes from quantum theory, attributed to Bell, Hardy and Kochen-Specker. We justified our use of effect algebras and presheaves by establishing relationships with earlier work by Abramsky and Brandenburger [1] and Hamilton, Isham and Butterfield [2].

We suppose a simple framework where Alice and Bob each have a measurement device with two settings which can emit 0 or 1, as the outcome of a measurement. To model this in classical probability theory we would consider a sample space S_A for Alice whose elements are functions $\{\mathrm{a}_0,\mathrm{a}_1\} \to \{0,1\}$, i.e., assignments of outcomes to measurements. Similarly we have a sample space S_B for Bob. We would then consider a joint probability distribution on S_A and S_B.

While we implicitly assume in this model that Alice and Bob cannot signal to each other, the classical model does include an assumption: Alice is able to record the outcome of the measurement in both settings. In reality, and in quantum physics, once Alice has recorded an outcome using one measurement setting, she cannot then know what the outcome would have been using the other setting. Effect algebras provide a way to describe a kind of probability distribution that takes this measure-only-once phenomenon into account. To this end, we define effect algebras E_A and E_B for Alice and Bob, which embed in their free Boolean completion B_A and B_B. The tensor product $E_\mathrm{A} \otimes E_\mathrm{B}$ is then the object of interest, since a map $E_\mathrm{A} \otimes E_\mathrm{B} \to [0,1]$ is a probability distribution on the different measurement contexts.

The non-locality 'paradox' is as follows: there are probability distributions in this effect algebraic sense, which are physically realizable, but cannot be explained in a classical probability theory without signaling.

The Bell paradox in terms of effect algebras and presheaves. As we show, the Bell scenario can be understood as a morphism of effect algebras $E \xrightarrow{t} [0,1]$, i.e., a generalized probability distribution. The paradox is that although this has a quantum realization, i.e., is physically realizable, in that it factors through $Proj(\mathscr{H})$, the projections on a Hilbert space $\mathscr{H}$, it has no explanation in classical probability theory, in that there it does not factor through a given Boolean algebra Ω. Informally:



(2)

Relationship with earlier sheaf-theoretic work on the Bell paradox. In [1], Abramsky and Brandenburger have studied Bell-type scenarios in terms of presheaves. We recover their results from our analysis in terms of generalized probability theory by noticing the embedding of effect algebras in the functor category $[\mathbf{Set_f} \to \mathbf{Set}]$. Using the fact that $T([0,1]) = D : \mathbf{Set_f} \to \mathbf{Set}$ and $B_\mathrm{A} \otimes B_\mathrm{B} \cong \mathscr{P}(O^X)$, the powerset of all functions from measurements to outcomes, so that in our example $O^X = 2^4 = 16$, we find

$T(B_\mathrm{A} \otimes B_\mathrm{B}) = T\mathscr{P}(16) = \mathbf{Set_f}(16,-)$. This leads to the diagram

$$\begin{array}{ccc} T(E_\mathrm{A} \otimes E_\mathrm{B}) & \xrightarrow{\quad Tt \quad} & D \\ & {\scriptstyle Ti}\searrow \quad \nearrow\!\!\!\!\!\!\not\;\; & \\ & \mathbf{Set_f}(16,-) & \end{array} \qquad \text{in the functor category } [\mathbf{Set_f} \to \mathbf{Set}]. \tag{3}$$

We can thus phrase Bell's paradox in the language of Grothendieck's sheaf theory. Since $i : (E_\mathrm{A} \otimes E_\mathrm{B}) \to (B_\mathrm{A} \otimes B_\mathrm{B})$ is a subalgebra and T preserves monos, $T(E_\mathrm{A} \otimes E_\mathrm{B})$ is a subpresheaf of $\mathbf{Set_f}(16,-)$, aka a 'sieve' on 16. A map $T(E_\mathrm{A} \otimes E_\mathrm{B}) \to D$ out of a sieve is called a 'compatible family', and a map $\mathbf{Set_f}(16,-) \to D$ amounts to a distribution in $D(16)$ (by the Yoneda lemma). Bell's paradox now states: "*the compatible family $T(t)$ has no amalgamation*".

We step even closer by recalling the slice category construction. This is a standard technique of categorical logic for working relative to a particular object. As we explain in the paper, the slice category $[\mathbf{Set_f} \to \mathbf{Set}]/\Omega$ is again a presheaf category. It is more-or-less the category used in [1]. Moreover, our non-factorization (2) transports to the slice category: Ω becomes terminal, and E is a subterminal object. Thus the non-factorization in diagram (2) can again be phrased in the sheaf-theoretic language of Abramsky and Brandenburger: 'the family t has no global section'.

Other paradoxes Alongside the Bell paradox we study two other paradoxes:

- The Hardy paradox is similar to the Bell paradox, except that it uses possibility rather than probability. We analyze this by replacing the unit interval $([0,1],+,0,1)$ by the PPCM $(\{0,1\},\vee,0,1)$ where $\vee$ is bitwise-or. Although this monoid is not an effect algebra, everything still works and we are able to recover the analysis of the Hardy paradox by Abramsky and Brandenburger.
- The Kochen-Specker paradox can be understood as saying that there is no PPCM morphism

$$\mathit{Proj}(\mathscr{H}) \to (\{0,1\},\otimes\!\!\!\!\!\vee,0,1) \tag{4}$$

  with dim. $\mathscr{H} \geq 3$ and where $\otimes\!\!\!\!\!\vee$ is like bitwise-or, except that $1 \otimes\!\!\!\!\!\vee 1$ is undefined. Now, the slice category $[\mathbf{Set_f} \to \mathbf{Set}]/\mathit{Proj}(\mathscr{H})$ is again a presheaf category, and it is more-or-less the presheaf category used by Hamilton, Isham and Butterfield. The non-existence of a homomorphism (4) transports to this slice category: $\mathit{Proj}(\mathscr{H})$ becomes the terminal object, and $(\{0,1\},\otimes\!\!\!\!\!\vee,0,1)$ becomes the so-called 'spectral presheaf'. We are thus able to rephrase the non-existence of a homomorphism (4) in the same way as Hamilton, Isham and Butterfield [2]: 'the spectral presheaf does not have a global section'.

Summary. We have exhibited a crucial adjunction between two general approaches to finite probability theory: effect algebras and presheaves. We have used this to analyze paradoxes of non-locality and contextuality. There are simple algebraic statements of these paradoxes in terms of partial commutative monoids, but these transport across the adjunction to statements about presheaves on $\mathbf{Set_f}$. By taking different slices of this presheaf category, we recover earlier analyses of the paradoxes.

References

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