A diagrammatic axiomatisation of the GHZ and W quantum states

Amar Hadzihasanovic
University of Oxford
amarh@cs.ox.ac.uk

In the context of categorical quantum mechanics, Coecke and Kissinger suggested that two 3-qubit states, GHZ and W, may be used as the building blocks of a new graphical calculus, aimed at a diagrammatic classification of multipartite qubit entanglement under stochastic local operations and classical communication (SLOCC). We present a full graphical axiomatisation of the relations between GHZ and W: the ZW calculus. This refines a version of the preexisting ZX calculus, while keeping its most desirable characteristics: undirectedness, a large degree of symmetry, and an algebraic underpinning. Through an explicit normalisation procedure, the ZW calculus was shown to be complete for the category of free abelian groups on a power of two generators ("qubits with integer coefficients").

1 Introduction

The classification of multipartite entanglement of n-qubit systems under *stochastic local operations* and *classical communication* (SLOCC) [15] is an open problem in quantum theory, with relevance to quantum computing and beyond [2]. SLOCC-inequivalence entails behavioural differences between quantum states, which condition the way they can be employed in quantum protocols, for communication or distributed computing. It is known that, for 3 qubits, there are only two SLOCC classes of maximally entangled qubits, containing the states (*modulo* normalisation) $|\mathrm{GHZ}\rangle := |000\rangle + |111\rangle$, $|\mathrm{W}\rangle := |100\rangle + |010\rangle + |001\rangle$, respectively [6]. For $n \geq 4$ qubits, however, there are infinite classes, and the existing, inductive classifications in super-classes give little insight about operational behaviour [11, 12].

By map-state duality - what is known in the context of quantum mechanics as the Choi-Jamiołkowski isomorphism [13] - a tripartite state may also be seen as a *binary operation*. In [4], Coecke and Kissinger showed that the GHZ and W states correspond to certain *Frobenius algebras* in the category of finite-dimensional complex Hilbert spaces; and that, in a particular sense, these are the only two possible kinds of Frobenius algebras in dimension 2.

Moreover, as quantum gates, together with single-qubit states, they are universal for quantum computing, which suggested they could be used as building blocks for a *compositional* classification of multipartite entanglement. In [5, 9], an axiomatisation of the relations between the GHZ and W algebras was started, with a graphical calculus, analogous to the ZX calculus [3, 1], in mind; but this was not brought to completion, and only results about universality and classification were obtained.

In [8], we presented the *ZW calculus*, a diagrammatic axiomatisation of the relations between the GHZ and W algebras, which incorporates a version of the ZX calculus and shares some of its best properties, such as featuring *undirected* diagrams, that are "as symmetrical as they look", and having a small number of graphical elements and axioms, described in terms of important algebraic structures and relations.

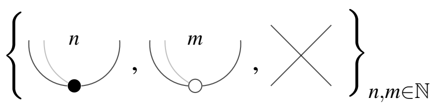
Submitted to:
QPL 2015

We proved that the ZW calculus is complete for the category $\mathbf{Ab}_{2,\text{free}}$ of free abelian groups on a power of two generators, by providing a normal form for string diagrams, and an explicit normalisation procedure.

Note. The full version of this article [8] has been accepted at the Thirtieth Annual IEEE/ACM Symposium on Logic in Computer Science (LICS) 2015.

2 The ZW calculus

The language of the ZW calculus can be described as the free self-dual, compact closed theory on the following set of generators:

$T_c :=$ 

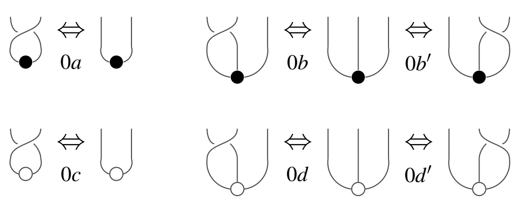
We interpret these diagrams in **Ab**, the monoidal category of abelian groups and homomorphisms, with monoidal product given by the tensor product of abelian groups; or rather, in its full subcategory $\mathbf{Ab}_{2,\text{free}}$, generated, under tensoring, by the free abelian group on two generators, $\mathbb{Z} \oplus \mathbb{Z}$. More informally, this can be seen as a category of "qubits with integer coefficients", which embed into ordinary qubits through the inclusion of integers into complex numbers, inducing a monoidal inclusion of $\mathbf{Ab}_{2,\text{free}}$ into **FHilb**, the category of finite Hilbert spaces.

Remark. This inclusion can be made into a †-functor, by endowing $\mathbf{Ab}_{2,\text{free}}$ with the only compatible †-category structure; since there are no complex phases, however, daggers are equal to duals (transposes) of morphisms.

The n-ary black vertex corresponds to the quantum state $|\mathrm{W}_n\rangle$, the n-ary generalisation of the W state. The n-ary white vertex, on the other hand, corresponds to the n-ary Z spider from the ZX calculus, with a π phase [3]. Save for this phase and normalisation, this is interpreted as the quantum state $|\mathrm{GHZ}_n\rangle$, the n-ary generalisation of the GHZ state [7]. A novel feature of this diagrammatic calculus is the presence of the "crossing", which induces a sign change on $|11\rangle$, alongside the usual symmetric braiding.

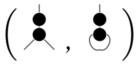
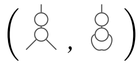
The following are the rules of the ZW calculus, to be added to the usual equations for braidings and dualities that determine the self-dual compact closed structure.

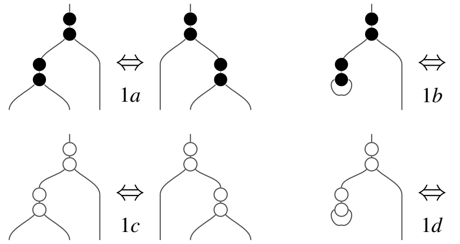
Rule 0. *The black and white vertices are symmetric.*



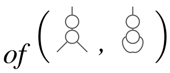
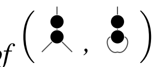
Remark. This rule allows us to treat the black and white vertices as vertices of an *undirected graph*; in particular, we can turn inputs into outputs, using the dualities, without worrying about which particular wire has been turned around.

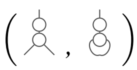
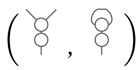
For instance, one can speak unambiguously of "the white vertex with 2 inputs and 1 output", and depict it as  .

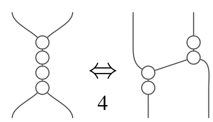
Rule 1.  *and*  *are monoids.*

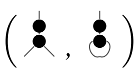
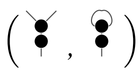


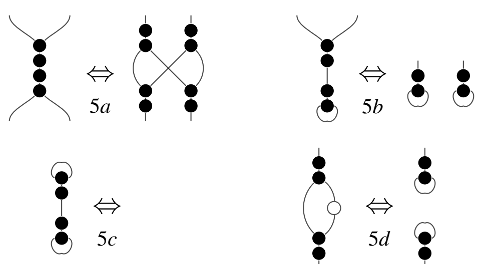
Rule 2.  *and*  *are involutions.*

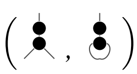
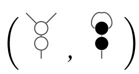
Rule 3.  *is an automorphism of* *, and*  *of* *.*

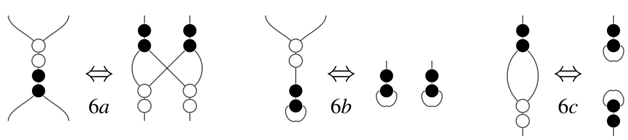
Rule 4.  *and*  *form a Frobenius algebra.*

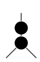


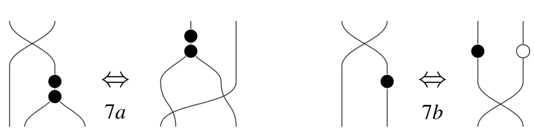
Rule 5.  *and*  *form a Hopf algebra with antipode* *.*



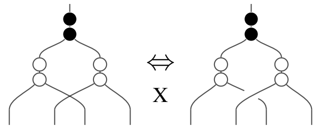
Rule 6.  *and*  *form a "Hopf algebra" with antipode* *.*



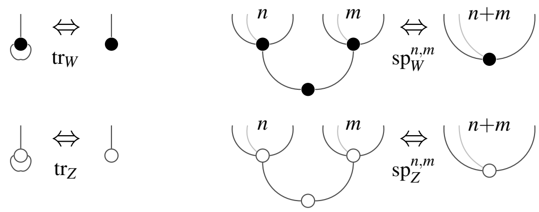
Rule 7.  *is an even map, while*  *is odd.*



Rule X. *The elimination rule for crossings.*



While these rules are complete on their own, the calculus can be greatly simplified by adding the *spider rules* (actually, rule schemata, for $n,m \in \mathbb{N}$), which, together with rules 2*a* and 2*b*, imply Rule 1.



Theorem (Completeness of the ZW calculus)**.** *The self-dual, compact closed category presented by the generators in T_c, subject to the relations described in Rules 0-7, X, and the spider rules, is monoidally equivalent to* $\mathbf{Ab}_{2,\text{free}}$.

3 Conclusions

One consequence that we can draw at once is that, under a suitable reinterpretation of the latter's diagrams, the ZW calculus contains the ZX calculus with π phases, and is, to all effects, a refinement of it. This follows from the fact that all gates representable in this calculus have integer coefficients. Moreover, the ZW calculus completes the axiomatisation of the GHZ/W calculus with additive inverses, as started in [5], and can be used to encode rational arithmetic as suggested there.

With little effort, we also obtain completeness results for mild extensions of the ZW calculus; one of them corresponds to the theory of *pure mobits* of [14].

While the completeness result may have a certain conceptual interest by itself, it is but one small step in a wider programme, which can be carried on in several directions.

Our normal form, devised for the sake of the completeness proof, has none of the advantages of the diagrammatic notation for states, such as representing their *separability* as topological disconnectedness; so it might be worth exploring some alternatives, tailored specifically to identifying the SLOCC class of a state. This may be done with the help of Quantomatic [10].

Other directions that are mentioned in [8] are extensions of the calculus with real numbers and/or complex phases, an investigation of fragments of the calculus - in particular the monochromatic fragment, consisting of black vertices and crossings only - and connections to the topology of morphisms in higher categories.

Understanding the compositional structure of multipartite entanglement is likely to involve an original interplay of algebra and geometry; monoidal categories, with their associated diagrammatic languages, might just provide the bridge that is needed.

References

[1] M. Backens (2013): *The ZX-calculus is complete for stabilizer quantum mechanics. arXiv preprint arXiv:1307.7025.*

[2] L. Borsten, D. Dahanayake, M.J. Duff, H. Ebrahim & W. Rubens (2009): *Black holes, qubits and octonions*. *Physics Reports* 471(3), pp. 113–219.

[3] B. Coecke & R. Duncan (2008): *Interacting quantum observables*. In: *Automata, Languages and Programming*, Springer, pp. 298–310.

[4] B. Coecke & A. Kissinger (2010): *The compositional structure of multipartite quantum entanglement*. In: *Automata, Languages and Programming*, Springer, pp. 297–308.

[5] B. Coecke, A. Kissinger, A. Merry & S. Roy (2011): *The GHZ/W-calculus contains rational arithmetic*. *arXiv preprint arXiv:1103.2812*.

[6] W. Dür, G. Vidal & J. I. Cirac (2000): *Three qubits can be entangled in two inequivalent ways*. *Physical Review A* 62(6), p. 062314.

[7] D. M. Greenberger, M. A. Horne, A. Shimony & A. Zeilinger (1990): *Bell's theorem without inequalities*. *Am. J. Phys* 58(12), pp. 1131–1143.

[8] A. Hadzihasanovic (2015): *A Diagrammatic Axiomatisation for Qubit Entanglement*. *arXiv preprint arXiv:1501.07082*.

[9] A. Kissinger (2012): *Pictures of Processes: Automated Graph Rewriting for Monoidal Categories and Applications to Quantum Computing*. Ph.D. thesis, University of Oxford.

[10] A. Kissinger, A. Merry, L. Dixon, R. Duncan, M. Soloviev, B. Frot, D. Quick & V. Zamdzhiev: *Quantomatic*. Software available on-line at http://sites.google.com/site/quantomatic/.

[11] L. Lamata, J. León, D. Salgado & E. Solano (2006): *Inductive classification of multipartite entanglement under stochastic local operations and classical communication*. *Physical Review A* 74(5), p. 052336.

[12] L. Lamata, J. León, D. Salgado & E. Solano (2007): *Inductive entanglement classification of four qubits under stochastic local operations and classical communication*. *Physical Review A* 75(2), p. 022318.

[13] V. Paulsen (2002): *Completely bounded maps and operator algebras*. 78, Cambridge University Press.

[14] B. Schumacher & M. D. Westmoreland (2012): *Modal quantum theory*. *Foundations of Physics* 42(7), pp. 918–925.

[15] F. Verstraete, J. Dehaene, B. De Moor & H. Verschelde (2002): *Four qubits can be entangled in nine different ways*. *Physical Review A* 65(5), p. 052112.