From the Kochen-Specker theorem to noncontextuality inequalities without assuming determinism

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We propose a general technique to go from a proof of the Kochen-Specker theorem, which is a statement about quantum theory, to a noncontextuality inequality that applies to any operational theory, even a nonquantum one.

Motivation: The Kochen-Specker (KS) theorem [1] stands out as a fundamental result about the impossibility of embedding the predictions of quantum mechanics in an underlying ontological model that satisfies a property we term *KS-noncontextuality*: namely, that an ontic state assigns a deterministic outcome to a projector, independent of the particular measurement basis (the *context*) this projector may appear in. Experimental testability of the KS theorem has been a subject of intense debate in the past [2].

We begin by outlining some obstacles to such a test. First of all, there is the problem that the standard notion of KS-noncontextuality holds only for projective measurements, and hence makes explicit reference to the formalism of quantum theory. If one seeks to implement a direct experimental test of whether nature admits of a noncontextual model, without presuming quantum theory, one must operationalize the notion of KS-noncontextuality. Second, there is the problem that even if one has such an operationalized notion, if it continues to apply only to measurements which are perfectly predictable on some preparations (the analogue of projective quantum measurements in a general operational theory) then it is still not applicable to any real experiment because no realistic measurement ever achieves the ideal of perfect predictability on any preparation.

It follows that any experimental test based on the notion of KS-noncontextuality will fall short of the benchmark that is set by another fundamental test of nonclassicality, namely, the violation of a Bell inequality [3, 4, 5]. Violating a Bell inequality allows one to infer that whatever operational theory governs the experiment, it does not admit of an ontological model that satisfies local causality. A notion of noncontextuality, like local causality, is an assumption that is meant to capture the spirit of *classicality*, but unlike local causality (which can only be applied to space-like separated systems), it is meant to apply in *any* experimental scenario. What one requires is a notion of noncontextuality that can be subjected to a direct experimental test, just as one can do for the hypothesis of local causality.

We use the generalization of KS-noncontextuality to the notion of noncontextuality for arbitrary operational theories proposed in Ref. [6]. This generalization views noncontextuality as a Leibnizian primitive, the *(ontological) identity of (operational) indiscernables*: if two experimental procedures are statistically indistinguishable at the operational level, then they must also be statistically indistinguishable at the ontological level. We term this property *universal noncontextuality*—universal in the sense that it applies to *all* experimental procedures, whether they correspond to preparations, transformations, or measurements. In particular, for preparation and measurement procedures, the corresponding notions of noncontextuality are preparation noncontextuality and measurement noncontextuality, respectively. We use these two notions to obtain a noncontextuality inequality based on the *KS-uncolourable* hypergraph of Fig. 1, which was originally devised in Ref. [7] as a so-called "state-independent" proof of the KS theorem.

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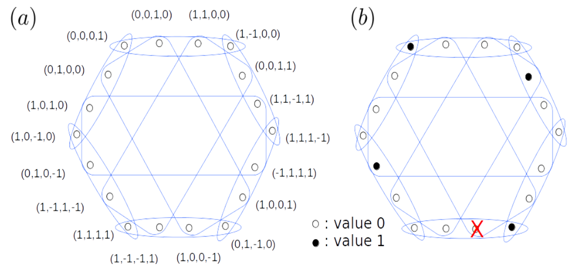

Figure 1: The 18 ray proof of the KS theorem from Ref. [7].

Operational theory and its ontological model: An operational theory is specified by $(\mathscr{P},\mathscr{M},p)$, where $\mathscr{P}$ is the set of preparation procedures, $\mathscr{M}$ is the set of measurement procedures, and $p(k|M,P) \in [0,1]$ denotes the probability that outcome $k \in \mathscr{K}$ occurs on implementing measurement procedure $M \in \mathscr{M}$ following a preparation procedure $P \in \mathscr{P}$ on a system.

An ontological model (Λ,μ,ξ) of an operational theory $(\mathscr{P},\mathscr{M},p)$ posits an ontic state space Λ such that a preparation procedure P is represented by a normalized distribution over Λ, $\mu(\lambda|P) \in [0,1]$ $(\lambda \in \Lambda)$ such that $\sum_{\lambda\in\Lambda}\mu(\lambda|P) = 1$ for all $P \in \mathscr{P}$, and the probability of occurrence of a measurement outcome $[k|M]$ for a given $\lambda \in \Lambda$ is specified by $\xi(k|M,\lambda) \in [0,1]$, where the measurement outcomes are assumed to be discrete. The following condition of empirical adequacy prescribes how the operational theory and its ontological model fit together:

$$p(k|M,P) = \sum_{\lambda\in\Lambda}\xi(k|M,\lambda)\mu(\lambda|P). \tag{1}$$

Operational equivalence: Two preparation procedures, P and P', are said to be operationally equivalent (denoted $P \simeq P'$) if no measurement procedure $M \in \mathscr{M}$ (with outcome set $\mathscr{K}$) yields different statistics for them, i.e.,

$$\forall M \in \mathscr{M}, \forall k \in \mathscr{K} : p(k|M,P) = p(k|M,P'). \tag{2}$$

Two measurement events, $[k|M]$ and $[k|M']$ (where M and M' are measurement procedures with outcome set $\mathscr{K}$ each, $k \in \mathscr{K}$), are said to be operationally equivalent (denoted $[k|M] \simeq [k|M']$) if no preparation procedure yields different statistics for them, i.e.,

$$\forall P \in \mathscr{P} : p(k|M,P) = p(k|M',P). \tag{3}$$

Noncontextuality: Preparation noncontextuality is the following assumption on the ontological model of an operational theory:

$$P \simeq P' \Rightarrow \mu(\lambda|P) = \mu(\lambda|P') \quad \forall\lambda \in \Lambda. \tag{4}$$

Measurement noncontextuality is the assumption that

$$[k|M] \simeq [k|M'] \Rightarrow \xi(k|M,\lambda) = \xi(k|M',\lambda) \quad \forall\lambda \in \Lambda. \tag{5}$$

Our noncontextuality inequality: We consider 9 measurements $\{M_i\}_{i=1}^{9}$ with 4 outcomes $k \in \{1,2,3,4\}$ each. The 36 measurement events $\{[k|M_i] : i \in \{1,\dots,9\}, k \in \{1,\dots,4\}\}$ are assumed to satisfy 18 operational equivalence relations, illustrated in Fig. 2. This leaves us with 18 equivalence classes of measurement events. We also consider 36 preparations, $\{P_{i,k} : i \in \{1,\dots,9\}, k \in \{1,\dots,4\}\}$, associated to the corresponding measurement events. The quantity of interest to us is $p(k|M_i,P_{i,k})$, the

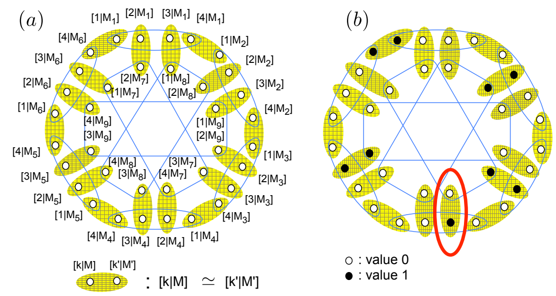

Figure 2: Operational equivalences between measurement events.

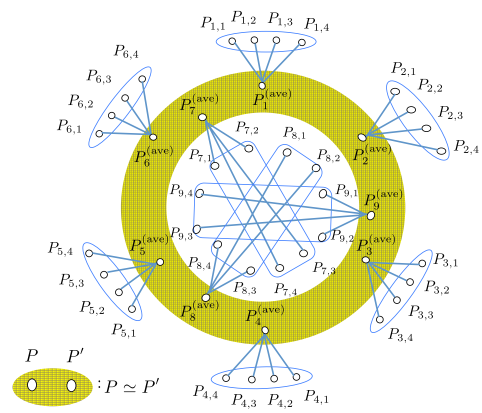

Figure 3: Operational equivalences between preparation procedures

probability that the measurement event $[k|M_i]$ occurs when measurement M_i follows a preparation $P_{i,k}$ on the system. We also define 9 effective preparations, denoted $\{P_i^{(\text{ave})}\}_{i=1}^9$, where $P_i^{(\text{ave})}$ is obtained by sampling $k \in \{1,\ldots,4\}$ uniformly at random, and then implementing $P_{i,k}$. These 9 effective preparations are assumed to be operationally equivalent to each other: $P_1^{(\text{ave})} \simeq P_2^{(\text{ave})} \simeq P_3^{(\text{ave})} \simeq P_4^{(\text{ave})} \simeq P_5^{(\text{ave})} \simeq P_6^{(\text{ave})} \simeq P_7^{(\text{ave})} \simeq P_8^{(\text{ave})} \simeq P_9^{(\text{ave})}$, as depicted in Fig. 3. The quantity that our inequality bounds is the average predictability of the measurement events with respect to the corresponding preparations:

$$A \equiv \frac{1}{36}\sum_{i=1}^{9}\sum_{k=1}^{4} p(k|M_i, P_{i,k}). \tag{6}$$

Given the operational equivalences in Figs. 2 and 3, it can be shown that the assumption of noncontextuality implies the following constraint on A:

$$A \leq \frac{5}{6}. \tag{7}$$

Discussion: In the accompanying paper [10], we also show how our noncontextuality inequality compares with an earlier proposal [8] for an experimentally testable inequality based on the construction of Ref. [7]. We argue that the inequality of Ref. [8] is not well-motivated and that its violation cannot be interpreted as revealing any interesting notion of nonclassicality in nature. Our inequality, on the other hand, is proven to provide a test of universal noncontextuality that is robust to noise.

Recall that the assumption of KS-noncontetuality incorporates an assumption of outcome determinism for sharp (i.e. projective) measurements. As argued elsewhere [6], as well as in the paper accompanying this submission [10], for ontological models of quantum theory, outcome determinism *can* be justified for sharp measurements, once preparation noncontextuality is assumed. However, generalizing this argument to the case of an arbitrary operational theory is problematic. For one, it requires a generalization of the notion of a sharp measurement, and it is not clear which of the many ways of achieving this generalization is appropriate (see [9] for examples of such generalizations). More importantly, even if one settles upon some choice, the requirement that the operational theory must include such measurements limits the scope of theories for which the notion of KS-noncontextuality is applicable. We circumvent this problem by demonstrating that operational equivalences and universal noncontextuality imply a restriction on the extent to which measurements can be presumed to be outcome-indeterministic in the ontological model, and this in turn implies a bound on the average predictability of the measurements, regardless of whether they are sharp or not. This allows us to obtain a noncontextuality inequality that makes sense for any operational theory without requiring a distinction between sharp and unsharp measurements in the theory. In particular, for the case of quantum theory where the sharp/unsharp distinction refers to projective/nonprojective measurements, our noncontextuality inequality also makes sense for nonprojective measurements, something that the traditional KS-noncontextuality inequalities fail to do.

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