Minimising the heat dissipation of information erasure

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Quantum state engineering and quantum computation rely on procedures that, up to some fidelity, prepare a quantum object in a pure state. If the object is initially in a statistical mixture, then this increases the largest eigenvalue of its probability spectrum. We refer to this as probabilistic information erasure. Such processes are said to occur within Landauer's framework if they rely on an interaction between the object and a thermal reservoir. Landauer's principle dictates that this must dissipate a minimum quantity of energy as heat, proportional to the entropy reduction that is incurred by the object, to the thermal reservoir. However, this lower bound is only reachable for some physical context, i.e, for a specific reservoir etc. To determine the achievable minimal heat dissipation of min-entropy reduction within a given physical context, we must explicitly optimise over all possible unitary operators that act on the composite system of object and reservoir. In this paper we characterise the equivalence class of such optimal unitary operators, using tools from majorisation theory, when we are restricted to finite dimensional Hilbert spaces.

1 Introduction

1.1 Information erasure and thermodynamics

In his attempt to exorcise Maxwell's demon [18, 16], Leo Szilard [26] conceived of an engine composed of a box that is in thermal contact with a reservoir at temperature T, and contains a single gas particle. By placing a partition in the middle of the box and determining on which side of this the particle is located, the Maxwellian demon can attach to said partition a weight-and-pulley system so that, as the gas expands, the weight is elevated. By ensuring that the partition moves without friction, and continuously adjusting the weight to make the process quasi-static, one may fully convert $k_B T \log(2)$ units of heat energy from the gas into work. Here, k_B is Boltzmann's constant and $\log(\cdot)$ is the natural logarithm. In order to save the second law of thermodynamics the engine must dissipate at least $k_B T \log(2)$ units of energy to the thermal reservoir as heat. While it was initially believed that this heat dissipation is due to the measurement act by the Maxwellian demon, following the work of Landauer, Penrose, and Bennet [14, 19, 2, 3] the responsible process was identified as the erasure of information in the demon's memory – the logically irreversible process of assigning a prescribed value to the memory, irrespective of its prior

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state. That the minimum heat dissipation required to erase one bit of information cannot be any smaller than $k_B T \log(2)$ is commonly known as Landauer's principle, and said minimum quantity as Landauer's limit. In general, Landauer's principle may be encapsulated by the Clausius inequality

$$\Delta Q \geqslant k_B T \Delta S, \tag{1}$$

where ΔQ is the heat dissipation to the thermal reservoir and ΔS is the entropy reduction in the object.

1.2 A quantum mechanical Landauer's principle

While thermodynamics is a phenomenological theory of macroscopic physics, statistical mechanics attempts to provide a microscopic picture in accordance with the known laws of motion. As the statistical postulate [12, 13] is distinct from the dynamical theory, statistical mechanics can be either classical or quantum. Indeed, recent years have been witness to a growing interest in thermodynamics and statistical mechanics in the quantum regime. This has included attempts to consider Landauer's principle quantum mechanically [21, 10, 8, 6, 25, 9, 11]. Most notable among such efforts is that of Reeb and Wolf [23], who provide a fully quantum statistical mechanical derivation of Landauer's principle by considering the process of reducing the entropy of a quantum object by its joint unitary evolution with a thermal reservoir. The only assumptions here are that, initially, the reservoir is in thermal equilibrium and hence described by the canonical distribution, and that the two systems are uncorrelated. For a reservoir with a Hilbert space of finite dimension $d_{\mathscr{R}}$, they arrive at an improved version of Landauer's inequality

$$\Delta Q \geqslant k_B T \left(\Delta S + \frac{2(\Delta S)^2}{\log^2(d_{\mathscr{R}} - 1) + 4} \right). \tag{2}$$

Moreover they demonstrate that, because unitary evolution does not change the rank of a density operator, a finite-dimensional reservoir can be used to fully purify an object if either the temperature is at absolute zero or the reservoir Hamiltonian is unbounded, with some of its eigenvalues being formally infinite; the latter case would result in an infinitely large ΔQ. If the reservoir's dimension is countably infinite, with infinitely many eigenvalues of its Hamiltonian being formally infinite also, one may always prepare the object in a pure state with a finite heat dissipation. This heat production in the reservoir can then be made arbitrarily close to $k_B T \Delta S$ by engineering the spectrum of the reservoir Hamiltonian.

1.3 The need for a context-dependent Landauer's principle

The study in [23] provides a lower bound of energy transferred to the thermal reservoir as heat dissipation, given that the object's entropy decreases by ΔS and that the reservoir's Hilbert space dimension is $d_{\mathscr{R}}$. The crucial point however is that this lower bound can be obtained for *some* physical context, but not all of them. By physical context, we mean the tuple $(\mathscr{H}_{\mathscr{O}}, \rho_{\mathscr{O}}, \mathscr{H}_{\mathscr{R}}, H_{\mathscr{R}}, T)$. Here $\mathscr{H}_{\mathscr{O}}$ and $\rho_{\mathscr{O}}$ are respectively the Hilbert space and state of the object, while $\mathscr{H}_{\mathscr{R}}$, $H_{\mathscr{R}}$, and T are respectively the Hilbert space, Hamiltonian, and temperature of the reservoir. The lower bound of Eq. (2) is achieved, by a swap map, when the Hilbert spaces of object and reservoir have the same dimension $d_{\mathscr{R}}$ and for a specific $\rho_{\mathscr{O}}$, $H_{\mathscr{R}}$ and T. Conversely, for a given physical context such inequalities prove less instructive. Indeed, if it is impossible to achieve the lower bound of Eq. (2) in a given experimental setup, in what sense can we consider this as the lowest possible heat dissipation due to information erasure? In this study, therefore, we aim to approach the problem of information erasure from the dual perspective: given a physical context, what is the minimum heat that must be dissipated in order to achieve a certain level of information erasure. This context-dependent Landauer's principle will be characterised by the equivalence class of unitary operators that achieve our task. Of course, this first requires a re-examination of what exactly we mean by information erasure.

1.4 Information erasure: pure state preparation and entropy reduction

Although erasing the information of an object leads to a reduction of its entropy, the two processes are not quantitatively the same. In quantum mechanics, erasure takes the form of pure state preparation; just as in classical mechanics erasure involves the many-to-one mapping on the information bearing degrees of freedom, then in quantum mechanics this translates naturally as the irreversible process of preparing the object in a pure state. If we wish to maximise the probability of preparing an object in a pure state, in general we need not minimise its entropy to do so; the only cases where the two coincide are when the object has a two-dimensional Hilbert space, or where we are able to fully purify the object and thereby take its entropy to zero. Consequently, our desired task can be stated as the minimisation of heat dissipation given probabilistic information erasure – that is to say, of minimising the amount of energy transferred to the thermal reservoir as heat if we require that the probability of preparing the object in a specific pure state $|\varphi_1\rangle$ be no smaller than $p_{\varphi_1}^{\max} - \delta$. Here $p_{\varphi_1}^{\max}$ is the maximum probability of information erasure that is permissible by the physical context, and $\delta \geqslant 0$ the error. We will refer to the equivalence class of unitary operators that achieve this as $[U_{\mathrm{opt}}(\delta)]$.

1.5 Layout of paper

In Sec. (2) we shall introduce the mathematical concepts, and notation, used throughout the paper. In Sec. (3) we shall characterise the equivalence class of unitary operators acting on the composite system of object and reservoir, as a result of which the object undergoes probabilistic information erasure and, given this, the reservoir gains the minimal quantity of heat. Here, we operate within the framework of Landauer: namely, that the object and reservoir are initially uncorrelated and where the composite system evolves unitarily. We demonstrate the tradeoff between probability of information erasure and minimal heat dissipation; an increase in probability of preparing the object in a defined pure state is accompanied by an increase in the minimal heat that must be dissipated to the thermal reservoir.

2 Mathematical prerequisites and notation

2.1 Majorisation theory

Here we shall introduce some useful concepts from the theory of majorisation [15]. Given a set of real numbers $\boldsymbol{a} = \{a_i\}_i$, where i belongs to an index set $I \subseteq \mathbb{N}$, we may construct the ordered sets $\boldsymbol{a}^\uparrow := \{a_i^\uparrow\}_i$ and $\boldsymbol{a}^\downarrow := \{a_i^\downarrow\}_i$ by permuting the elements in $\boldsymbol{a}$. The non-decreasing set $\boldsymbol{a}^\uparrow$ is defined such that for all $i, j \in I$ where $i < j$, we have $a_i^\uparrow \leqslant a_j^\uparrow$. Conversely the non-increasing set $\boldsymbol{a}^\downarrow$ is defined such that for all

$i, j \in I$ where $i < j$, we have $a_i^{\downarrow} \geqslant a_j^{\downarrow}$. The set $\boldsymbol{a}$ is said to be weakly majorised by $\boldsymbol{b}$ from below, denoted $\boldsymbol{a} \prec_w \boldsymbol{b}$, if and only if for every $k \in I$, $\sum_{i=1}^{k} b_i^{\downarrow} \geqslant \sum_{i=1}^{k} a_i^{\downarrow}$. Conversely, $\boldsymbol{a}$ is said to be weakly majorised by $\boldsymbol{b}$ from above, denoted $\boldsymbol{a} \prec^w \boldsymbol{b}$, if and only if for every $k \in I$, $\sum_{i=1}^{k} a_i^{\uparrow} \geqslant \sum_{i=1}^{k} b_i^{\uparrow}$. The stronger condition of $\boldsymbol{a}$ being majorised by $\boldsymbol{b}$, denoted $\boldsymbol{a} \prec \boldsymbol{b}$, is satisfied if both $\boldsymbol{a} \prec_w \boldsymbol{b}$ and $\boldsymbol{a} \prec^w \boldsymbol{b}$ (or alternatively, if $\sum_i a_i = \sum_i b_i$). A sufficient condition for $\boldsymbol{a} \prec_w \boldsymbol{b}$ is if for all i, $a_i^{\downarrow} \leqslant b_i^{\downarrow}$. We now introduce a theorem that will be central to many results in the paper.

Theorem 2.1.1. *For two sets of real numbers $\boldsymbol{a}$ and $\boldsymbol{b}$, of the same cardinality N, their dot-product obeys the relation*

$$\boldsymbol{a}^{\downarrow} \cdot \boldsymbol{b}^{\uparrow} \prec_w \boldsymbol{a} \cdot \boldsymbol{b} \prec_w \boldsymbol{a}^{\downarrow} \cdot \boldsymbol{b}^{\downarrow}.$$

For a proof we refer to Theorem II.4.2 in [4]. This leads to the simple corollary:

Corollary 2.1.1. *Consider the sets $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ and $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$, such that $\boldsymbol{a}_1 \prec \boldsymbol{a}_2$ and $\boldsymbol{b}_1 \prec \boldsymbol{b}_2$. It follows from Theorem 2.1.1 that $\boldsymbol{a}_1^{\downarrow} \cdot \boldsymbol{b}_1^{\downarrow} \prec_w \boldsymbol{a}_2^{\downarrow} \cdot \boldsymbol{b}_2^{\downarrow}$, and $\boldsymbol{a}_2^{\downarrow} \cdot \boldsymbol{b}_2^{\uparrow} \prec_w \boldsymbol{a}_1^{\downarrow} \cdot \boldsymbol{b}_1^{\uparrow}$.*

2.2 Finite-dimensional quantum mechanics

A Hilbert space $\mathscr{H}$ of finite dimension d is isomorphic to $\mathbb{C}^d$, which we denote as $\mathscr{H} \simeq \mathbb{C}^d$. We define $\mathscr{L}(\mathscr{H})$ as the space of linear operators on Hilbert space $\mathscr{H}$, $\mathscr{L}_s(\mathscr{H})$ the subspace of self-adjoint operators, and $\mathscr{S}(\mathscr{H})$ the subspace of positive operators of unit trace – namely, the state space on $\mathscr{H}$. The entropy of a state $\rho \in \mathscr{S}(\mathscr{H})$ is given by the von Neumann entropy defined as $S(\rho) := -\mathrm{tr}[\rho \log(\rho)]$. The entropy of $\rho \in \mathscr{S}(\mathscr{H})$, relative to $\sigma \in \mathscr{S}(\mathscr{H})$, is given by the relative entropy $S(\rho\|\sigma) := \mathrm{tr}[\rho(\log(\rho) - \log(\sigma))]$. The mutual information between two quantum systems A and B, with Hilbert spaces $\mathscr{H}_A$ and $\mathscr{H}_B$ respectively, is defined as $I(A:B)_\rho := S(\rho_A) + S(\rho_B) - S(\rho)$. Here $\rho \in \mathscr{S}(\mathscr{H}_A \otimes \mathscr{H}_B)$ is the state of the composite system, and $\rho_A := \mathrm{tr}_B[\rho] \in \mathscr{S}(\mathscr{H}_A)$ and $\rho_B := \mathrm{tr}_A[\rho] \in \mathscr{S}(\mathscr{H}_B)$ the marginal states of the two subsystems, given by the partial trace.

3 Information erasure within Landauer's framework

3.1 The setup

We consider a system composed of an object, $\mathscr{O}$, with Hilbert space $\mathscr{H}_{\mathscr{O}} \simeq \mathbb{C}^{d_{\mathscr{O}}}$ and reservoir, $\mathscr{R}$, with Hilbert space $\mathscr{H}_{\mathscr{R}} \simeq \mathbb{C}^{d_{\mathscr{R}}}$. Let the Hamiltonian of the reservoir be the self-adjoint operator $H_{\mathscr{R}} \in \mathscr{L}_s(\mathscr{H}_{\mathscr{R}})$ given in diagonal form as $H_{\mathscr{R}} = \sum_{m=1}^{d_{\mathscr{R}}} \lambda_m^{\uparrow} |\xi_m\rangle\langle\xi_m|$, where $\boldsymbol{\lambda}^{\uparrow} := \{\lambda_m^{\uparrow}\}_m$ is a non-decreasing set of energy eigenvalues. Similarly, the object Hamiltonian is denoted $H_{\mathscr{O}} \in \mathscr{L}_s(\mathscr{H}_{\mathscr{O}})$. Let the initial state of the system be given as $\rho = \rho_{\mathscr{O}} \otimes \rho_{\mathscr{R}}(\beta)$, where $\rho_{\mathscr{O}} \in \mathscr{S}(\mathscr{H}_{\mathscr{O}})$ and $\rho_{\mathscr{R}}(\beta) \in \mathscr{S}(\mathscr{H}_{\mathscr{R}})$. We define $\rho_{\mathscr{O}} := \sum_{l=1}^{d_{\mathscr{O}}} o_l^{\downarrow} |\varphi_l\rangle\langle\varphi_l|$ and $\rho_{\mathscr{R}}(\beta) := e^{-\beta H_{\mathscr{R}}}/\mathrm{tr}[e^{-\beta H_{\mathscr{R}}}] \equiv \sum_{m=1}^{d_{\mathscr{R}}} r_m^{\downarrow} |\xi_m\rangle\langle\xi_m|$, such that $\boldsymbol{o}^{\downarrow} := \{o_l^{\downarrow}\}_l$ and $\boldsymbol{r}^{\downarrow} := \{r_m^{\downarrow}\}_m$ are non-increasing probability sets. These representations are unique if and only if there are no degeneracies in the probability distributions. Here, $\rho_{\mathscr{R}}(\beta)$ is referred to as the thermal state of the reservoir, at the inverse temperature $\beta := (k_B T)^{-1} \in (0, \infty)$. For simplicity, we write the initial state ρ in the equivalent form

$$\rho = \sum_{l=1}^{d_{\mathscr{O}}} \sum_{m=1}^{d_{\mathscr{R}}} o_l^{\downarrow} r_m^{\downarrow} |\varphi_l\rangle\langle\varphi_l| \otimes |\xi_m\rangle\langle\xi_m| \equiv \sum_{n=1}^{d_{\mathscr{O}} d_{\mathscr{R}}} p_n^{\downarrow} |\psi_n\rangle\langle\psi_n|, \tag{3}$$

where the non-increasing set $\boldsymbol{p}^{\downarrow} := \{p_n^{\downarrow}\}_n$ is the ordered permutation of $\{o_l^{\downarrow} r_m^{\downarrow}\}_{l,m}$, and $\{|\psi_n\rangle \in \mathscr{H}_{\mathscr{O}} \otimes \mathscr{H}_{\mathscr{R}}\}_n$ the associated permutation of $\{|\varphi_l\rangle \otimes |\xi_m\rangle\}_{l,m}$. Assuming the total system is closed, it will evolve according to a unitary operator $U \in \mathscr{L}(\mathscr{H}_{\mathscr{O}} \otimes \mathscr{H}_{\mathscr{R}})$ to give the final state

$$\rho' := U\rho U^{\dagger} = \sum_{n=1}^{d_{\mathscr{O}} d_{\mathscr{R}}} p_n^{\downarrow} U|\psi_n\rangle\langle\psi_n| U^{\dagger}. \tag{4}$$

The marginal states of ρ' are $\rho'_{\mathscr{O}} := \mathrm{tr}_{\mathscr{R}}[\rho']$ and $\rho'_{\mathscr{R}} := \mathrm{tr}_{\mathscr{O}}[\rho']$. We define the reduction in entropy of $\mathscr{O}$ as $\Delta S := S(\rho_{\mathscr{O}}) - S(\rho'_{\mathscr{O}})$. The total average energy consumption of the erasure protocol is

$$\begin{aligned} \Delta E := \mathrm{tr}[(H_{\mathscr{O}} + H_{\mathscr{R}})(\rho' - \rho)] &= \mathrm{tr}[H_{\mathscr{O}}(\rho'_{\mathscr{O}} - \rho_{\mathscr{O}})] + \mathrm{tr}[H_{\mathscr{R}}(\rho'_{\mathscr{R}} - \rho_{\mathscr{R}}(\beta)], \\ &= \Delta W + \Delta Q. \end{aligned} \tag{5}$$

A positive ΔE implies that the process requires energy from a power supply, or battery. Conversely, a negative ΔE implies that the process produces energy that can, in turn, be stored in said battery. Here, ΔW is the energy change in the object, and ΔQ the energy change in the reservoir. As shown in [7, 23], these terms can also be written as

$$\beta \Delta W = S(\rho'_{\mathscr{O}} \| \rho_{\mathscr{O}}(\beta)) - S(\rho_{\mathscr{O}} \| \rho_{\mathscr{O}}(\beta)) - \Delta S, \tag{6}$$

$$\beta \Delta Q = \Delta S + I(\mathscr{O} : \mathscr{R})_{\rho'} + S(\rho'_{\mathscr{R}} \| \rho_{\mathscr{R}}(\beta)). \tag{7}$$

As we are only interested in cases where ΔS is positive, we can infer from the non-negativity of the relative entropy and mutual information that ΔQ is always positive for information erasure, even though ΔW may be negative. By construction the reservoir is a system which, after its utility in the erasure process, remains maximally passive and the energy stored therein cannot be used to do work afterwards. It is in this sense that we may interpret ΔQ as heat [22, 24]. As such, minimising ΔQ in an erasure protocol is tantamount to minimising the waste of potential energy – that is to say, the energy stored in the battery in addition to the free energy of the object. Of course, a finite-dimensional reservoir will not in general remain in a thermal state after its joint evolution with the object.

3.2 Maximising the probability of information erasure

As the pure state we wish to prepare the object in is arbitrary up to local unitary operations, for simplicity we choose this to be $|\varphi_1\rangle$. The probability of finding $\rho'_{\mathscr{O}}$ in the state $|\varphi_1\rangle$ is defined as

$$\begin{aligned} p(\varphi_1|\rho'_{\mathscr{O}}) := \langle\varphi_1|\rho'_{\mathscr{O}}|\varphi_1\rangle &= \sum_{n=1}^{d_{\mathscr{O}} d_{\mathscr{R}}} p_n^{\downarrow} \langle\psi_n| U^{\dagger}(|\varphi_1\rangle\langle\varphi_1| \otimes \mathbb{1}_{\mathscr{R}}) U |\psi_n\rangle, \\ &= \sum_{n=1}^{d_{\mathscr{O}} d_{\mathscr{R}}} p_n^{\downarrow} g_n(U) \equiv \boldsymbol{p}^{\downarrow} \cdot \boldsymbol{g}(\boldsymbol{U}), \end{aligned} \tag{8}$$

where $\boldsymbol{g(U)}$ is a vector of objects $g_n(U) := \langle \psi_n | U^\dagger (|\varphi_1\rangle\langle\varphi_1| \otimes \mathbb{1}_{\mathscr{R}}) U | \psi_n \rangle$.

Lemma 3.2.1. *The maximum probability of information erasure is $p^{\max}_{\varphi_1} = \sum_{m=1}^{d_{\mathscr{R}}} p^\downarrow_m$. The equivalence class of unitary operators that achieve this, denoted $[U^g_{\text{maj}}]$, is characterised by the rule*

$$\text{for all } m \in \{1, \dots, d_{\mathscr{R}}\}\ ,\ U^g_{\text{maj}} |\psi_m\rangle = |\varphi_1\rangle \otimes |\xi'_m\rangle ,$$

where $\{|\xi'_m\rangle\}_m$ is an arbitrary orthonormal basis in $\mathscr{H}_{\mathscr{R}}$.

Proof. By Theorem 2.1.1 we know that $\boldsymbol{p}^\downarrow \cdot \boldsymbol{g(U)} \prec_w \boldsymbol{p}^\downarrow \cdot \boldsymbol{g}^\downarrow\boldsymbol{(U)}$. Let U^g_{maj} be a member of an equivalence class of unitary operators such that $\boldsymbol{g(U^g_{\text{maj}})} = \boldsymbol{g}^\downarrow\boldsymbol{(U^g_{\text{maj}})}$ and $\boldsymbol{g}^\downarrow\boldsymbol{(U)} \prec \boldsymbol{g}^\downarrow\boldsymbol{(U^g_{\text{maj}})}$ for all $U \in \mathscr{L}(\mathscr{H}_{\mathscr{O}} \otimes \mathscr{H}_{\mathscr{R}})$. Therefore, by Corollary 2.1.1 we get $\boldsymbol{p}^\downarrow \cdot \boldsymbol{g}^\downarrow\boldsymbol{(U)} \prec_w \boldsymbol{p}^\downarrow \cdot \boldsymbol{g}^\downarrow\boldsymbol{(U^g_{\text{maj}})}$, and hence $p(\varphi_1|\rho'_{\mathscr{O}})$ is maximised by U^g_{maj}. Because $g_n(U) \in [0,1]$ for all n, and $\sum_n g_n(U) = d_{\mathscr{R}}$, the first $d_{\mathscr{R}}$ elements in $\boldsymbol{g}^\downarrow\boldsymbol{(U^g_{\text{maj}})}$ must be one, and the rest zero. □

The fact that $p^{\max}_{\varphi_1}$ in general cannot be brought to unity has been reported in [23, 27, 1, 29]. A necessary and sufficient condition for $p^{\max}_{\varphi_1}$ to be greater than $p(\varphi_1|\rho_{\mathscr{O}}) := o^\downarrow_1$ is that $o^\downarrow_2 r^\downarrow_1$ be greater than $o^\downarrow_1 r^\downarrow_{d_{\mathscr{R}}}$. Otherwise, we would have $p^{\max}_{\varphi_1} = \sum_{m=1}^{d_{\mathscr{R}}} o^\downarrow_1 r^\downarrow_m = o^\downarrow_1$. This implies that for probabilistic information erasure, we require that

$$\frac{o^\downarrow_1}{o^\downarrow_2} < \frac{r^\downarrow_1}{r^\downarrow_{d_{\mathscr{R}}}} = e^{\beta(\lambda^\uparrow_{d_{\mathscr{R}}} - \lambda^\uparrow_1)} \leqslant e^{2\beta \|H_{\mathscr{R}}\|}. \tag{9}$$

Similar arguments were made in [23], although there the focus was on providing a bound on the smallest eigenvalue of $\rho'_{\mathscr{O}}$ that could be obtained. The more pure the initial state of the object is, therefore, the larger $\lambda^\uparrow_{d_{\mathscr{R}}} - \lambda^\uparrow_1$ must be to further purify it. Of course if the object is maximally mixed then, so long as $H_{\mathscr{R}}$ is not proportional to $\mathbb{1}$, we may increase its purity by some degree, small though it may be.

3.3 Minimising the heat dissipation

We may always write the post-transformation marginal state of the reservoir as

$$\rho'_{\mathscr{R}} = \sum_{m=1}^{d_{\mathscr{R}}} r'^\downarrow_m(U) |\xi'_m\rangle\langle\xi'_m|, \tag{10}$$

with $\boldsymbol{r'^\downarrow(U)} := \{r'^\downarrow_m(U)\}_m$ a non-increasing set of probabilities and $\{|\xi'_m\rangle\}_m$ an arbitrary orthonormal basis in $\mathscr{H}_{\mathscr{R}}$. Because $\rho_{\mathscr{R}}(\beta)$ is fixed, minimising ΔQ is achieved by minimising the average energy of this state, given as

$$\text{tr}[H_{\mathscr{R}} \rho'_{\mathscr{R}}] = \sum_{m=1}^{d_{\mathscr{R}}} r'^\downarrow_m(U) \langle\xi'_m|H_{\mathscr{R}}|\xi'_m\rangle \equiv \boldsymbol{r'^\downarrow(U)} \cdot \boldsymbol{\lambda'}, \tag{11}$$

where $\boldsymbol{\lambda'}$ is a set of objects $\lambda'_m := \langle\xi'_m|H_{\mathscr{R}}|\xi'_m\rangle$. To determine how ΔQ can be minimised, we first provide a recursive proof to show that the set of eigenvalues $\boldsymbol{\lambda}$ majorises all possible $\boldsymbol{\lambda'}$.

Lemma 3.3.1. *$\boldsymbol{\lambda}' \prec \boldsymbol{\lambda}$ for all orthonormal bases $\{|\xi_m'\rangle \in \mathscr{H}_{\mathscr{R}}\}_m$.*

Proof. To show this, it is sufficient to show that $\sum_m \lambda_m = \sum_m \lambda_m'$ and $\lambda' \prec^w \lambda$ for all $\{|\xi_m'\rangle\}_m$. The first condition is trivial, as $\sum_m \lambda_m' = \mathrm{tr}[H_{\mathscr{R}}]$ and is independent of $\{|\xi_m'\rangle\}_m$. To show that $\lambda' \prec^w \lambda$, it is sufficient to prove that for all m and $\{|\xi_m'\rangle\}_m$, $\lambda_m^{\uparrow} \leqslant \lambda_m'^{\uparrow}$. This can be done by showing that the minimal value attainable by λ_1' is λ_1 and, given this constraint, the minimal value attainable by λ_2' is λ_2, and so on. One may always write $|\xi_m'\rangle = \alpha_m |\xi_m\rangle + \beta_m \left|\xi_m^{\perp}\right\rangle$ where $\left|\xi_m^{\perp}\right\rangle$ is the orthogonal complement to $|\xi_m\rangle$ in $\mathscr{H}_{\mathscr{R}}$. Consequently, we have $\lambda_m' = |\alpha_m|^2\langle \xi_m|H_{\mathscr{R}}|\xi_m\rangle + |\beta_m|^2\langle \xi_m^{\perp}|H_{\mathscr{R}}|\xi_m^{\perp}\rangle$. It follows from the Ky Fan principle [4] that $\langle \xi_1^{\perp}|H_{\mathscr{R}}|\xi_1^{\perp}\rangle \geqslant \langle \xi_1|H_{\mathscr{R}}|\xi_1\rangle =: \lambda_1$. Therefore we know that λ_1' is minimised by setting $|\alpha_1|^2 = 1$. In the next step, the fact that $\langle \xi_1'|\xi_2'\rangle = 0$ and that our previous step sets $|\xi_1'\rangle = |\xi_1\rangle$ implies that $\langle \xi_1|\xi_2^{\perp}\rangle = 0$. This in turn implies that $\langle \xi_2^{\perp}|H_{\mathscr{R}}|\xi_2^{\perp}\rangle \geqslant \langle \xi_2|H_{\mathscr{R}}|\xi_2\rangle =: \lambda_2$, so that $\langle \xi_2'|H_{\mathscr{R}}|\xi_2'\rangle$ is minimised by setting $|\alpha_2|^2 = 1$. This argument can be made recursively for all m. □

Now we are able to characterise the equivalence class of unitary operators that minimise ΔQ.

Lemma 3.3.2. *ΔQ is minimised by the equivalence class of unitary operators $[U_{\mathrm{maj}}^f]$ characterised by the rule*

$$\text{for all } m \in \{1,\ldots,d_{\mathscr{R}}\} \text{ and } n \in \{(m-1)d_{\mathscr{O}}+1,\ldots,md_{\mathscr{O}}\}\ ,\ U_{\mathrm{maj}}^f|\psi_n\rangle = |\varphi_l^m\rangle \otimes |\xi_m\rangle\,,$$

with the set $\{\left|\varphi_l^m\right\rangle | l \in \{1,\ldots,d_{\mathscr{O}}\}\}$ forming an orthonormal basis in $\mathscr{H}_{\mathscr{O}}$ for each m.

Proof. By Corollary 2.1.1 and Lemma 3.3.1, $\boldsymbol{r}'^{\downarrow}(\boldsymbol{U}) \cdot \boldsymbol{\lambda}^{\uparrow} \prec_w \boldsymbol{r}'^{\downarrow}(\boldsymbol{U}) \cdot \boldsymbol{\lambda}'$. Therefore $\mathrm{tr}[H_{\mathscr{R}}\rho_{\mathscr{R}}']$ is minimal when for all m, $|\xi_m'\rangle = |\xi_m\rangle$. In such a case, we have

$$\begin{aligned} r_m'^{\downarrow}(U) := \langle \xi_m|\rho_{\mathscr{R}}'|\xi_m\rangle &= \sum_{n=1}^{d_{\mathscr{O}}d_{\mathscr{R}}} p_n^{\downarrow}\langle \psi_n|U^{\dagger}(\mathbb{1}_{\mathscr{O}} \otimes |\xi_m\rangle\langle\xi_m|)U|\psi_n\rangle, \\ &= \sum_{n=1}^{d_{\mathscr{O}}d_{\mathscr{R}}} p_n^{\downarrow} f_n(U,m) = \boldsymbol{p}^{\downarrow} \cdot \boldsymbol{f}(\boldsymbol{U},\boldsymbol{m}), \end{aligned} \tag{12}$$

where $\boldsymbol{f}(\boldsymbol{U},\boldsymbol{m})$ is a set of objects $f_n(U,m) := \langle \psi_n|U^{\dagger}(\mathbb{1}_{\mathscr{O}} \otimes |\xi_m\rangle\langle\xi_m|)U|\psi_n\rangle$. Let U_{maj}^f be a member of the equivalence class of unitary operators such that $\boldsymbol{r}'^{\downarrow}(\boldsymbol{U}) \prec \boldsymbol{r}'^{\downarrow}(\boldsymbol{U}_{\mathrm{maj}}^{\boldsymbol{f}})$ for all $U \in \mathscr{L}(\mathscr{H}_{\mathscr{O}} \otimes \mathscr{H}_{\mathscr{R}})$. By Corollary 2.1.1 it would then follow that $\boldsymbol{r}'^{\downarrow}(\boldsymbol{U}_{\mathrm{maj}}^{\boldsymbol{f}}) \cdot \boldsymbol{\lambda}^{\uparrow} \prec_w \boldsymbol{r}'^{\downarrow}(\boldsymbol{U}) \cdot \boldsymbol{\lambda}^{\uparrow}$, resulting in the minimisation of $\mathrm{tr}[H\rho_{\mathscr{R}}']$ and hence ΔQ. To find $\boldsymbol{r}'^{\downarrow}(\boldsymbol{U}_{\mathrm{maj}}^{\boldsymbol{f}})$, we first need to maximise $r_1'^{\downarrow}(U)$ and then, given this constraint, maximise $r_2'^{\downarrow}(U)$, and so on. This, in turn, is achieved by choosing U_{maj}^f so that $\boldsymbol{f}(\boldsymbol{U}_{\mathrm{maj}}^{\boldsymbol{f}},\boldsymbol{1}) = \boldsymbol{f}^{\downarrow}(\boldsymbol{U}_{\mathrm{maj}}^{\boldsymbol{f}},\boldsymbol{1})$ and $\boldsymbol{f}^{\downarrow}(\boldsymbol{U}_{\mathrm{maj}}^{\boldsymbol{f}},\boldsymbol{1}) \succ \boldsymbol{f}^{\downarrow}(\boldsymbol{U},\boldsymbol{1})$ for all U. Note that for each m, $f_n(U,m) \in [0,1]$ for all n, and $\sum_n f_n(U,m) = d_{\mathscr{O}}$. Hence, the first $d_{\mathscr{O}}$ entries of $\boldsymbol{f}^{\downarrow}(\boldsymbol{U}_{\mathrm{maj}}^{\boldsymbol{f}},\boldsymbol{1})$ are taken to one and the rest to zero. Because of the constraint posed by the orthogonality of the vectors $\{U|\psi_n\rangle\}_n$, however, the first $d_{\mathscr{O}}$ elements of $\boldsymbol{f}(\boldsymbol{U}_{\mathrm{maj}}^{\boldsymbol{f}},\boldsymbol{2})$ must be zero, and to maximise $r_2'^{\downarrow}(U)$ the best we can do is to only take the second $d_{\mathscr{O}}$ entries of $\boldsymbol{f}(\boldsymbol{U}_{\mathrm{maj}}^{\boldsymbol{f}},\boldsymbol{2})$ to one, with the rest being zero. This argument is then made recursively for all m. □

3.4 Minimal heat dissipation conditional on maximising the probability of information erasure

It should now be clear that the two objectives of maximising the probability of information erasure, and minimising the heat dissipation, cannot be achieved simultaneously: $[U^g_{\text{maj}}] \cap [U^f_{\text{maj}}] = \{\emptyset\}$. The two tasks are in some sense complementary, and there will be a tradeoff between them. Here, we shall prioritise; a unitary operator will be chosen such that it maximises the probability of information erasure and, given this constraint, minimises the heat dissipation. In other words, we find the equivalence class of unitary operators $[U_{\text{opt}}(0)] \subset [U^g_{\text{maj}}]$ that minimise ΔQ. The purpose of the zero in parentheses will become apparent in Sec. (3.5). To this end we first divide the set of probabilities $\boldsymbol{p}^{\downarrow}$ to form the non-increasing set of cardinality $d_{\mathscr{R}}$, denoted $\Pi^{\downarrow}_0$, and the non-increasing sets of cardinality $d_{\mathscr{O}} - 1$, denoted $\{\Pi^{\downarrow}_m | m \in \{1, \dots, d_{\mathscr{R}}\}\}$, defined as

$$\begin{aligned} \Pi^{\downarrow}_0 &:= \{p^{\downarrow}_m | m \in \{1, \dots, d_{\mathscr{R}}\}\}, \\ \Pi^{\downarrow}_{m \geqslant 1} &:= \{p^{\downarrow}_{d_{\mathscr{R}} + (m-1)(d_{\mathscr{O}}-1)+l} | l \in \{1, \dots, d_{\mathscr{O}} - 1\}\}. \end{aligned} \tag{13}$$

We refer to the m^{th} element of $\Pi^{\downarrow}_0$ as $\Pi^{\downarrow}_0(m)$, and the l^{th} element of $\Pi^{\downarrow}_{m \geqslant 1}$ as $\Pi^{\downarrow}_{m \geqslant 1}(l)$.

Theorem 3.4.1. *The equivalence class of unitary operators that maximise the probability of information erasure and, given this constraint, minimise the heat dissipation, is denoted as $[U_{\text{opt}}(0)]$. This is characterised by the rules*

$$U_{\text{opt}}(0) : \begin{cases} |\psi_n\rangle \mapsto |\varphi_1\rangle \otimes |\xi_m\rangle & \text{if } p^{\downarrow}_n = \Pi^{\downarrow}_0(m), \\ |\psi_n\rangle \mapsto |\varphi^m_l\rangle \otimes |\xi_m\rangle & \text{if } p^{\downarrow}_n = \Pi^{\downarrow}_m(l) \text{ and } m \geqslant 1, \end{cases}$$

where, for all m, each member of the orthonormal set $\{|\varphi^m_l\rangle\}_l$ is orthogonal to $|\varphi_1\rangle$.

Proof. The first line conforms with the conditions imposed by Lemma 3.2.1 and, as such, results in $p(\varphi_1|\rho'_{\mathscr{O}}) = p^{\max}_{\varphi_1}$. However, here we are restricted to the case $|\xi'_m\rangle = |\xi_m\rangle$ for all m, thereby minimising the contribution to heat dissipation by Corollary 2.1.1 and Lemma 3.3.1. The second line, by virtue of not affecting $p(\varphi_1|\rho'_{\mathscr{O}})$, is evidently allowed for a unitary operator in the equivalence class $[U^g_{\text{maj}}]$. This rule takes the $d_{\mathscr{R}}$ largest remaining probabilities to states $|\varphi^1_l\rangle \otimes |\xi_1\rangle$, thereby maximising the probability associated with $|\xi_1\rangle$, and so on for the other states $|\xi_m\rangle$. By the same line of reasoning as in Lemma 3.3.2, therefore, the contribution to heat dissipation from this line is minimal. □

We now make the following observations:

(a) If we choose $|\varphi^m_l\rangle = |\varphi_{l+1}\rangle$ for all m, and such that $\{|\varphi_l\rangle\}_l$ are the eigenvectors of the object Hamiltonian $H_{\mathscr{O}}$ in increasing order of energy, then $U_{\text{opt}}(0)$ would also ensure that the purification to the ground state $|\varphi_1\rangle$ would be done in such a way that $p(\varphi_i|\rho'_{\mathscr{O}}) \geqslant p(\varphi_j|\rho'_{\mathscr{O}})$ for all $i < j$; the object is brought to a passive state, although in general with more energy than if it was "cooled" [22, 20]. We refer to this as *passive information erasure*, and the resultant equivalence class of unitary operators as $[U^{\text{p}}_{\text{opt}}(0)]$. These unitary operators will result in the smallest possible ΔE, conditional on first maximising the probability of information erasure, and then minimising the heat dissipation; that is to say, $[U^{\mathrm{p}}_{\mathrm{opt}}(0)]$ minimises ΔW for all unitary operators in the equivalence class $[U_{\mathrm{opt}}(0)]$. The state $\rho' = U^{\mathrm{p}}_{\mathrm{opt}}(0)\rho U^{\mathrm{p}}_{\mathrm{opt}}(0)^\dagger$ for such a case is depicted in matrix representation in Fig. (2).

(b) Since the desired task is the maximisation of $p(\varphi_1|\rho'_{\mathscr{O}})$, we need not maximise ΔS because this will lead to a greater amount of heat dissipation than necessary, as per Eq. (7). The only cases where maximisation of $p(\varphi_1|\rho'_{\mathscr{O}})$ necessarily leads to the maximisation of ΔS are when: (i) $p^{\max}_{\varphi_1} = 1$; and (ii) where $\mathscr{H}_{\mathscr{O}} \simeq \mathbb{C}^2$. In case (i) the entropy of the object is brought to zero, so ΔS is trivially maximised. In case (ii), we note that if $\boldsymbol{o}_1^\downarrow \succ \boldsymbol{o}_2^\downarrow$, where $\boldsymbol{o}_1^\downarrow$ and $\boldsymbol{o}_2^\downarrow$ are the probability spectra of $\rho^1_{\mathscr{O}}$ and $\rho^2_{\mathscr{O}}$ respectively, then $S(\rho^1_{\mathscr{O}}) \leqslant S(\rho^2_{\mathscr{O}})$. If we maximise $p(\varphi_1|\rho'_{\mathscr{O}})$ in the case of $\mathscr{O}$ being a two-level system, this will necessarily minimise $p(\varphi_2|\rho'_{\mathscr{O}})$. This in turn will result in the probability spectrum of $\rho'_{\mathscr{O}}$ to majorise all possible spectra. Consequently, this will minimise $S(\rho'_{\mathscr{O}})$ and hence maximise ΔS.

However, one can always say that maximising the probability of information erasure requires that we maximise the increase in min-entropy, $S_{\min}$, defined as

$$S_{\min}(\rho) := \min_i\{-\log(p_i)\}, \tag{14}$$

where $\{p_i\}_i$ is the probability spectrum of ρ [28]. The min-entropy is clearly given by the largest probability, and to maximise its increase, we must maximise the largest probability of the system; this is the definition of maximising the probability of information erasure.

(c) The only instance where $\mathscr{H}_{\mathscr{O}} \simeq \mathscr{H}_{\mathscr{R}} \simeq \mathbb{C}^d$, and $U^{\mathrm{p}}_{\mathrm{opt}}(0)$ for passive, maximally probable information erasure is a swap operation, is when $d = 2$. For larger dimensions, this is no longer the case.

(d) It is evident that the spectrum of $\rho'_{\mathscr{R}}$ is is non-decreasing with respect to the energy levels of its Hamiltonian, but that its spectrum is majorised by that of $\rho_{\mathscr{R}}(\beta)$. As such, by Corollary 2.1.1, $\Delta Q \geqslant 0$. This conforms with Landauer's principle that information erasure must dissipate heat.

3.5 The tradeoff between probability of information erasure and minimal heat dissipation

It may be the case that one does not need to maximise the probability of information erasure, but simply requires that $p(\varphi_1|\rho'_{\mathscr{O}}) \geqslant p^{\max}_{\varphi_1} - \delta$, with the error $\delta \in [0, p^{\max}_{\varphi_1} - o_1^\downarrow]$. The question would therefore be: how will the minimal achievable ΔQ be affected by varying δ, and how may we characterise the equivalence class of unitary operators $[U^{\mathrm{p}}_{\mathrm{opt}}(\delta)]$ in such a case? The answer for the extremal cases is trivial; when $\delta = p^{\max}_{\varphi_1} - o_1^\downarrow$, then $[U^{\mathrm{p}}_{\mathrm{opt}}(\delta)] = \mathbb{1}$ and $\Delta Q = 0$, while $[U^{\mathrm{p}}_{\mathrm{opt}}(\delta)]$ for $\delta = 0$ reduces to the case discussed in Sec. (3.4), wherein $\Delta Q \geqslant 0$. To answer the question for the intermediate values of δ, we first make the following observations:

(a) For any value of ΔQ, $p(\varphi_1|\rho'_{\mathscr{O}})$ is maximised when the eigenvectors of $\rho'_{\mathscr{O}}$ that have support on $|\varphi_1\rangle$ are given by the set $\{|\varphi_l\rangle\}_l$. This follows from Corollary 2.1.1, which implies that $p(\varphi_1|\rho'_{\mathscr{O}}) = \sum_l o_l^{\prime\downarrow}|\langle\varphi_1|\varphi'_l\rangle|^2 \leqslant o_1^{\prime\downarrow}$, where $\rho'_{\mathscr{O}} = \sum_l o_l^{\prime\downarrow}|\varphi'_l\rangle\langle\varphi'_l|$.

(b) For any value of $p(\varphi_1|\rho'_{\mathscr{O}})$, ΔQ is minimised when the eigenvectors of $\rho'_{\mathscr{R}}$ are given by the set $\{|\xi_m\rangle\}_m$. This follows from Lemma 3.3.1.

Observations (a) and (b), together, show that the optimal case will require that, for all n, $U|\psi_n\rangle = \sum_l \sqrt{\gamma_l^n}|\varphi_l\rangle \otimes |\xi_l^n\rangle$. Here $\gamma_l^n \geqslant 0$ are the Schmidt coefficients, and $|\xi_l^n\rangle = e^{\mathrm{i}\phi_l^n}\sigma_n|\xi_l\rangle$ with σ_n a permutation on the set $\{|\xi_l\rangle\}_l$ and $\phi_l^n \in [0, 2\pi)$ a phase.

Consider now the following algorithm for sequential swaps within 2-dimensional subspaces of $\mathscr{H}_{\mathscr{O}} \otimes \mathscr{H}_{\mathscr{R}}$.

Sequential swap algorithm

Step (1): Denote the probability of state $|\varphi_l\rangle \otimes |\xi_m\rangle$ as $p_{l,m}$. Set $i = 2$ and $m = d_{\mathscr{R}}$.

Step (2): Sequentially swap $|\varphi_1\rangle \otimes |\xi_i\rangle$ with the vectors $\{|\varphi_l\rangle \otimes |\xi_m\rangle\}_l$, with l running from $d_{\mathscr{O}}$ down through to 2, only if $p_{1,i} < p_{l,m}$.

Step (3): If $m > 1$, set $m = m - 1$ and go back to Step (2). Else, proceed to Step (4).

Step (4): If $i < d_{\mathscr{R}}$, set $i = i + 1$, $m = d_{\mathscr{R}}$, and go back to Step (2). Else, terminate.

We first wish to show that the sequence of unitary operators produced by this algorithm will give $\{U_{\mathrm{opt}}^p(\delta_j^{\downarrow})\}_j$ for a discrete, non-increasing sequence of errors $\boldsymbol{\delta}^{\downarrow}$, and that this will be accompanied by a non-decreasing sequence of heat $\boldsymbol{\Delta Q}^{\uparrow}$.

Lemma 3.5.1. *The sequential swap algorithm produces a non-increasing sequence of errors, $\boldsymbol{\delta}^{\downarrow} := \{\delta_j^{\downarrow}\}_j$, commensurate with a non-decreasing sequence of heat, $\boldsymbol{\Delta Q}^{\uparrow} := \{\Delta Q_j^{\uparrow}\}_j$, such that the resultant state $\rho'_{\mathscr{O}}$ is always passive.*

Proof. For every iteration of Step (2), each swap operation increases $p(\varphi_1|\rho'_{\mathscr{O}})$, so we obtain the non-increasing sequence of errors $\boldsymbol{\delta}^{\downarrow}$ by construction. Furthermore, each swap increases $p(\xi_i|\rho'_{\mathscr{R}})$, while decreasing $p(\xi_m|\rho'_{\mathscr{R}})$. To show that this always leads to an increase in heat by Corollary 2.1.1, we must show that, for each swap, $i > m$. Every swap in each iteration of Step (2) effects a permutation on the set $\{p_{1,i}, p_{2,m}, \dots, p_{d_{\mathscr{O}},m}\}$. Initially, $p_{1,i} = o_1^{\downarrow} r_i^{\downarrow}$. We note that if $o_1^{\downarrow} r_i^{\downarrow} < o_l^{\downarrow} r_m^{\downarrow}$ with $l \geqslant 2$, then by necessity $i > m$. As such, the swaps for the first iteration of Step (2), that involve state $|\varphi_1\rangle \otimes |\xi_2\rangle$ and lead to a permutation in $\{p_{1,2}, p_{2,1}, \dots, p_{d_{\mathscr{O}},1}\}$, result in a decrease in $p(\xi_1|\rho'_{\mathscr{R}})$ and an increase in $p(\xi_2|\rho'_{\mathscr{R}})$, which indeed leads to a non-decreasing sequence of heat. And so on recursively for all i. To show that $\rho'_{\mathscr{O}}$ is always passive, we need to show that after each swap, $\sum_m p_{i,m} \geqslant \sum_m p_{j,m}$ for all $i < j$. This follows from the fact that $\{p_{i,m}\}_i$ are always in non-increasing order, and that $\{p_{i,m}\}_{i\geqslant 2} \geqslant \{p_{i,m'}\}_{i\geqslant 2}$ if $m < m'$. □

Now, we wish to show that the non-decreasing sequence of heat $\boldsymbol{\Delta Q}^{\uparrow}$ is optimal for the associated non-increasing sequence of errors $\boldsymbol{\delta}^{\downarrow}$.

Theorem 3.5.1. *If an error δ can be achieved using the sequential swap algorithm, the consequent heat dissipation will be optimal. Achieving the same δ with the presence of entanglement in the vectors $\{U|\psi_n\rangle\}_n$ will either increase ΔQ, make $\rho'_{\mathscr{O}}$ less passive, or both.*

Proof. By Corollary 2.1.1, Lemma 3.3.1 and Lemma 3.5.1, the heat dissipation due to the sequential swap algorithm is minimal if we are restricted to swap operations. If we are not restricted to performing swap operations, we could also achieve the same error δ by allowing for entanglement in the vectors $\{U\,|\psi_n\rangle\}_n$. To show that this will result in a greater amount of heat dissipation, it is sufficient to show that doing so would increase $p_{i,m}$ and decrease $p_{i,m'}$, where $m > m'$. Likewise, we may show that this would make $\rho'_{\mathscr{O}}$ less passive by demonstrating that the process would increase $p_{i,m}$ and decrease $p_{j,m}$, where $i > j$.

Start with $\rho = U^p_{\text{opt}}(0)\rho U^p_{\text{opt}}(0)^\dagger$, with $p^{j_{\max}}_{1,d_{\mathscr{R}}} = p^{\downarrow}_{d_{\mathscr{R}}}$, and $p^{j_{\max}}_{2,1} = p^{\downarrow}_{d_{\mathscr{R}}+1}$. Here we have $\delta^{\downarrow}_{j_{\max}} = 0$. The first step of the sequential swap algorithm, run backwards, gives us $p^{j_{\max}-1}_{1,d_{\mathscr{R}}} = p^{\downarrow}_{d_{\mathscr{R}}+1}$ and $p^{j_{\max}-1}_{2,1} = p^{\downarrow}_{d_{\mathscr{R}}}$, with $\delta^{\downarrow}_{j_{\max}-1} = p^{\downarrow}_{d_{\mathscr{R}}} - p^{\downarrow}_{d_{\mathscr{R}}+1}$. All other values are the same as before. Now instead have $U\,|\psi_{d_{\mathscr{R}}}\rangle = \sqrt{\gamma}|\varphi_1\rangle\otimes|\xi_{d_{\mathscr{R}}}\rangle + \sqrt{1-\gamma}|\varphi_i\rangle\otimes|\xi_m\rangle$ and $U\,\big|\psi_{d_{\mathscr{R}}+(m-1)d_{\mathscr{O}}+i}\big\rangle = \sqrt{1-\gamma}|\varphi_1\rangle\otimes|\xi_{d_{\mathscr{R}}}\rangle - \sqrt{\gamma}|\varphi_i\rangle\otimes|\xi_m\rangle$, with all other $U\,|\psi_n\rangle$ defined by $U^p_{\text{opt}}(0)$. With some choice of γ, i, m, we can obtain $p_{1,d_{\mathscr{R}}} = \gamma p^{\downarrow}_{d_{\mathscr{R}}} + (1-\gamma)p^{\downarrow}_{d_{\mathscr{R}}+(m-1)d_{\mathscr{O}}+i} = p^{\downarrow}_{d_{\mathscr{R}}+1}$ and hence the same value of $\delta^{\downarrow}_{j_{\max}-1}$. This, however, will lead to $p_{2,1} = p^{\downarrow}_{d_{\mathscr{R}}+1} \leqslant p^{j_{\max}-1}_{2,1}$ and $p_{i,m} = (1-\gamma)p^{\downarrow}_{d_{\mathscr{R}}} + \gamma p^{\downarrow}_{d_{\mathscr{R}}+(m-1)d_{\mathscr{O}}+i} \geqslant p^{j_{\max}-1}_{i,m}$. If $i = 2$ and $m \geqslant 2$, this will result in a larger ΔQ than $\Delta Q^{\uparrow}_{j_{\max}-1}$. Conversely, if $m = 1$ and $i \geqslant 3$, this will make $\rho_{\mathscr{O}}$ less passive than with the sequential swap algorithm. If both $i \geqslant 3$ and $m \geqslant 2$, then both ΔQ will be larger and $\rho'_{\mathscr{O}}$ less passive. The same line of reasoning would apply for entanglement of higher Schmidt-rank. □

Fig. (3) depicts this process for the case where $\mathscr{H}_{\mathscr{O}} \simeq \mathscr{H}_{\mathscr{R}} \simeq \mathbb{C}^3$, with $\rho_{\mathscr{O}} = \frac{1}{3}\mathbb{1}_{\mathscr{O}}$. Here the diagonal entries of the density operator ρ' are shown in each column, with the first column from the right representing the initial state, and the final column representing the case of passive, maximally probable information erasure. The algorithm for reducing error by increasing heat moves from right to left, as shown by the arrows. The elements surrounded by dashed circles, and coloured in red, are those which must be swapped to decrease δ, with the resultant diagonal elements of the new state shown to the left.

To allow for a continuous change in δ, we need to generalise the swap operation to an entangling swap. That is to say, for the vectors $|\varphi_1\rangle\otimes|\xi_i\rangle$ and $|\varphi_l\rangle\otimes|\xi_m\rangle$, and the real number $\gamma \in [0,1]$, we define

$$\text{SW}_\gamma : \begin{cases} |\varphi_1\rangle\otimes|\xi_i\rangle \mapsto \sqrt{1-\gamma}|\varphi_1\rangle\otimes|\xi_i\rangle + \sqrt{\gamma}|\varphi_l\rangle\otimes|\xi_m\rangle\,, \\ |\varphi_l\rangle\otimes|\xi_m\rangle \mapsto \sqrt{\gamma}|\varphi_1\rangle\otimes|\xi_i\rangle - \sqrt{1-\gamma}|\varphi_l\rangle\otimes|\xi_m\rangle\,. \end{cases} \tag{15}$$

Therefore, $\text{SW}_0 = \mathbb{1}$ and as $\gamma \to 1$, SW_γ converges to the swap operation. Hence, for any error $\delta \in (\delta^{\downarrow}_j, \delta^{\downarrow}_{j+1})$, the optimal unitary operator $U^p_{\text{opt}}(\delta)$ would be given by following the algorithm for discrete errors up to $\delta^{\downarrow}_j$, and then replacing the swap operation which would give the error $\delta^{\downarrow}_{j+1}$ with the entangling swap operation defined above. This will ensure for a continuous decrease in δ and a continuous increase in ΔQ.

4 Conclusions

In quantum mechanics, information erasure is the irreversible process of preparing an object in a predefined pure state; the probability of information erasure is defined as the probability of measuring the object to be in said state, subsequent to the process of information erasure. Landauer's framework for information erasure consists of an object of information erasure, and a reservoir prepared in a thermal state, which are initially uncorrelated. The erasure process is then implemented by an appropriate choice of a global unitary operator acting on the composite system. Landauer's principle states that *some* physical context exists in which an entropy reduction of ΔS in an object will cost $k_B T \Delta S$ units of energy as heat dissipation to the thermal reservoir – this theoretical limit is referred to as Landauer's limit. The physical context of information erasure is described by the Hamiltonian and temperature of the thermal reservoir, as well as the initial state of the object of information erasure. For most physical contexts, however, not only will the heat dissipation be generally greater than Landauer's limit, but not all values of ΔS may even be realisable.

To this end we have developed a context-dependent, dynamical variant of Landauer's principle. We used techniques from majorisation theory to characterise the equivalence class of unitary operators that bring the probability of information erasure to a desired value and minimise the consequent heat dissipation to the thermal reservoir. We demonstrated that there is a tradeoff between the probability of information erasure and the minimal heat dissipation, with a continuous increase in one being accompanied by a continuous increase in the other. Furthermore, we showed that except for the cases where the object is a two-level system, or when we are able to fully erase the object's information, we may maximise the probability of information erasure without also minimising the object's entropy; this allows for a more energy-efficient procedure for information erasure.

The primary question we have not addressed in this study, and shall leave for future work, is the inclusion of time-dynamics into what we consider as the physical context; the optimal unitary operator for information erasure is considered here as a bijection between orthonormal basis sets. In most realistic settings, however, one is restricted in the Hamiltonians they can establish between the object and reservoir. As such, the optimal unitary operator may not always be reachable, resulting in a smaller maximal probability of information erasure, a larger minimal heat dissipation, or both. Furthermore, an interesting question to address is the number of times that we must switch between the Hamiltonians, that generate the unitary group, in order to obtain the optimal unitary operator, and how this would scale with the reservoir's dimension. This would provide a link between this work and the third law of thermodynamics [17] from a control-theoretic [5] viewpoint.

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A Figures

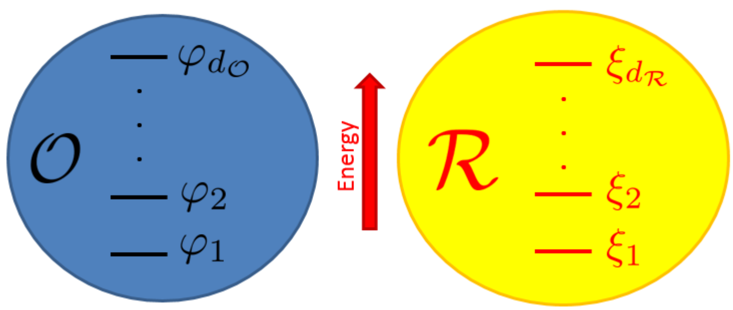

Figure 1: The object $\mathscr{O}$ with Hilbert space $\mathscr{H}_\mathscr{O} \simeq \mathbb{C}^{d_\mathscr{O}}$ and thermal reservoir $\mathscr{R}$ with Hilbert space $\mathscr{H}_\mathscr{R} \simeq \mathbb{C}^{d_\mathscr{R}}$. The eigenbasis of the reservoir Hamiltonian $H_\mathscr{R}$ is $\{|\xi_m\rangle\}_m$, with the vector numbering being in order of increasing energy. The eigenbasis with respect to which the object is initially diagonal is $\{|\varphi_n\rangle\}_n$.

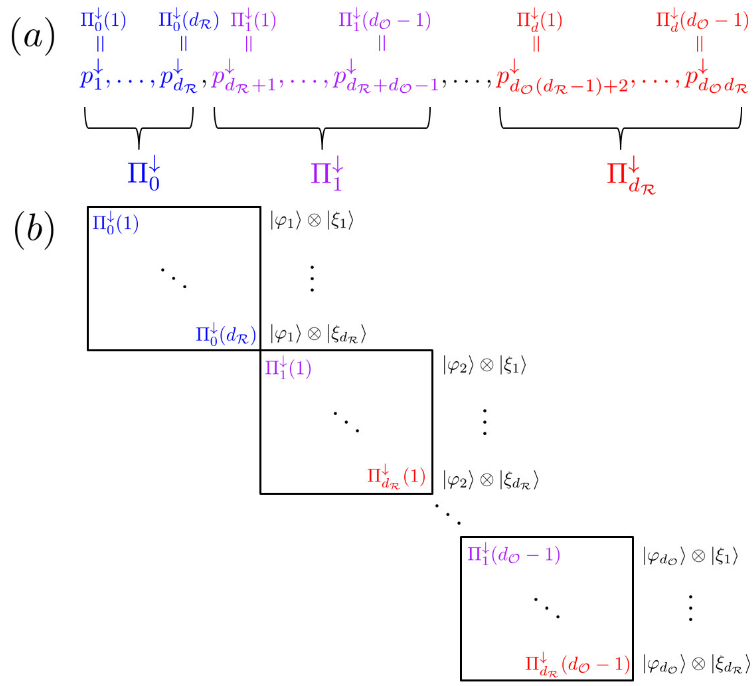

Figure 2: (Colour online) (a) The partitioning of $\boldsymbol{p}^\downarrow$, the decreasing set of eigenvalues of ρ, into the sets $\Pi_0^\downarrow$ and $\Pi_{m\geqslant 1}^\downarrow$. (b) The density operator $\rho' := U_{\text{opt}}^{\text{p}}(0)\rho U_{\text{opt}}^{\text{p}}(0)^\dagger$, in matrix representation, where $U_{\text{opt}}^{\text{p}}(0)$ is the optimal unitary operator for passive, maximally probable information erasure. The post-transformation marginal state of the object, $\rho'_\mathscr{O}$, is the most passive, given the constraints: (i) $p(\varphi_1|\rho'_\mathscr{O}) = p_{\varphi_1}^{\max}$; and (ii) ΔQ is minimal given (i).

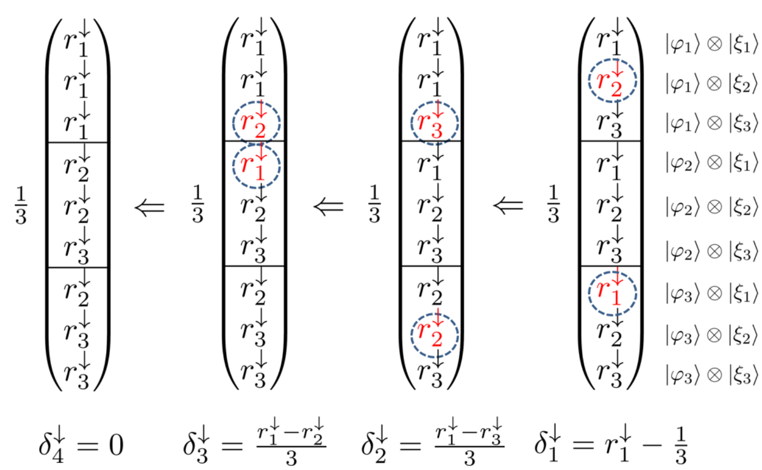

Figure 3: (Colour online) The diagonal elements of $\rho' := U_{\text{opt}}^{\text{p}}(\delta)\rho U_{\text{opt}}^{\text{p}}(\delta)^\dagger$, for $\rho = \frac{1}{3}\mathbb{1}_\mathscr{O}\otimes\rho_\mathscr{R}(\beta)$, resulting in $p(\varphi_1|\rho'_\mathscr{O}) = p_{\varphi_1}^{\max} - \delta$, where ΔQ is minimised and $\rho'_\mathscr{O}$ is as passive as possible given this constraint. Here $\mathscr{H}_\mathscr{O} \simeq \mathscr{H}_\mathscr{R} \simeq \mathbb{C}^3$, and $\{\delta_j^\downarrow\}_j$ is a non-increasing sequence of errors. The elements inside a dashed circle (red online) are those which must be swapped to move from $\delta_j^\downarrow$ to $\delta_{j+1}^\downarrow$.