Total and Partial Computation in Categorical Quantum Foundations

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This paper uncovers the fundamental relationship between total and partial computation in the form of an equivalence of certain categories. This equivalence involves on the one hand *effectuses*, which are categories for *total* computation, introduced by Jacobs for the study of quantum/effect logic. On the other hand, it involves what we call *FinPACs with effects*; they are finitely partially additive categories equipped with effect algebra structures, serving as categories for *partial* computation. It turns out that the Kleisli category of the lift monad $(-)+1$ on an effectus is always a FinPAC with effects, and this construction gives rise to the equivalence. Additionally, *state-and-effect triangles* over FinPACs with effects are presented.

1 Introduction

An *effectus* is a category with a final object 1 and finite coproducts $(+,0)$ satisfying certain assumptions (see Definition 2.2), introduced recently by Jacobs [5], which provides a suitable setting for quantum/effect logic and computation. In an effectus, arrows $\omega\colon 1 \to X$ are *states* on X, and $p\colon X \to 1+1$ are *predicates*. They turn out to form a *convex set* and a so-called *effect module*, respectively. Arrows $f\colon X \to Y$ are seen as computation, inducing state and predicate transformers. The situation is summarised in a *state-and-effect triangle*, see §2.2 for an overview of effectuses.

Motivating examples of effectuses, which model quantum computation and logic, are given by C^*-algebras with (completely) positive unital maps, and by W^*-algebras with normal (completely) positive unital maps. Other effectuses include the category **Set** of sets for a classical setting, and the Kleisli category $\mathcal{K}\ell(\mathcal{D})$ of the distribution monad $\mathcal{D}$ for a probabilistic setting. As seen in these examples, computation modelled by an effectus is *total* (or terminating) but not *partial* (or non-terminating). Indeed, arrows in an effectus always induce 'terminating' predicate transformers in the sense that they preserve the truth predicates. We need models of partial computation in some cases, however, since programs do not necessarily terminate in general. Moreover, such models often have richer structures such as complete partial orders, which allow us to interpret loop and recursion. For instance, the category of sets and *partial* functions are enriched over complete partial orders, and so is the category of W^*-algebras and normal (completely) positive *subunital* maps [2, 11].

The present paper studies partial computation in effectuses via the *lift monad* (a.k.a. maybe monad), which is a common technique in categorical semantics of computation, going back to Moggi [9]. We switch from an effectus $\mathbf{B}$ to the Kleisli category of the lift monad $(-)+1$ on $\mathbf{B}$, which we denote by $\mathbf{B}_{+1}$. An arrow $X \to Y$ in $\mathbf{B}_{+1}$ is $X \to Y+1$ in $\mathbf{B}$, seen as a partial computation from X to Y. This simple idea makes a lot of sense for any effectus, leading us to the main results of this paper as follows.

- For an effectus $\mathbf{B}$, the Kleisli category $\mathbf{B}_{+1}$ of the lift monad is a *finitely partially additive category (FinPAC)*, which is a finite variant of Arbib and Manes' partially additive category (PAC) [1, 8].

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Table 1: Examples of effectuses and corresponding FinPACs with effects.

effectus	total computation	partial computation	FinPAC with effects
$\mathbf{B}$	$X \to Y$	$X \to Y+1$ (in $\mathbf{B}$)	$\mathbf{B}_{+1}$
Set	function	partial function	**Pfn**
$\mathcal{K}\ell(\mathcal{D})$	$X \to \mathcal{D}Y$ (stochastic relation)	$X \to \widehat{\mathcal{D}}Y$ (substochastic relation)	$\mathcal{K}\ell(\widehat{\mathcal{D}})$
$(\mathbf{Cstar}_{\mathrm{PU}})^{\mathrm{op}}$	positive unital map	positive subunital map	$(\mathbf{Cstar}_{\mathrm{PSU}})^{\mathrm{op}}$

The homsets $\mathbf{B}_{+1}(X,1) = \mathbf{B}(X,1+1)$ are the sets of predicates and form effect algebras. The category $\mathbf{B}_{+1}$ is what we call a *FinPAC with effects*, which has an effect algebra structure related to the partially additive structure in an appropriate manner (see Definition 4.4).

- On the other hand, if $\mathbf{C}$ is a FinPAC with effects, then the subcategory $\mathbf{C}_{\mathrm{t}}$ with "total" arrows is an effectus. Moreover, the two constructions $(-)_{+1}$ and $(-)_{\mathrm{t}}$ are inverses of each other up to isomorphism. Categorically, we obtain a 2-equivalence of the 2-categories of effectuses and FinPACs with effects.

$$\begin{pmatrix}\text{effectuses}\\ [\textit{total}\text{ computation}]\end{pmatrix} \underset{(-)_{\mathrm{t}}}{\overset{(-)_{+1}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \begin{pmatrix}\text{FinPACs with effects}\\ [\textit{partial}\text{ computation}]\end{pmatrix} \tag{1}$$

See Table 1 for examples of effectuses and corresponding FinPACs with effects. This equivalence characterises the Kleisli categories $\mathbf{B}_{+1}$ of the lift monad on effectuses as FinPACs with effects, and effectuses as the 'total' subcategories $\mathbf{C}_{\mathrm{t}}$ of FinPACs with effects.

We additionally present two type of state-and-effect triangles over a FinPAC with effects. One triangle is rather simple and easy, involving generalised effect modules and subconvex sets. Another triangle is obtained by an application of the above 2-equivalence to a state-and-effect triangle over an effectus, but only under an additional 'normalisation' condition. This also contains a slight improvement of a known result on effectuses with normalisation, via division in effect monoids.

The paper is organised as follows. We first give preliminaries in the next section. Section 3 introduces FinPACs. In §4 we study partial computation in effectuses and FinPACs with effects, and then in §5 we prove categorical equivalence of effectuses and FinPACs with effects. Section 6 presents state-and-effect triangles over FinPACs with effects.

2 Preliminaries

2.1 Partial commutative monoids, (generalised) effect modules and (sub)convex sets

A *partial commutative monoid (PCM)* is a set M with a partial binary 'sum' operation $\ovee\colon M \times M \rightharpoonup M$ and a 'zero' element $0 \in M$ subject to $(x \ovee y) \ovee z \simeq x \ovee (y \ovee z)$, $x \ovee y \simeq y \ovee x$ and $x \ovee 0 \simeq x$, where $\simeq$ denotes the Kleene equality: if either side is defined, then so is the other, and they are equal. We write $x \perp y$ if $x \ovee y$ is defined, and we say elements $x_1, \dots, x_n$ are *orthogonal* if $x_1 \ovee x_2 \ovee \cdots \ovee x_n$ is defined. Any PCM carries a preorder via $x \le y \Leftrightarrow \exists z. x \ovee z = y$, with 0 as a bottom (a least element). A *generalised effect algebra (GEA)* is a PCM that is positive ($x \ovee y = 0 \Rightarrow x = y = 0$) and cancellative ($x \ovee y = x \ovee z \Rightarrow y = z$). In a GEA the preorder $\le$ above is a partial order, and we have a 'partial difference' given by $x \ominus y = z \Leftrightarrow x = y \ovee z$. An *effect algebra* is a GEA that has a top (a greatest element), which is denoted by 1. Any element x in an effect algebra has an orthocomplement $x^\perp := 1 \ominus x$, i.e. a unique element such that $x \ovee x^\perp = 1$. Homomorphisms of PCMs and GEAs preserve $\ovee$ and 0, and those of effect algebras additionally preserve 1. An *effect monoid* is an effect algebra M with a 'multiplication' PCM-bihomomorphism $\cdot\colon M \times M \to M$ satisfying $1 \cdot r = r = r \cdot 1$ and $(r \cdot s) \cdot t = r \cdot (s \cdot t)$. A *partial commutative module (PCMod)* over an effect monoid M is a PCM E with a 'scalar multiplication' PCM-bihomomorphism $\bullet\colon M \times E \to E$ satisfying $1 \bullet x = x$ and $r \bullet (s \bullet x) = (r \cdot s) \bullet x$. A *generalised effect module (GEMod)* and an *effect module* are respectively a GEA and an effect algebra that are at the same time a PCMod. Homomorphisms of them are required to preserve the scalar multiplication.

For an effect monoid M, we denote by $\mathcal{D}_M$ and $\widehat{\mathcal{D}}_M$ respectively the (finite, discrete) distribution and subdistribution monads over M on the category **Set**. For a set X, the set $\mathcal{D}_M X$ consists of formal convex sums $|x_1\rangle r_1 + \cdots + |x_n\rangle r_n$ where $x_i \in X$ and $r_i \in M$ with $\bigovee_i r_i = 1$, while $\widehat{\mathcal{D}}_M X$ consists of $|x_1\rangle r_1 + \cdots + |x_n\rangle r_n$ with $\bigovee_i r_i \le 1$ (which holds automatically as long as $r_1, \ldots, r_n$ are orthogonal). Here the 'ket' notation $|x\rangle$ is just syntactic sugar to distinguish formal sums from elements $x \in X$. A *convex set* (resp. *subconvex set*) over M is an Eilenberg-Moore algebra of $\mathcal{D}_M$ (resp. $\widehat{\mathcal{D}}_M$), which is a set X with an operation mapping a formal sum $\sum_i |x_i\rangle r_i \in \mathcal{D}_M X$ (resp. $\widehat{\mathcal{D}}_M X$) to an actual sum $\bigovee_i x_i r_i \in X$. For an effect monoid M we write $\mathbf{EMod}_M$ and $\mathbf{GEMod}_M$ for the categories of effect modules and GEMod's over M, and $\mathbf{Conv}_M = \mathcal{EM}(\mathcal{D}_M)$ and $\mathbf{SConv}_M = \mathcal{EM}(\widehat{\mathcal{D}}_M)$ for the categories of convex and subconvex sets over M. The following dualities are fundamental.

Proposition 2.1. *There are the following adjunctions, obtained by "homming into M".*

$$(\mathbf{EMod}_M)^{\mathrm{op}} \underset{\mathbf{Conv}(-,M)}{\overset{\mathbf{EMod}(-,M)}{\rightleftarrows}} \mathbf{Conv}_M \qquad (\mathbf{GEMod}_M)^{\mathrm{op}} \underset{\mathbf{SConv}(-,M)}{\overset{\mathbf{GEMod}(-,M)}{\rightleftarrows}} \mathbf{SConv}_M$$

Proof. The left-hand adjunction is shown in [5, Proposition 6]. The right-hand one is shown in [10, Appendix B] for $M = [0,1]$, and the proof is easily generalised. ∎

We say a PCMod E over an effect monoid M is *subconvex* if $r_1 \bullet x_1, \cdots, r_n \bullet x_n$ are orthogonal for any $x_1, \cdots, x_n \in E$ and for orthogonal $r_1, \cdots, r_n \in M$. A subconvex PCMod is indeed a subconvex set via the subconvex sum $\bigovee_i r_i \bullet x_i$. The category **PCM** of PCMs is symmetric monoidal closed via a tensor product representing bihomomorphisms [6]. Therefore, **PCM**-enriched categories are well-defined. Specifically, a category is **PCM**-enriched if each homset is a PCM and the composition is a PCM-bihomomorphism.

2.2 Effectuses

Several assumptions on a category were identified by Jacobs [5] for the study of quantum/effect logic and computation. A category that satisfies the most basic assumption [5, Assumption 1] was later given the name 'effectus' in [7, Definition 12] with a slight change of the joint monicity requirement. In this paper we also use the term 'effectus' but follow the requirements of [5].

Definition 2.2. An *effectus* is a category with a final object 1 and finite coproducts $(+, 0)$ satisfying:

- squares of the following form (E) and (K') are pullbacks;

$$\begin{array}{ccc} A+X & \xrightarrow{\mathrm{id}+f} & A+Y \\ {\scriptstyle g+\mathrm{id}}\downarrow & \text{(E)} & \downarrow{\scriptstyle g+\mathrm{id}} \\ B+X & \xrightarrow[\mathrm{id}+f]{} & B+Y \end{array} \qquad \begin{array}{ccc} A & = & A \\ {\scriptstyle \kappa_1}\downarrow & \text{(K')} & \downarrow{\scriptstyle \kappa_1} \\ A+X & \xrightarrow[\mathrm{id}+f]{} & A+Y \end{array}$$

- for each X, the two arrows $[\rhd_1, \kappa_2], [\rhd_2, \kappa_2]\colon (X+X)+1 \to X+1$ are jointly monic, where the 'partial projections' $\rhd_i\colon X+X \to X+1$ are given by $\rhd_1 = [\kappa_1, \kappa_2 \circ !_X]$ and $\rhd_2 = [\kappa_2 \circ !_X, \kappa_1]$.[1]

In an effectus $\mathbf{B}$, a *state* on an object X is an arrow $\omega\colon 1 \to X$; a *predicate* on X is $p\colon X \to 1+1$; and a *scalar* is $r\colon 1 \to 1+1$. For a state ω and a predicate p, the validity probability is given by the *abstract Born rule* $(\omega \vDash p) := p \circ \omega\colon 1 \to 1+1$. We write $\mathit{Stat}(X) = \mathbf{B}(1,X)$ and $\mathit{Pred}(X) = \mathbf{B}(X,1+1)$ for the sets of states and predicates respectively.

It turns out that the set of scalars $M := \mathbf{B}(1,1+1)$ is an effect monoid, and $\mathit{Stat}(X)$ and $\mathit{Pred}(X)$ are a convex set and an effect module over M respectively. In particular, $\mathit{Pred}(X)$ is a poset with a top (truth) 1_X and a bottom (falsum) 0_X. We refer to [5] for the details, but later in §6 we will come to this point with a different approach. An arrow $f\colon X \to Y$ in $\mathbf{B}$ induces a state transformer $\mathit{Stat}(f)\colon \mathit{Stat}(X) \to \mathit{Stat}(Y)$ by $\mathit{Stat}(f)(\omega) = f \circ \omega$, and a (backward) predicate transformer $\mathit{Pred}(f)\colon \mathit{Pred}(Y) \to \mathit{Pred}(X)$ by $\mathit{Pred}(f)(p) = p \circ f$, making Stat and Pred functors in a *state-and-effect triangle* shown in Figure 1.

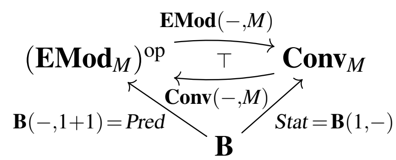


Figure 1: State-and-effect triangle

The dual adjunction $(\mathbf{EMod}_M)^{\mathrm{op}} \rightleftarrows \mathbf{Conv}_M$ from Proposition 2.1 expresses the duality between predicates and states. By "currying" the abstract Born rule $\vDash\colon \mathit{Stat}(X) \times \mathit{Pred}(X) \to M$ we obtain maps α_X and β_X in the bijective correspondence of the dual adjunction:

$$\frac{\alpha_X\colon \mathit{Pred}(X) \longrightarrow \mathbf{Conv}_M(\mathit{Stat}(X), M) \text{ in } \mathbf{EMod}_M}{\overline{\beta_X\colon \mathit{Stat}(X) \longrightarrow \mathbf{EMod}_M(\mathit{Pred}(X), M) \text{ in } \mathbf{Conv}_M}}$$

These maps α and β are natural transformations filling the triangle.

A motivating example of an effectus, which models quantum computation and logic, is the opposite $(\mathbf{Cstar}_{\mathrm{PU}})^{\mathrm{op}}$ of the category of (unital) C^*-algebras and positive unital (PU) maps. Note that an initial object in $\mathbf{Cstar}_{\mathrm{PU}}$ is the set of complex numbers $\mathbb{C}$, and finite products are given by the cartesian products of underlying sets with coordinatewise operations; they are a final object and finite coproducts in the opposite. Then, states on a C^*-algebra A are PU-maps $\omega\colon A \to \mathbb{C}$, which coincide with the standard definition of 'states' in operator theory. Predicates on A are PU-maps $f\colon \mathbb{C} \times \mathbb{C} \to A$, which are in bijective correspondence with elements $p \in A$ with $0 \le p \le 1$, via $p = f(1,0)$ and $f(\lambda,\rho) = \lambda p + \rho(1-p)$. Such elements $p \in A$ with $0 \le p \le 1$ are called *effects* and thought of as "unsharp" predicates, which include "sharp" projections. Scalars are effects in the complex numbers $\mathbb{C}$, i.e. real numbers between 0 and 1. Then the abstract Born rule is the usual Born rule $(\omega \vDash p) := \omega(p) \in [0,1]$. The sum $\ovee$ of effects p,q is defined if $p+q \le 1$; and in that case $p \ovee q = p+q$. With an obvious scalar multiplication, effects form an effect module. The convex structure of states $\omega\colon A \to \mathbb{C}$ is given in a pointwise manner. One has similar examples of effectuses given by C^*-algebras with completely positive unital maps, and W^*-algebras with normal (completely) positive unital maps.

Another example of an effectus is the category $\mathbf{Set}$ of sets and functions, which models classical computation and logic. States $1 \to X$ are simply elements $x \in X$, while predicates $X \to 1+1 \cong 2$ are subsets $P \subseteq X$ as usual. The set of scalars is the two element set $\{0,1\}$, and then the abstract Born rule is the membership relation $(x \vDash P) = (x \in P)$. The Kleisli category $\mathcal{K}\ell(\mathcal{D})$ of the distribution monad $\mathcal{D} = \mathcal{D}_{[0,1]}$ over $[0,1]$ is an effectus for a probabilistic setting. States $1 \to X$ are functions $1 \to \mathcal{D}(X)$, hence probability distributions $\omega \in \mathcal{D}(X)$. Predicates $X \to 1+1$ are functions $X \to \mathcal{D}(1+1) \cong [0,1]$,

[1] The 'n-ary' joint monicity requirement used in [5, Assumption 1] turns out to be equivalent to the binary one used here. In fact, the joint monicity of $[\rhd_1, \kappa_2], [\rhd_2, \kappa_2]\colon (1+1)+1 \to 1+1$ is sufficient, which we omit for space reasons.

thus 'fuzzy' predicates $p \in [0,1]^X$. The set of scalars is the unit interval $[0,1]$, and the abstract Born rule is given by the expectation value $(\omega \vDash p) = \sum_x p(x)\omega(x)$.

Further explanation and examples are found in [5]. We quote the following result for later use.

Lemma 2.3 ([5, Lemma 10])**.** *In an effectus, coproducts are disjoint; in particular, coprojections κ_i are monic. Moreover, squares of the form (K) on the right, which strengthens (K'), are pullbacks.*

$$\begin{array}{ccc} A & \xrightarrow{\;g\;} & B \\ {\scriptstyle \kappa_1}\downarrow & \text{(K)} & \downarrow{\scriptstyle \kappa_1} \\ A+X & \xrightarrow[g+f]{} & B+Y \end{array}$$

■

3 Finitely partially additive categories (FinPACs)

Here we introduce a notion of *finitely partially additive category (FinPAC)*, which is a finite variant of Arbib and Manes' *partially additive category (PAC)* [1, 8]. First we need some preliminary definitions. A category has *zero arrows* if there is a family of 'zero arrows' $0_{XY}\colon X \to Y$ such that $0_{WY} \circ f = 0_{XW} = g \circ 0_{XZ}$ for all $f\colon X \to Y$ and $g\colon Z \to W$. Such a family is unique if exists (indeed, $0_{XY} = 0_{XY} \circ 0'_{XX} = 0'_{XY}$). If a category has a zero object 0, then it has zero arrows $X \to 0 \to Y$. The converse is also true when the category has an initial (or final) object, see e.g. [8, §2.2.19]. For a coproduct $\coprod_{i \in I} X_i$ in a category with zero arrows, *partial projections* (or *quasi projections*) are arrows $\rhd_i\colon \coprod_{i \in I} X_i \to X_i$ given by $\rhd_i \circ \kappa_i = \mathrm{id}_{X_i}$ and $\rhd_i \circ \kappa_j = 0_{X_j X_i}$ $(j \neq i)$. Now we give the definition:

Definition 3.1 (cf. [1, §3.3])**.** A *finitely partially additive category* (*FinPAC* for short) is a category **C** with finite coproducts $(+, 0)$ which is **PCM**-enriched and satisfies the following two axioms.

- (Compatible sum axiom) Arrows $f, g\colon X \to Y$ are orthogonal (in the PCM $\mathbf{C}(X,Y)$) whenever f and g are *compatible* in the sense that there exists a 'bound' $b\colon X \to Y+Y$ such that $f = \rhd_1 \circ b$ and $g = \rhd_2 \circ b$.
- (Untying axiom) If $f, g\colon X \to Y$ are orthogonal, then $\kappa_1 \circ f, \kappa_2 \circ g\colon X \to Y+Y$ are orthogonal too.

Note that **C** has zero arrows, i.e. zero elements 0_{XY} of the PCMs $\mathbf{C}(X,Y)$.

For any objects Y_1 and Y_2 in a FinPAC, arrows $\kappa_1 \circ \rhd_1, \kappa_2 \circ \rhd_2\colon Y_1 + Y_2 \to Y_1 + Y_2$ are compatible via a bound $\kappa_1 + \kappa_2$, hence orthogonal. Then we obtain $\kappa_1 \circ \rhd_1 \ovee \kappa_2 \circ \rhd_2 = \mathrm{id}_{Y_1+Y_2}$ because $(\kappa_1 \circ \rhd_1 \ovee \kappa_2 \circ \rhd_2) \circ \kappa_i = \kappa_i$. For any arrow $f\colon X \to Y_1 + Y_2$, therefore, one has a 'decomposition' $f = (\kappa_1 \circ \rhd_1 \ovee \kappa_2 \circ \rhd_2) \circ f = \kappa_1 \circ f_1 \ovee \kappa_2 \circ f_2$, where $f_i = \rhd_i \circ f\colon X \to Y_i$. In particular, it follows that the two partial projections $\rhd_i\colon Y_1 + Y_2 \to Y_i$ are jointly monic: namely, $\rhd_i \circ f = \rhd_i \circ g$ for $i = 1,2$ implies $f = g$. Now we see the **PCM**-enrichment of a FinPAC is canonical and unique, as is the case for a PAC (cf. [8, Theorem 3.2.18]).

Proposition 3.2. *In a FinPAC, arrows $f, g\colon X \to Y$ are orthogonal if and only if they are compatible. In that case, a bound $b\colon X \to Y+Y$ for f and g is unique and $f \ovee g = \nabla \circ b$, where $\nabla = [\mathrm{id}, \mathrm{id}]$ is the codiagonal.*

Proof. Assume that f and g are orthogonal. By untying, $\kappa_1 \circ f, \kappa_2 \circ g\colon X \to Y+Y$ are orthogonal, and then let $b = \kappa_1 \circ f \ovee \kappa_2 \circ g$. It is easy to see that f and g are compatible via b. A bound $b\colon X \to Y+Y$ is unique since $\rhd_1$ and $\rhd_2$ are jointly monic. Finally, $\nabla \circ b = \nabla \circ (\kappa_1 \circ f \ovee \kappa_2 \circ g) = f \ovee g$. ■

In a FinPAC, not very surprisingly, the n-ary version of the compatible sum and the untying axiom hold, proved by induction with a small trick; see Lemma A.1. Therefore, we can prove the decomposition property for finite coproducts, in the same way as the binary case above.

Lemma 3.3. *Let $\coprod_i Y_i$ be a finite coproduct in a FinPAC. An arrow $f\colon X \to \coprod_i Y_i$ is uniquely decomposed as $f = \bigovee_i \kappa_i \circ f_i$, where $f_i = \rhd_i \circ f\colon X \to Y_i$. Thus, in particular, the partial projections $\rhd_i\colon \coprod_i Y_i \to Y_i$ are jointly monic.* ■

Finally, we give a characterisation of FinPACs.

Theorem 3.4 (cf. [1, §5.3])**.** *A category is a FinPAC if and only if it has finite coproducts and zero arrows, and satisfies the following two conditions for each object X:*

- *the two partial projections $\rhd_1, \rhd_2 \colon X+X \to X$ are jointly monic;*
- *the square on the right is a pullback.*

$$\begin{array}{ccc} (X+X)+X & \xrightarrow{\nabla+\mathrm{id}} & X+X \\ {\scriptstyle \rhd_1}\downarrow & & \downarrow{\scriptstyle \rhd_1} \\ X+X & \xrightarrow{\quad\nabla\quad} & X \end{array}$$

Proof. For the 'if' direction, we define the partial sum $\otimes$ in the manner of Proposition 3.2. The complete proof is deferred to Appendix A. ■

4 Partial computation in effectuses and FinPACs with effects

Recall from Definition 2.2 that an effectus $\mathbf{B}$ has a final object 1 and finite coproducts $(+,0)$. Therefore we have the *lift monad* $(-)+1$ on $\mathbf{B}$. The unit is the first coprojection $\kappa_1 \colon X \to X+1$; the multiplication is the cotuple $[\mathrm{id}_{X+1}, \kappa_2] \colon (X+1)+1 \to X+1$; and the Kleisli extension of $f \colon X \to Y+1$ is the cotuple $[f, \kappa_2] \colon X+1 \to Y+1$. Its Kleisli category, seen as a category for partial computation, plays an important role in this paper. Hence we reserve a few notations for it.

We denote by $\mathbf{B}_{+1}$ the Kleisli category of the lift monad on $\mathbf{B}$. Namely, $\mathbf{B}_{+1}$ has the same objects as $\mathbf{B}$, and arrows given by $\mathbf{B}_{+1}(X,Y) = \mathbf{B}(X, Y+1)$. We write $f \colon X \rightharpoonup Y$ ('harpoon' arrows) for arrows in $\mathbf{B}_{+1}$, and $g \mathbin{\hat{\circ}} f = [g, \kappa_2] \circ f$ for the composition in $\mathbf{B}_{+1}$. We denote the canonical functor $\mathbf{B} \to \mathbf{B}_{+1}$ by $\widehat{(-)}$; namely $\widehat{X} = X$ and $\widehat{f} = \kappa_1 \circ f$. Then $\widehat{\mathrm{id}}_X$ denotes the identity $\kappa_1 \colon X \rightharpoonup X$ in $\mathbf{B}_{+1}$. The Kleisli category $\mathbf{B}_{+1}$ has all finite coproducts, which are inherited from $\mathbf{B}$ in the way the functor $\widehat{(-)} \colon \mathbf{B} \to \mathbf{B}_{+1}$ preserves the coproducts on the nose. In other words, a coproduct in $\mathbf{B}_{+1}$ is a coproduct $\coprod_i X_i$ in $\mathbf{B}$ with coprojections $\widehat{\kappa}_i \colon X_i \rightharpoonup \coprod_i X_i$, where κ_i are coprojections in $\mathbf{B}$. For arrows f, g in $\mathbf{B}_{+1}$, we write $f \mathbin{\hat{+}} g = [\widehat{\kappa}_1 \mathbin{\hat{\circ}} f, \widehat{\kappa}_2 \mathbin{\hat{\circ}} g]$ in order to distinguish it from $f+g$ in $\mathbf{B}$. The base category $\mathbf{B}$ is understood as the 'total' part of $\mathbf{B}_{+1}$ via $\widehat{(-)} \colon \mathbf{B} \to \mathbf{B}_{+1}$, see also Proposition 4.6.

The category $\mathbf{B}_{+1}$ has zero arrows $0_{XY} \colon X \rightharpoonup Y$ given by $X \xrightarrow{!_X} 1 \xrightarrow{\kappa_2} Y+1$ in $\mathbf{B}$. Hence $\mathbf{B}_{+1}$ has partial projections $\rhd_i \colon X_1 + X_2 \rightharpoonup X_i$. There are certain pullbacks involving them.

Lemma 4.1. *For a ('total') arrow $f \colon X \to Y$ in an effectus $\mathbf{B}$, the square on the right is a pullback in $\mathbf{B}_{+1}$.*

$$\begin{array}{ccc} X+A & \xrightharpoonup{\widehat{f} \mathbin{\hat{+}} \widehat{\mathrm{id}}} & Y+A \\ {\scriptstyle \rhd_1}\downharpoonleft & & \downharpoonleft{\scriptstyle \rhd_1} \\ X & \xrightharpoonup{\quad\widehat{f}\quad} & Y \end{array}$$

Proof. The square is a pullback in $\mathbf{B}_{+1}$ if and only if the left-hand square below is pullback in $\mathbf{B}$.

$$\begin{array}{ccc} (X+A)+1 & \xrightarrow{[\widehat{f} \mathbin{\hat{+}} \widehat{\mathrm{id}}, \kappa_2]} & (Y+A)+1 \\ {\scriptstyle [\rhd_1, \kappa_2]}\downarrow & & \downarrow{\scriptstyle [\rhd_1, \kappa_2]} \\ X+1 & \xrightarrow{\quad[\widehat{f}, \kappa_2]\quad} & Y+1 \end{array} \qquad \begin{array}{ccc} X+(A+1) & \xrightarrow{f+\mathrm{id}} & Y+(A+1) \\ {\scriptstyle \mathrm{id}+!}\downarrow \ \lrcorner & & \downarrow{\scriptstyle \mathrm{id}+!} \\ X+1 & \xrightarrow{\quad f+\mathrm{id}\quad} & Y+1 \end{array}$$

Up to isomorphism, it coincides with the right-hand pullback (E). ■

Theorem 4.2. *For an effectus $\mathbf{B}$, the category $\mathbf{B}_{+1}$ is a FinPAC.*

Proof. We use Theorem 3.4, a characterisation of a FinPAC. We have already seen that $\mathbf{B}_{+1}$ has finite coproducts and zero arrows. Note that the arrows $\rhd_i \colon X+X \to X+1$ in Definition 2.2 are precisely partial projections $\rhd_i \colon X+X \rightharpoonup X$ in $\mathbf{B}_{+1}$. Then, the joint monicity requirement of an effectus is equivalent to say that $\rhd_1$ and $\rhd_2$ are jointly monic in $\mathbf{B}_{+1}$, as required for FinPACs. The pullback condition is proved by taking $f = \nabla \colon X+X \to X$ in Lemma 4.1. ■

In particular, $\mathbf{B}_{+1}$ is **PCM**-enriched, and hence each homset is a PCM. For the sets of predicates $\mathit{Pred}(X) = \mathbf{B}(X, 1+1)$, Jacobs showed a similar, but stronger result.

Proposition 4.3 ([5, Proposition 12]). *Let* $\mathbf{B}$ *be an effectus. For each* $X \in \mathbf{B}$*, the homset* $\mathbf{B}(X, 1+1) = \mathbf{B}_{+1}(X, 1)$ *is an effect algebra with the top* $1_X := \kappa_1 \circ !_X = \widehat{!}_X$*.* ■

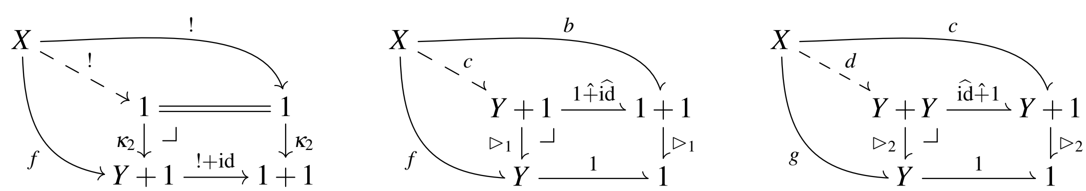
Note that the PCM structure on $\mathbf{B}(X, 1+1)$ given by Jacobs [5, Definition 11] coincides with ours (see Proposition 3.2). Crucially, the category $\mathbf{B}_{+1}$ is not only equipped with both the partially additive and the effect algebra structure, but satisfies further conditions that relate them. We give a name to such categories, since they will turn out to characterise $\mathbf{B}_{+1}$.

Definition 4.4. A *FinPAC with effects* is a FinPAC $\mathbf{C}$ with a special object $I \in \mathbf{C}$ such that hom-PCMs $\mathbf{C}(X, I)$ are effect algebras for all $X \in \mathbf{C}$, satisfying the two conditions below. We write 1_X and 0_X $(= 0_{XI})$ for the top and the bottom of $\mathbf{C}(X, I)$.

1. $1_Y \circ f = 0_X$ implies $f = 0_{XY}$ for all $f\colon X \to Y$.
2. $1_Y \circ f \perp 1_Y \circ g$ implies $f \perp g$ for all $f, g\colon X \to Y$.

Theorem 4.5. *Let* $\mathbf{B}$ *be an effectus. Then* $(\mathbf{B}_{+1}, 1)$ *is a FinPAC with effects.*

Proof. We check the two requirement. 1) Assume that $1_Y \hat{\circ} f = 0_X$, i.e. $(!_Y + \mathrm{id}) \circ f = \kappa_2 \circ !_X$. Using a pullback (K) with the symmetry of coproducts, we obtain $f = \kappa_2 \circ !_X = 0_{XY}$ as in the left diagram below.



2) Let $b\colon X \rightharpoonup 1+1$ be a bound for $1_Y \hat{\circ} f$ and $1_X \hat{\circ} g$. Note that $1_Y = \widehat{!}_Y$ is a 'total' arrow. Then we use a pullback of Lemma 4.1 and obtain a mediating map $c\colon X \rightharpoonup Y+1$ as in the middle diagram above. Using a similar pullback given by the symmetry of coproducts, we obtain $d\colon X \rightharpoonup Y+Y$ as in the right diagram above. Then it is straightforward to check d is a bound for f and g. ■

In a FinPAC with effects $(\mathbf{C}, I)$, we call an arrow $p\colon X \to I$ a *predicate* on X, and write $\mathit{Pred}(X) = \mathbf{C}(X, I)$, which is by definition an effect algebra. When $\mathbf{C} = \mathbf{B}_{+1}$, this definition coincides with predicates in the effectus $\mathbf{B}$, since $\mathbf{B}_{+1}(X, 1) = \mathbf{B}(X, 1+1)$. For an arrow $f\colon X \to Y$ in $\mathbf{C}$, we call $1_Y \circ f \in \mathit{Pred}(X)$ the *domain predicate* of f and write $\mathrm{Dp}(f) = 1_Y \circ f$. We then have a PCM-homomorphism $\mathrm{Dp}\colon \mathbf{C}(X, Y) \to \mathit{Pred}(X)$. We say an arrow $f\colon X \to Y$ in $\mathbf{C}$ is *total* if $\mathrm{Dp}(f) = 1_X$. It is easy to see that all objects of $\mathbf{C}$ with total arrows form a subcategory of $\mathbf{C}$, which is denoted by $\mathbf{C}_{\mathrm{t}}$.

Proposition 4.6. *For an effectus* $\mathbf{B}$*, the functor* $\widehat{(-)}\colon \mathbf{B} \to \mathbf{B}_{+1}$ *restricts to an isomorphism* $\mathbf{B} \cong (\mathbf{B}_{+1})_{\mathrm{t}}$*.*

Proof. Note that $\widehat{(-)}$ is faithful, since coprojections $\kappa_1\colon X \to X+1$ are monic in $\mathbf{B}$ by Lemma 2.3. It suffices to show that an arrow $f\colon X \rightharpoonup Y$ in $\mathbf{B}_{+1}$ is total if and only if $f = \widehat{g}$ for some $g \in X \to Y$ in $\mathbf{B}$. If $f = \widehat{g}$ then $\mathrm{Dp}(f) = 1_Y \hat{\circ} f = \widehat{!}_Y \hat{\circ} \widehat{g} = \widehat{!_Y \circ g} = 1_X$. Conversely, assume that $\mathrm{Dp}(f) = 1_X$, i.e. $(!_Y + \mathrm{id}) \circ f = \kappa_1 \circ !_X$. Using a pullback (K), we obtain $g\colon X \to Y$ with $f = \kappa_1 \circ g = \widehat{g}$. ■

We list several basic properties of a FinPAC with effects.

Lemma 4.7. *In a FinPAC with effects* $(\mathbf{C}, I)$*, the following hold.*

1. *For* $f\colon X \to Y$*,* $\mathrm{Dp}(f) = 0_X$ *if and only if* $f = 0_{XY}$

2. *For $f_1,\dots,f_n\colon X\to Y$, $\mathrm{Dp}(f_1),\dots,\mathrm{Dp}(f_n)$ are orthogonal if and only if $f_1,\dots,f_n$ are orthogonal. In that case $\mathrm{Dp}(\bigovee_i f_i)=\bigovee_i \mathrm{Dp}(f_i)$.*
3. *For $f\colon X\to Y$ and $g\colon Y\to Z$, $\mathrm{Dp}(g\circ f)=\mathrm{Dp}(g)\circ f$*
4. *For $f\colon X\to Y$ and $g\colon Y\to Z$, $\mathrm{Dp}(g\circ f)\le \mathrm{Dp}(f)$. If g is total then $\mathrm{Dp}(g\circ f)=\mathrm{Dp}(f)$.*
5. *Any split mono is total. In particular, any isomorphism is total.*
6. *Coprojections κ_i are split monic and hence total.*
7. *$1_I=\mathrm{id}_I\colon I\to I$.*

Proof. 1. The 'if' direction holds because Dp is homomorphism, while 'only if' holds by definition.

2. The binary case is easy like the previous, and then the n-ary case follows by induction.

3. $\mathrm{Dp}(h\circ f)=1_Z\circ h\circ f=\mathrm{Dp}(h)\circ f$

4. $\mathrm{Dp}(h\circ f)=\mathrm{Dp}(h)\circ f\le 1_Y\circ f=\mathrm{Dp}(f)$. We have equality when h is total.

5. If $g\circ f=\mathrm{id}$, then $\mathrm{Dp}(f)\ge \mathrm{Dp}(g\circ f)=\mathrm{Dp}(\mathrm{id})=1$.

6. It is split monic as $\rhd_i\circ\kappa_i=\mathrm{id}$.

7. Note that $\mathrm{Dp}(\mathrm{id}_I)\perp \mathrm{Dp}(\mathrm{id}_I^{\perp})$ and $\mathrm{Dp}(\mathrm{id}_I)=1_I$. Then $\mathrm{Dp}(\mathrm{id}_I^{\perp})=0_I$ and hence $\mathrm{id}_I^{\perp}=0_{II}=0_I$. Namely $\mathrm{id}_I=1_I$. ∎

In a FinPAC with effects, we have the decomposition property (Lemma 3.3) as a more explicit bijective correspondence involving domain predicates.

Lemma 4.8. *Let $\coprod_i Y_i$ be a finite coproduct in a FinPAC with effects. We have the following bijective correspondence.*

$$\frac{f\colon X\longrightarrow \coprod_i Y_i}{\text{a family } (f_i\colon X\longrightarrow Y_i)_i \text{ where } (\mathrm{Dp}(f_i))_i \text{ is orthogonal (so that } \bigovee_i \mathrm{Dp}(f_i)\le 1_X)}$$

They are related in $f_i=\rhd_i\circ f$ and $f=\bigovee_i \kappa_i\circ f_i$. Moreover one has $\mathrm{Dp}(f)=\bigovee_i\mathrm{Dp}(f_i)$. In particular, f is total if and only if $\bigovee_i\mathrm{Dp}(f_i)=1_X$.

Proof. Given $f\colon X\to\coprod_i Y_i$, we have the decomposition $f=\bigovee_i\kappa_i\circ f_i$, where $f_i=\rhd_i\circ f$. The family $(\mathrm{Dp}(f_i))_i$ is orthogonal because $\mathrm{Dp}(f_i)=\mathrm{Dp}(\kappa_i\circ f_i)$. Conversely, if $(\mathrm{Dp}(f_i))_i$ is orthogonal for arrows $f_i\colon X\to Y_i$, then $(\kappa_i\circ f_i)_i$ is orthogonal. Hence we have the sum $f=\bigovee_i\kappa_i\circ f_i$. It is easy to see that the correspondence is bijective. Finally, $\mathrm{Dp}(f)=\bigovee_i\mathrm{Dp}(\kappa_i\circ f_i)=\bigovee_i\mathrm{Dp}(f_i)$. ∎

Lemma 4.9. *Let $(\mathbf{C},I)$ be a FinPAC with effects. Coproducts $\coprod_i X_i$ in $\mathbf{C}$ restrict to $\mathbf{C}_{\mathrm{t}}$, so that $\mathbf{C}_{\mathrm{t}}$ has all finite coproducts. Moreover, I is final in $\mathbf{C}_{\mathrm{t}}$, and we have an isomorphism $(\mathbf{C}_{\mathrm{t}})_{+1}\cong\mathbf{C}$, which is identity on objects, and maps $f\colon X\to Y+I$ to $\rhd_1\circ f\colon X\to Y$.*

Proof. Since coprojections are total, the coproduct diagram is in $\mathbf{C}_{\mathrm{t}}$. Let $f_i\colon X_i\to Y$ be total arrows. Then the cotuple $[f_i]_i\colon\coprod_i X_i\to Y$ is total, i.e. $\mathrm{Dp}([f_i]_i)=1_{\coprod_i X_i}$ since $\mathrm{Dp}([f_i]_i)\circ\kappa_i=\mathrm{Dp}([f_i]_i\circ\kappa_i)=\mathrm{Dp}(f_i)=1_{X_i}=\mathrm{Dp}(\kappa_i)=1_{\coprod_i X_i}\circ\kappa_i$ for all i. The object I is final in $\mathbf{C}_{\mathrm{t}}$, because $\mathbf{C}_{\mathrm{t}}(X,I)=\{1_X\}$.

It is easy to see the mapping $f\mapsto\rhd_1\circ f$ is functorial. To prove $(\mathbf{C}_{\mathrm{t}})_{+1}\cong\mathbf{C}$, it suffices to show the functor is full and faithful. Let $f\in\mathbf{C}(X,Y)$. Using Lemma 4.8, we obtain $g\colon X\to Y+I$ by $g=\kappa_1\circ f\ovee\kappa_2\circ\mathrm{Dp}(f)^{\perp}$. Then g is total and $\rhd_1\circ g=f$. Suppose that $h\colon X\to Y+I$ is a total arrow with $\rhd_1\circ h=f$. Consider the decomposition $h=\kappa_1\circ h_1\ovee\kappa_2\circ h_2$. We have $h_1=f$ and $1=\mathrm{Dp}(h_1)\ovee\mathrm{Dp}(h_2)=\mathrm{Dp}(f)\ovee h_2$, so that $h_2=\mathrm{Dp}(f)^{\perp}$. Hence $h=g$. ∎

Theorem 4.10. *Let $(\mathbf{C},I)$ be a FinPAC with effects. The subcategory $\mathbf{C}_{\mathrm{t}}$ is an effectus.*

Proof. The joint monicity requirement is equivalent to say that partial projections are jointly monic in $(\mathbf{C}_{\mathrm{t}})_{+1}$, which holds because $(\mathbf{C}_{\mathrm{t}})_{+1} \cong \mathbf{C}$. We prove that the squares (E) and (K') are pullbacks. **(E)** Let $\alpha\colon Z \to A+Y$ and $\beta\colon Z \to B+X$ be total arrows with $(g+\mathrm{id})\circ\alpha = (\mathrm{id}+f)\circ\beta$. By postcomposing partial projections $\rhd_i$ to $h := (g+\mathrm{id})\circ\alpha = (\mathrm{id}+f)\circ\beta$, we obtain $h_1 = g\circ\alpha_1 = \beta_1$ and $h_2 = \alpha_2 = f\circ\beta_2$, where $\alpha_i = \rhd_i\circ\alpha$, $\beta_i = \rhd_i\circ\beta$ and $h_i = \rhd_i\circ h$. Note that $\mathrm{Dp}(\alpha_1) = \mathrm{Dp}(g\circ\alpha_1) = \mathrm{Dp}(h_1)$ and $\mathrm{Dp}(\beta_2) = \mathrm{Dp}(f\circ\beta_2) = \mathrm{Dp}(h_2)$. By Lemma 4.8, we can define a total arrow $\gamma\colon Z \to A+X$ by $\gamma = \kappa_1\circ\alpha_1 \ovee \kappa_2\circ\beta_2$. As desired, $(\mathrm{id}+f)\circ\gamma = \kappa_1\circ\alpha_1 \ovee \kappa_2\circ f\circ\beta_2 = \kappa_1\circ\alpha_1 \ovee \kappa_2\circ\alpha_2 = \alpha$, and $(g+\mathrm{id})\circ\gamma = \beta$ similarly. To see the uniqueness, assume that $\gamma\colon Z \to A+X$ satisfies $(\mathrm{id}+f)\circ\gamma = \alpha$ and $(g+\mathrm{id})\circ\gamma = \beta$. Then one has $\rhd_1\circ\gamma = \alpha_1$ and $\rhd_2\circ\gamma = \alpha_2$. Hence the joint monicity of partial projections implies the uniqueness. **(K')** Let $\alpha\colon Z \to A$ and $\beta\colon Z \to A+X$ be total arrows with $\kappa_1\circ\alpha = (\mathrm{id}+f)\circ\beta$. By postcomposing partial projections $\rhd_i$ to $\kappa_1\circ\alpha = (\mathrm{id}+f)\circ\beta$, we obtain $\alpha = \beta_1$ and $0 = f\circ\beta_2$, where $\beta_i = \rhd_i\circ\beta$. Then $\beta_2 = 0$ since $\mathrm{Dp}(\beta_2) = \mathrm{Dp}(f\circ\beta_2) = 0$. Now we have $\beta = \kappa_1\circ\beta_1 \ovee \kappa_2\circ\beta_2 = \kappa_1\circ\alpha$ as desired. ■

Example 4.11. Recall from §2.2 three examples of effectuses $(\mathbf{Cstar}_{\mathrm{PU}})^{\mathrm{op}}$, **Set** and $\mathcal{K}\ell(\mathcal{D})$. By Theorem 4.5, the Kleisli categories of the lift monads on these effectuses are FinPACs with effects (see also Table 1 in §1).

The Kleisli category $((\mathbf{Cstar}_{\mathrm{PU}})^{\mathrm{op}})_{+1}$ is isomorphic to the opposite $(\mathbf{Cstar}_{\mathrm{PSU}})^{\mathrm{op}}$ of the category of C^*-algebras and positive subunital[2] (PSU) maps. Indeed, we have the bijective correspondence shown on the right, via $g(x) = f(x,0)$ and $f(x,\lambda) = g(x) + \lambda(1-g(1))$. Predicates are PSU-maps $\mathbb{C} \to A$, which are easily identified with effects $p \in [0,1]_A := \{p \in A \mid 0 \le p \le 1\}$. Then the domain predicate of a PSU-map $f\colon A \to B$ is identified with $f(1) \in [0,1]_B$. By Lemma 4.7.2, the sum $f \ovee g$ of PSU-maps $f,g\colon A \to B$ is defined precisely when $f(1) \perp g(1)$ in $[0,1]_B$, namely $f(1)+g(1) \le 1$. In that case the sum is defined pointwise: $(f \ovee g)(x) = f(x)+g(x)$. Note that C^*-algebras with completely positive subunital maps and W^*-algebras with normal (completely) positive subunital maps work in exactly the same way. The latter is especially important for semantics of quantum programming languages [2, 11].

$$\frac{f\colon A\times\mathbb{C} \longrightarrow B \quad \text{PU-map}}{g\colon A \longrightarrow B \quad \text{PSU-map}}$$

For the classical example, it is well-known that $\mathbf{Set}_{+1} \cong \mathbf{Pfn}$, where **Pfn** is the category of sets and partial functions. The domain predicate $\mathrm{Dp}(f)$ of a partial function $f\colon X \rightharpoonup Y$ is identified with its domain of definition $\mathrm{dom}(f) \subseteq X$. The sum of $f,g\colon X \rightharpoonup Y$ is defined precisely if $\mathrm{dom}(f) \perp \mathrm{dom}(g)$ in $\mathcal{P}(X)$, i.e. $\mathrm{dom}(f)\cap\mathrm{dom}(g) = \varnothing$. In that case $f \ovee g$ is defined on $\mathrm{dom}(f)\cup\mathrm{dom}(g)$ in an obvious way.

For the probabilistic example, we have $\mathcal{K}\ell(\mathcal{D})_{+1} \cong \mathcal{K}\ell(\widehat{\mathcal{D}})$, where $\widehat{\mathcal{D}} = \widehat{\mathcal{D}}_{[0,1]}$ is the subdistribution monad over $[0,1]$. This is due to the natural bijections $\widehat{\mathcal{D}}(X) \cong \mathcal{D}(X+1)$. The domain predicate $\mathrm{Dp}(f)$ of a map $f\colon X \to Y$ in $\mathcal{K}\ell(\widehat{\mathcal{D}})$, i.e. a function $f\colon X \to \widehat{\mathcal{D}}(Y)$ is identified with the 'fuzzy' predicate $p \in [0,1]^X$ given by $p(x) = \sum_y f(x)(y)$. The sum of functions $f,g\colon X \to \widehat{\mathcal{D}}Y$ is defined if and only if $\sum_y f(-)(y) \perp \sum_y g(-)(y)$ in $[0,1]^X$, that is, $\sum_y f(x)(y) + \sum_y g(x)(y) \le 1$ for all $x \in X$. In that case, the sum is defined by $(f \ovee g)(x)(y) = f(x)(y) + g(x)(y)$.

5 Categorical equivalence of effectuses and FinPACs with effects

In the previous section we showed that for an effectus $\mathbf{B}$, the category $\mathbf{B}_{+1}$ with $1 \in \mathbf{B}_{+1}$ is a FinPAC with effects; and for a FinPAC with effects $(\mathbf{C}, I)$, the subcategory $\mathbf{C}_{\mathrm{t}}$ is an effectus. Moreover we have isomorphisms $\mathbf{B} \cong (\mathbf{B}_{+1})_{\mathrm{t}}$ and $\mathbf{C} \cong (\mathbf{C}_{\mathrm{t}})_{+1}$. We immediately obtain a characterisation of effectuses.

Corollary 5.1. *A category* $\mathbf{B}$ *is an effectus if and only if there is a FinPAC with effects* $(\mathbf{C}, I)$ *such that* $\mathbf{B} \cong \mathbf{C}_{\mathrm{t}}$. ■

[2] A positive map $g\colon A \to B$ is *subunital* if $g(1) \le 1$.

The results are most naturally presented in terms of (2-)categorical equivalence.

Definition 5.2. We define a (strict) 2-category **Eff** of effectuses as follows. An object is an effectus **B**. An arrow $F\colon \mathbf{A} \to \mathbf{B}$ is a functor that preserves the final object and finite coproducts. A 2-cell $\alpha\colon F \Rightarrow G$ is a natural transformation that is monoidal w.r.t. $(+,0)$. We also define a 2-category **FPE** of FinPACs with effects as follows. An object is a FinPAC with effects $(\mathbf{C}, I)$. An arrow $F\colon (\mathbf{C}, I_{\mathbf{C}}) \to (\mathbf{D}, I_{\mathbf{D}})$ is a functor $F\colon \mathbf{C} \to \mathbf{D}$ that preserves finite coproducts and "preserves the truth" in the sense that $1_{FI_{\mathbf{C}}}\colon FI_{\mathbf{C}} \to I_{\mathbf{D}}$ is an isomorphism, and $1_{FI} \circ F1_X = 1_{FX}$ for all $X \in \mathbf{C}$. A 2-cell $\alpha\colon F \Rightarrow G$ is a natural transformation that is monoidal w.r.t. $(+,0)$, and satisfies $1_{GI} \circ \alpha_I = 1_{FI}$.

Theorem 5.3. *The mappings* $\mathbf{B} \mapsto \mathbf{B}_{+1}$ *and* $\mathbf{C} \mapsto \mathbf{C}_{\mathrm{t}}$ *extend to 2-functors* $(-)_{+1}\colon \mathbf{Eff} \to \mathbf{FPE}$ *and* $(-)_{\mathrm{t}}\colon \mathbf{FPE} \to \mathbf{Eff}$ *respectively. Moreover, they form a 2-equivalence of 2-categories* $\mathbf{Eff} \simeq \mathbf{FPE}$*.*

Proof. The essential part is already done. The rest, checking functoriality and naturality, is mostly routine. We defer the details to Appendix B. ■

6 State-and-effect triangles over FinPACs with effects

Let $(\mathbf{C}, I)$ be a FinPAC with effects. Recall that $\mathit{Pred}(X) = \mathbf{C}(X, I)$ is the set of predicates on X. We call an arrow $\omega\colon I \to X$ a *substate* on X, an arrow $r\colon I \to I$ a *scalar*. We write $\mathit{SStat}(X) = \mathbf{C}(I, X)$ for the set of substates on X, and let $M = \mathbf{C}(I, I)$ be the set of scalars.

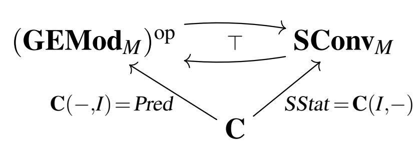
Proposition 6.1. *Let* $(\mathbf{C}, I)$ *be a FinPAC with effects.*

1. *The effect algebra* $M = \mathbf{C}(I, I)$ *is an effect monoid with the composition* $\circ$ *as a multiplication.*
2. *For each* $X \in \mathbf{C}$*, the effect algebra* $\mathrm{Pred}(X) = \mathbf{C}(X, I)$ *is an effect module over* M*, with the composition* $\circ$ *as a scalar multiplication.*
3. *For each* $X \in \mathbf{C}$*, the PCM* $\mathit{SStat}(X) = \mathbf{C}(I, X)$ *is a PCMod over* M *with the composition* $\circ$ *as a (right) scalar multiplication. Moreover it is subconvex.*

Proof. Straightforward, but note that $1_I = \mathrm{id}_I \in M$. To see $\mathit{SStat}(X)$ is subconvex, use $\mathrm{Dp}(\omega \circ r) \leq r$. ■

Combining the dual adjunction $(\mathbf{GEMod}_M)^{\mathrm{op}} \rightleftarrows \mathbf{SConv}_M$ from Proposition 2.1, we obtain a state-and-effect triangle. We use the category $\mathbf{GEMod}_M$ of GEMod's because induced predicate transformers do not necessarily preserve the truth predicates.

Theorem 6.2. *For a FinPAC with effects* $(\mathbf{C}, I)$*, the hom-functors* $\mathbf{C}(-, I)$ *and* $\mathbf{C}(I, -)$ *give rise to the functors in the diagram on the right, constituting a state-and-effect triangle.*



Proof. It is easy to check that the precomposition $\mathbf{C}(f, I)$ and the postcomposition $\mathbf{C}(I, f)$ are desired homomorphisms. ■

Examples of this type of state-and-effect triangles have already appeared in [4, 10], but the general construction is new. Substates in the quantum example $(\mathbf{Cstar}_{\mathrm{PSU}})^{\mathrm{op}}$ are PSU-maps $\omega\colon A \to \mathbb{C}$. In the classical example **Pfn**, substates on X are either elements $x \in X$ or the 'bottom'. In the probabilistic example $\mathcal{K}\ell(\widehat{\mathcal{D}})$, substates are subdistributions $\omega \in \widehat{\mathcal{D}}(X)$.

As is the case for effectuses (§2.2), there is an *abstract Born rule* given by $(\omega \vDash p) := p \circ \omega \in M$ for $\omega\colon I \to X$ and $p\colon X \to I$. The map $\vDash\colon \mathit{SStat}(X) \times \mathit{Pred}(X) \to M$ is an appropriate bihomomorphism, so that by "currying", we obtain the following maps α_X and β_X in the bijective correspondence of the dual adjunction.

$$\frac{\alpha_X : \mathit{Pred}(X) \longrightarrow \mathbf{SConv}_M(\mathit{SStat}(X), M) \text{ in } \mathbf{GEMod}_M}{\beta_X : \mathit{SStat}(X) \longrightarrow \mathbf{GEMod}_M(\mathit{Pred}(X), M) \text{ in } \mathbf{SConv}_M}$$

These maps α and β give natural transformations which fill the state-and-effect triangle

In a FinPAC with effects, a *state* on X is a substate $\omega : I \to X$ with $\mathrm{Dp}(\omega) = 1$ (i.e. a total substate), and the set of states is denoted by $\mathit{Stat}(X) = \mathbf{C}_{\mathrm{t}}(I, X)$. This definition accords with states in an effectus, since $(\mathbf{B}_{+1})_{\mathrm{t}}(1, X) \cong \mathbf{B}(1, X)$. The set $\mathit{Stat}(X)$ is a subset of $\mathit{SStat}(X)$ that is closed under convex sum, hence $\mathit{Stat}(X)$ is a convex set, giving a functor $\mathit{Stat} : \mathbf{C}_{\mathrm{t}} \to \mathbf{Conv}_M$. On the other hand, we obtain a functor $\mathit{Pred} : \mathbf{C}_{\mathrm{t}} \to (\mathbf{EMod}_M)^{\mathrm{op}}$ as a restriction of $\mathit{Pred} : \mathbf{C} \to (\mathbf{GEMod}_M)^{\mathrm{op}}$, since predicate transformers induced by total arrows preserve the truth predicates. This is an alternative way to obtain a state-and-effect triangle over an effectus shown in Figure 1 (cf. [5]).

In what follows, we will focus on a FinPAC with effects satisfying 'normalisation' (of states). A FinPAC with effects $(\mathbf{C}, I)$ satisfies *normalisation* if for each object X and for each substate $\omega \in \mathit{SStat}(X)$ that is nonzero ($\omega \neq 0_{IX}$), there exists a unique state $\tilde{\omega} \in \mathit{Stat}(X)$ such that $\omega = \tilde{\omega} \circ \mathrm{Dp}(\omega)$. An effectus $\mathbf{B}$ satisfies *normalisation* if the corresponding FinPAC with effects $(\mathbf{B}_{+1}, 1)$ satisfies normalisation. An effectus with normalisation was introduced and studied in [7], where the most results are restricted to the case when the set of scalars M is the unit interval $[0,1]$. In fact, if an effectus or FinPAC with effects satisfies normalisation, then the scalars are already 'good' enough to take away the restriction $M = [0,1]$.

Definition 6.3. An effect monoid M has *division* if for all $s, t \in M$ with $s \leq t$ and $t \neq 0$, there exists unique 'quotient' $q \in M$ such that $q \cdot t = s$. The quotient q is denoted by s/t. We call such an effect monoid a *division effect monoid*.

Proposition 6.4. *If a FinPAC with effects $(\mathbf{C}, I)$ satisfies normalisation, then the effect monoid of scalars $M = \mathbf{C}(I, I)$ has division.*

Proof. Let $s, t \in M$ be scalars with $s \leq t$ and $t \neq 0$. Let $s' = t \ominus s$, so that $s \ovee s' = t$. Let $\omega = \kappa_1 \circ s \ovee \kappa_2 \circ s' : I \to I + I$, which is nonzero because $\mathrm{Dp}(\omega) = s \ovee s' = t \neq 0$. By normalisation there is a state $\tilde{\omega} : I \to I + I$ with $\omega = \tilde{\omega} \circ \mathrm{Dp}(\omega) = \tilde{\omega} \circ t$. Then $s = \rhd_1 \circ \omega = \rhd_1 \circ \tilde{\omega} \circ t$. Therefore $\rhd_1 \circ \tilde{\omega}$ is a desired quotient. To see the uniqueness of the quotient, assume that $q \in M$ satisfies $s = q \circ t$. Then $s' = t \ominus s = t \ominus (q \circ t) = q^{\perp} \circ t$. Let $\omega_q = \kappa_1 \circ q \ovee \kappa_2 \circ q^{\perp} : I \to I + I$, which is a state and $\omega_q \circ t = \kappa_1 \circ q \circ t \ovee \kappa_2 \circ q^{\perp} \circ t = \kappa_1 \circ s \ovee \kappa_2 \circ s' = \omega$. By the uniqueness of normalisation, we obtain $\omega_q = \tilde{\omega}$. Therefore $\rhd_1 \circ \tilde{\omega} = \rhd_1 \circ \omega_q = q$. ∎

The division indeed satisfies desired properties, see Lemmas C.1 and C.2. It allows us to obtain the following result, by generalising $M = [0,1]$ to any division effect monoid.

Theorem 6.5 ([7, Corollary 19])**.** *Let $\mathbf{B}$ be an effectus satisfying normalisation. Then, all the categories and the functors in the state-and-effect triangle over $\mathbf{B}$ (Figure 1) are objects and arrows in $\mathbf{Eff}$.* ∎

Note that, unlike [7], we simply use $\mathbf{Conv}_M$ rather than the category of *cancellative* convex sets. This is because we use a weaker variant of the joint monicity requirement in Definition 2.2, and $\mathbf{Conv}_M$ is indeed an effectus in our sense; see Proposition C.3. Furthermore, it is straightforward to check the following.

Lemma 6.6. *Let M be a division effect monoid. The unit and the counit of the adjunction $(\mathbf{EMod}_M)^{\mathrm{op}} \rightleftarrows \mathbf{Conv}_M$ are 2-cells in $\mathbf{Eff}$. Namely, $(\mathbf{EMod}_M)^{\mathrm{op}} \rightleftarrows \mathbf{Conv}_M$ is an adjunction in the 2-category $\mathbf{Eff}$.* ∎

In the light of the 2-equivalence $\mathbf{Eff} \simeq \mathbf{FPE}$, we obtain a corresponding state-and-effect triangle over a FinPAC with effects.

Corollary 6.7. *Let $(\mathbf{C}, I)$ be a FinPAC with effects satisfying normalisation. We have a state-and-effect triangle on the right, where the categories, the functors and the adjunction are in $\mathbf{FPE}$.*

$$\begin{array}{ccc} ((\mathbf{EMod}_M)^{\mathrm{op}})_{+1} & \underset{\top}{\rightleftarrows} & (\mathbf{Conv}_M)_{+1} \\ {\scriptstyle \mathit{Pred}_{+1}} \nwarrow & & \nearrow {\scriptstyle \mathit{Stat}_{+1}} \\ & (\mathbf{C}_{\mathrm{t}})_{+1} \cong \mathbf{C} & \end{array}$$

∎

7 Conclusions

We studied partial computation in effectuses, giving a fundamental equivalence of effectuses and FinPACs with effects. Despite the equivalence, FinPACs with effects sometimes have an advantage over effectuses, because they have richer structures such as the finitely partially additive structure. For instance, an *instrument* map $\mathsf{instr}_p\colon X \to X + \cdots + X$ for an 'n-test' $p\colon X \to 1 + \cdots + 1$ in an effectus allow us to perform a (quantum) measurement, with n outcomes [5, Assumption 2]. Switching to a FinPAC with effects, we can decompose such an instrument map to n 'partial' endomaps $X \to X$, which give a simpler formulation. The details will be elaborated in a subsequent paper.

Recently the author and his colleagues studied *quotient–comprehension chains* [3] which are related to such instrument maps and measurement. It is worth noting that many examples of quotient–comprehension chains are given by FinPACs with effects, including a quantum setting via W^*-algebras. An important future work is thus to give a categorical axiomatisation of such a quotient–comprehension chain in the effectus / 'FinPAC with effects' framework.

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A Omitted proofs in Section 3

We write $[n] = \{1, \dots, n\}$ for the n element set.

Lemma A.1. *In a FinPAC the following hold.*

1. *A family* $(f_i\colon X \to Y)_{i\in[n]}$ *is orthogonal whenever* $(f_i)_{i\in[n]}$ *is compatible in the sense that there exists a 'bound'* $b\colon X \to n\cdot Y$ *such that* $f_i = \rhd_i \circ b$.
2. *If a family* $(f_i\colon X \to Y)_{i\in[n]}$ *is orthogonal, then a family* $(\kappa_i \circ f_i\colon X \to n\cdot Y)_{i\in[n]}$ *is orthogonal too.*

Proof. **1.** We prove the following stronger statement by induction on n.

- If a family $(f_i\colon X \to Y)_{i\in[n]}$ is compatible via a bound $b\colon X \to n\cdot Y$, then it is orthogonal and $\bigovee_{i\in[n]} f_i = \nabla \circ b$.

The base case ($n = 0$) is trivial. To show the induction step, let $(f_i\colon X \to Y)_{i\in[n+1]}$ be a compatible family via a bound $b\colon X \to (n+1)\cdot Y$. Let $\alpha\colon (n+1)\cdot Y \to n\cdot Y + Y$ be the canonical associativity isomorphism. Then it is easy to see that $(f_i)_{i\in[n]}$ is compatible via $\rhd_1 \circ \alpha \circ b\colon X \to n\cdot Y$. By the induction hypothesis $(f_i)_{i\in[n]}$ is orthogonal and $\bigovee_{i\in[n]} f_i = \nabla \circ \rhd_1 \circ \alpha \circ b$. Note that

$$\rhd_1 \circ (\nabla + \mathrm{id}) \circ \alpha \circ b = \nabla \circ \rhd_1 \circ \alpha \circ b = \bigovee_{i\in[n]} f_i$$
$$\rhd_2 \circ (\nabla + \mathrm{id}) \circ \alpha \circ b = \rhd_2 \circ \alpha \circ b = \rhd_{n+1} \circ b = f_{n+1} \ .$$

Therefore $\bigovee_{i\in[n]} f_i \perp f_{n+1}$ via $(\nabla + \mathrm{id}) \circ \alpha \circ b$, so that $(f_i)_{i\in[n+1]}$ is orthogonal. Moreover we have

$$\bigovee_{i\in[n+1]} f_i = \Big(\bigovee_{i\in[n]} f_i\Big) \ovee f_{n+1} = \nabla \circ (\nabla + \mathrm{id}) \circ \alpha \circ b = \nabla \circ b \ .$$

2. We prove it by induction on n. The base case $n = 0$ is trivial. Let $(f_i\colon X \to Y)_{i\in[n+1]}$ be an orthogonal family. Then n arrows $f_1 \ovee f_{n+1}, f_2, \dots, f_n$ are orthogonal. By the induction hypothesis,

$$\kappa_1 \circ (f_1 \ovee f_{n+1}) = \kappa_1 \circ f_1 \ovee \kappa_1 \circ f_{n+1}, \kappa_2 \circ f_2, \dots, \kappa_n \circ f_n \colon X \to n\cdot Y$$

are orthogonal. This implies that $\bigovee_{i\in[n]} \kappa_i \circ f_i$ and $\kappa_1 \circ f_{n+1}$ are orthogonal. By the untying axiom, $\kappa_1 \circ \bigovee_{i\in[n]} \kappa_i \circ f_i = \bigovee_{i\in[n]} \kappa_1 \circ \kappa_i \circ f_i$ and $\kappa_2 \circ \kappa_1 \circ f_{n+1}$ are orthogonal. It follows that

$$\kappa_1 \circ \kappa_1 \circ f_1, \dots, \kappa_1 \circ \kappa_n \circ f_n, \kappa_2 \circ \kappa_1 \circ f_{n+1} \colon X \to n\cdot Y + n\cdot Y$$

are orthogonal. Let $\alpha\colon n\cdot Y + Y \to (n+1)\cdot Y$ be the associativity isomorphism. Then

$$\alpha \circ (\mathrm{id} + \rhd_1) \circ \kappa_1 \circ \kappa_i \circ f_i = \alpha \circ \kappa_1 \circ \kappa_i \circ f_i = \kappa_i \circ f_i$$
$$\alpha \circ (\mathrm{id} + \rhd_1) \circ \kappa_2 \circ \kappa_1 \circ f_{n+1} = \alpha \circ \kappa_2 \circ \rhd_1 \circ \kappa_1 \circ f_{n+1} = \kappa_{n+1} \circ f_{n+1}$$

Therefore $(\kappa_i \circ f_i\colon X \to (n+1)\cdot Y)_{i\in[n+1]}$ is orthogonal. ■

Proof of Theorem 3.4. (Only if) In a FinPAC, the partial projections are jointly monic by Lemma 3.3. To show the pullback condition, let $f, g\colon Y \to X + X$ be arrows with $\nabla \circ f = \rhd_1 \circ g$. Let $f_i = \rhd_i \circ f$ and $g_i = \rhd_i \circ g$ ($i = 1, 2$). Using Lemma 3.2, one has $f_1 \perp f_2$, $g_1 \perp g_2$, and $f_1 \ovee f_2 = \nabla \circ f = \rhd_1 \circ g = g_1$, so that f_1, f_2, g_2 are orthogonal. By (ternary) untying, $\kappa_1 \circ f_1, \kappa_2 \circ f_2, \kappa_3 \circ g_2\colon Y \to X + X + X$ are orthogonal. Writing $\alpha\colon X + X + X \to (X + X) + X$ for the associativity isomorphism, define $h\colon Y \to (X + X) + X$ by

$$\begin{aligned} h &= \alpha \circ (\kappa_1 \circ f_1 \ovee \kappa_2 \circ f_2 \ovee \kappa_3 \circ g_2) \\ &= \kappa_1 \circ \kappa_1 \circ f_1 \ovee \kappa_1 \circ \kappa_2 \circ f_2 \ovee \kappa_2 \circ g_2 \\ &= \kappa_1 \circ f \ovee \kappa_2 \circ g_2 \qquad (\text{note } f = \kappa_1 \circ f_1 \ovee \kappa_2 \circ f_2) \ . \end{aligned}$$

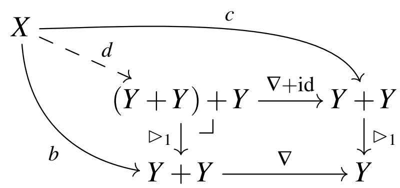
Then $\rhd_1 \circ h = f$ easily, and

$$(\nabla + \mathrm{id}) \circ h = \kappa_1 \circ \nabla \circ f \ovee \kappa_2 \circ g_2 = \kappa_1 \circ g_1 \ovee \kappa_2 \circ g_2 = g \ .$$

Hence h is a desired mediating map. To see the uniqueness, let $k\colon Y \to (X+X)+X$ be an arrow with $\rhd_1 \circ k = f$ and $(\nabla+\mathrm{id}) \circ k = g$. Then $\rhd_2 \circ k = \rhd_2 \circ (\nabla + \mathrm{id}) \circ k = \rhd_2 \circ g$. Such k must be unique because partial projections $\rhd_1$ and $\rhd_2$ are jointly monic.

(If) Assume that a category $\mathbf{C}$ satisfies the given conditions. The joint monicity of partial projections allows us to define the partial sum $\ovee$ on homsets $\mathbf{C}(X,Y)$ in the way of Proposition 3.2. We show that $\mathbf{C}$ is **PCM**-enriched with zero arrows as neutral elements.

Associativity: Let $f,g,h \in \mathbf{C}(X,Y)$ be arrows with $f \perp g$ (i.e. compatible) via $b\colon X \to Y+Y$, and $f \ovee g \perp h$ via $c\colon X \to Y+Y$. By definition we have $\nabla \circ b = f \ovee g = \rhd_1 \circ c$, so that we obtain a mediating map d as in the diagram:



Then, it is straightforward to check that

$$g \perp h \ \text{ via } \ X \xrightarrow{d} (Y+Y)+Y \xrightarrow{\rhd_2+\mathrm{id}} Y+Y \ \ ; \ \text{ and } \quad f \perp g \ovee h \ \text{ via } \ X \xrightarrow{d} (Y+Y)+Y \xrightarrow{[\mathrm{id},\kappa_2]} Y+Y \ \ .$$

Finally we have

$$f \ovee (g \ovee h) = \nabla \circ [\mathrm{id}, \kappa_2] \circ d = [\nabla, \mathrm{id}] \circ d = \nabla \circ (\nabla + \mathrm{id}) \circ d = \nabla \circ c = (f \ovee g) \ovee h \ .$$

Commutativity: Let $f,g \in \mathbf{C}(X,Y)$ be arrows with $f \perp g$ via $b\colon X \to Y+Y$. Then it is easy to see that $g \perp f$ via $[\kappa_2,\kappa_1] \circ b\colon X \to Y+Y$, and that $g \ovee f = \nabla \circ [\kappa_2, \kappa_1] \circ b = \nabla \circ b = f \ovee g$.

Zero: For $f \in \mathbf{C}(X,Y)$, we have $0_{XY} \perp f$ via $\kappa_2 \circ f\colon X \to Y+Y$, and $0_{XY} \ovee f = \nabla \circ \kappa_2 \circ f = f$.

Therefore $\mathbf{C}(X,Y)$ is a PCM for each $X,Y \in \mathbf{C}$. We need to show that the composition $\circ\colon \mathbf{C}(Y,Z) \times \mathbf{C}(X,Y) \to \mathbf{C}(X,Z)$ is a PCM-bihomomorphism. Let $f \in \mathbf{C}(X,Y)$ and $h,k \in \mathbf{C}(Y,Z)$ be arrows with $h \perp k$ via $b\colon Y \to Z+Z$. Then $h \circ f \perp k \circ f$ via $b \circ f\colon X \to Z+Z$, and $h \circ f \ovee k \circ f = \nabla \circ b \circ f = (h \ovee k) \circ f$. We also have $0 \circ f = 0$. Hence $(-) \circ f$ is a PCM-homomorphism. Next, let $h \in \mathbf{C}(Y,Z)$ and $f,g \in \mathbf{C}(X,Y)$ be arrows with $f \perp g$ via $b\colon X \to Y+Y$. Then $h \circ f \perp h \circ g$ via $(h+h) \circ b\colon X \to Z+Z$, and $h \circ f \ovee h \circ f = \nabla \circ (h+h) \circ b = h \circ \nabla \circ b = h \circ (f \ovee g)$. We also have $h \circ 0 = 0$, and hence $h \circ (-)$ is a PCM-homomorphism.

We have shown that $\mathbf{C}$ is **PCM**-enriched. The compatibility sum axiom holds by definition. To show the untying axiom, let $f,g\colon X \to Y$ be arrows with $f \perp g$ via $b\colon X \to Y+Y$. Then $\kappa_1 \circ f, \kappa_2 \circ g\colon X \to Y+Y$ are compatible via $(\kappa_1 + \kappa_2) \circ b\colon X \to (Y+Y)+(Y+Y)$. ■

B Proof of a 2-equivalence of the 2-categories of effectuses and FinPACs with effects

First note the condition when a natural transformation is monoidal w.r.t. $(+,0)$.

Remark B.1. A natural transformation $\alpha\colon F \to G$ is monoidal w.r.t. $(+,0)$ if the following diagrams commute.

$$\begin{array}{ccc} FX+FY & \xrightarrow{\alpha_X+\alpha_Y} & GX+GY \\ {\scriptstyle [F\kappa_1,F\kappa_2]}\downarrow & & \downarrow{\scriptstyle [G\kappa_1,G\kappa_2]} \\ F(X+Y) & \xrightarrow{\alpha_{X+Y}} & G(X+Y) \end{array} \qquad \begin{array}{ccc} 0 & \searrow^{\text{!`}} & \\ {\scriptstyle \text{!`}}\downarrow & & \\ F0 & \xrightarrow{\alpha_0} & G0 \end{array}$$

Obviously, the right diagram commutes automatically. The left diagram commutes if and only if $\alpha_{X+Y} \circ F\kappa_1 = G\kappa_1 \circ \alpha_X$ and $\alpha_{X+Y} \circ F\kappa_2 = G\kappa_2 \circ \alpha_Y$.

Let $F\colon \mathbf{A} \to \mathbf{B}$ be a functor between effectuses in $\mathbf{Eff}$, i.e. a functor that preserves 1 and $(0,+)$. Then, the canonical arrow $FX+1 \to F(X+1)$ in the diagram below is an isomorphism.

$$\begin{array}{ccccc} FX & \xrightarrow{\kappa_1} & FX+1 & \xleftarrow{\kappa_2} & 1 \\ & \searrow_{F\kappa_1} & \downarrow & & \uparrow{\scriptstyle \cong} \\ & & F(X+1) & \xleftarrow{F\kappa_2} & F1 \end{array}$$

We denote its inverse by $l_{F,X}\colon F(X+1) \to FX+1$ or simply by l_X. Note the following equations.

$$l_{F,X} \circ F\kappa_1 = \kappa_1 \tag{2}$$

$$l_{F,X} \circ F\kappa_2 = \kappa_2 \circ !_{F1} \tag{3}$$

Lemma B.2. *Let* $F\colon \mathbf{A} \to \mathbf{B}$ *be a functor between effectuses in* $\mathbf{Eff}$*. Then we have a functor* $F_{+1}\colon \mathbf{A}_{+1} \to \mathbf{B}_{+1}$ *which is a "lifting" of* F *in the sense that the following diagram commutes.*

$$\begin{array}{ccc} \mathbf{A}_{+1} & \xrightarrow{F_{+1}} & \mathbf{B}_{+1} \\ {\scriptstyle \widehat{(-)}}\uparrow & & \uparrow{\scriptstyle \widehat{(-)}} \\ \mathbf{A} & \xrightarrow{F} & \mathbf{B} \end{array}$$

Proof. We define F_{+1} by $F_{+1}X = FX$ and $F_{+1}(f) = l_{F,Y} \circ Ff\colon FX \to FY+1$ for $f\colon X \rightharpoonup Y$ in $\mathbf{A}_{+1}$, i.e. $f\colon X \to Y+1$ in $\mathbf{A}$. For $h\colon X \to Y$ in $\mathbf{A}$, using (2),

$$F_{+1}\widehat{h} = l_Y \circ F\kappa_1 \circ Fh = \kappa_1 \circ Fh = \widehat{Fh} \ .$$

Therefore F_{+1} is a lifting of F. Taking $h = \mathrm{id}_X$ we obtain $F_{+1}(\widehat{\mathrm{id}}_X) = \widehat{\mathrm{id}}_{FX}$. Let $f\colon X \rightharpoonup Y$ and $g\colon Y \rightharpoonup Z$ be arrows in $\mathbf{A}_{+1}$. Note that we have

$$\begin{aligned} l_Z \circ F[g,\kappa_2] \circ F\kappa_1 &= l_Z \circ Fg \\ &= [l_Z \circ Fg, \kappa_2] \circ \kappa_1 \\ &= [l_Z \circ Fg, \kappa_2] \circ l_Y \circ F\kappa_1 && \text{by (2)} \\ l_Z \circ F[g,\kappa_2] \circ F\kappa_2 &= l_Z \circ F\kappa_2 \\ &= \kappa_2 \circ !_{F1} && \text{by (3)} \\ &= [l_Z \circ Fg, \kappa_2] \circ \kappa_2 \circ !_{F1} \\ &= [l_Z \circ Fg, \kappa_2] \circ l_Y \circ F\kappa_2 && \text{by (3)} \ . \end{aligned}$$

Because F preserves finite coproducts and hence $FY \xrightarrow{F\kappa_1} F(Y+1) \xleftarrow{F\kappa_2} F1$ is a coproduct in $\mathbf{B}$, we obtain $l_Z \circ F[g,\kappa_2] = [l_Z \circ Fg, \kappa_2] \circ l_Y$. Then,

$$F_{+1}(g \mathbin{\hat{\circ}} f) = l_Z \circ F[g,\kappa_2] \circ Ff = [l_Z \circ Fg, \kappa_2] \circ l_Y \circ Ff = F_{+1}g \mathbin{\hat{\circ}} F_{+1}f \ .$$

Therefore F_{+1} is a functor. ∎

Lemma B.3. *The mapping* $\mathbf{B} \mapsto (\mathbf{B}_{+1}, 1)$ *for an effectus* $\mathbf{B}$ *gives rise to a 2-functor* $(-)_{+1} \colon \mathbf{Eff} \to \mathbf{FPE}$.

Proof. Recall $(\mathbf{B}_{+1}, 1)$ is a FinPAC with effects by Theorem 4.5. For an arrow $F \colon \mathbf{A} \to \mathbf{B}$ in $\mathbf{Eff}$, we have a functor $F_{+1} \colon \mathbf{A}_{+1} \to \mathbf{B}_{+1}$ by Lemma B.2. Since F_{+1} is a lifting of F, the functor F_{+1} preserves finite coproducts as F does. The arrow $1_{F1} = \widehat{!}_{F1} \colon F1 \rightharpoonup 1$ is an isomorphism because $!_{F1} \colon F1 \to 1$ is an isomorphism. Since $F_{+1} 1_X = F_{+1}\widehat{!}_X = \widehat{F!_X}$ is total, we have $1_{F1} \mathbin{\hat{\circ}} F_{+1} 1_X = 1_{FX}$. Therefore F_{+1} is an arrow in $\mathbf{FPE}$.

Let $F, G \colon \mathbf{A} \to \mathbf{B}$ be arrows and $\alpha \colon F \Rightarrow G$ a 2-cell in $\mathbf{Eff}$. Note the equation

$$(\alpha_X + \mathrm{id}_1) \circ l_{F,X} = l_{G,X} \circ \alpha_{X+1} \quad , \tag{4}$$

which holds because

$$\begin{aligned}
(\alpha_X + \mathrm{id}) \circ l_{F,X} \circ F\kappa_1 &= (\alpha_X + \mathrm{id}) \circ \kappa_1 && \text{by (2)} \\
&= \kappa_1 \circ \alpha_X \\
&= l_{G,X} \circ G\kappa_1 \circ \alpha_X && \text{by (2)} \\
&= l_{G,X} \circ \alpha_{X+1} \circ F\kappa_1 && \text{since } \alpha \text{ is } (+,0)\text{-monoidal} \\
(\alpha_X + \mathrm{id}) \circ l_{F,X} \circ F\kappa_2 &= (\alpha_X + \mathrm{id}) \circ \kappa_2 \circ !_{F1} && \text{by (3)} \\
&= \kappa_2 \circ !_{F1} \\
&= \kappa_2 \circ !_{G1} \circ \alpha_1 \\
&= l_{G,X} \circ G\kappa_2 \circ \alpha_1 && \text{by (3)} \\
&= l_{G,X} \circ \alpha_{X+1} \circ F\kappa_2 && \text{since } \alpha \text{ is } (+,0)\text{-monoidal} \quad .
\end{aligned}$$

We define $\alpha_{+1} \colon F_{+1} \Rightarrow G_{+1}$ by $(\alpha_{+1})_X = \widehat{\alpha}_X \colon FX \rightharpoonup GX$. It is natural: for $f \colon X \rightharpoonup Y$ in $\mathbf{A}_{+1}$,

$$\begin{aligned}
(\alpha_{+1})_Y \mathbin{\hat{\circ}} F_{+1} f &= \widehat{\alpha}_Y \mathbin{\hat{\circ}} (l_{F,Y} \circ Ff) \\
&= (\alpha_Y + \mathrm{id}_1) \circ l_{F,Y} \circ Ff \\
&= l_{G,Y} \circ \alpha_{X+1} \circ Ff && \text{by (4)} \\
&= l_{G,Y} \circ Gf \circ \alpha_X && \text{by naturality of } \alpha \\
&= (l_{G,Y} \circ Gf) \mathbin{\hat{\circ}} \widehat{\alpha}_X \\
&= G_{+1} f \mathbin{\hat{\circ}} (\alpha_{+1})_X \quad .
\end{aligned}$$

It is monoidal with respect to $(+,0)$:

$$\begin{aligned}
(\alpha_{+1})_{X+Y} \mathbin{\hat{\circ}} F_{+1}\widehat{\kappa}_1 &= \widehat{\alpha}_{X+Y} \mathbin{\hat{\circ}} \widehat{F\kappa_1} \\
&= (\alpha_{X+Y} \circ F\kappa_1)\widehat{\ } \\
&= (G\kappa_1 \circ \alpha_X)\widehat{\ } && \text{since } \alpha \text{ is } (+,0)\text{-monoidal} \\
&= \widehat{G\kappa_1} \mathbin{\hat{\circ}} \widehat{\alpha}_X \\
&= G_{+1}\widehat{\kappa}_1 \mathbin{\hat{\circ}} (\alpha_{+1})_X
\end{aligned}$$

and similarly we have $(\alpha_{+1})_{X+Y} \mathbin{\hat{\circ}} F_{+1}\widehat{\kappa}_2 = G_{+1}\widehat{\kappa}_2 \mathbin{\hat{\circ}} (\alpha_{+1})_Y$. The arrow $(\alpha_{+1})_1 = \widehat{\alpha}_1 \colon F1 \rightharpoonup G1$ is total, hence $1_{G1} \mathbin{\hat{\circ}} (\alpha_{+1})_1 = 1_{F1}$. Therefore α_{+1} is a 2-cell in $\mathbf{FPE}$.

We then check that $(-)_{+1} \colon \mathbf{Eff}(\mathbf{A}, \mathbf{B}) \to \mathbf{FPE}(\mathbf{A}_{+1}, \mathbf{B}_{+1})$ is a (1-)functor. For the identity $\mathrm{id}_F \colon F \Rightarrow F$ we have $((\mathrm{id}_F)_{+1})_X = \widehat{\mathrm{id}}_X \colon FX \rightharpoonup FX$, so that $(\mathrm{id}_F)_{+1} = \mathrm{id}_{F_{+1}} \colon F_{+1} \Rightarrow F_{+1}$. Let $\alpha \colon F \Rightarrow G$ and $\beta \colon G \Rightarrow H$ be 2-cells in $\mathbf{Eff}$. Then

$$((\beta \circ \alpha)_{+1})_X = (\beta_X \circ \alpha_X)\widehat{\ } = \widehat{\beta}_X \mathbin{\hat{\circ}} \widehat{\alpha}_X = (\beta_{+1})_X \mathbin{\hat{\circ}} (\alpha_{+1})_X = (\beta_{+1} \circ \alpha_{+1})_X \quad .$$

Therefore $(\beta \circ \alpha)_{+1} = \beta_{+1} \circ \alpha_{+1}$.

Now we show that $(-)_{+1}$ is a 2-functor. For the identity functor $\mathrm{id}_{\mathbf{B}} \colon \mathbf{B} \to \mathbf{B}$, it is easy to see the canonical isomorphism $l_{\mathrm{id}_{\mathbf{B}},X} \colon X+1 \to X+1$ is the identity, so that $(\mathrm{id}_{\mathbf{B}})_{+1} = \mathrm{id}_{\mathbf{B}_{+1}}$. Let $F \colon \mathbf{A} \to \mathbf{B}$ and $G \colon \mathbf{B} \to \mathbf{C}$ be arrows in **Eff**. Note the equation

$$l_{GF,X} = l_{G,FX} \circ G l_{F,X} \quad , \tag{5}$$

which holds because

$$\begin{aligned} l_{G,FX} \circ G l_{F,X} \circ GF\kappa_1 &= l_{G,FX} \circ G\kappa_1 = \kappa_1 = l_{GF,X} \circ GF\kappa_1 \\ l_{G,FX} \circ G l_{F,X} \circ GF\kappa_2 &= l_{G,FX} \circ G\kappa_2 \circ G!_{F1} = \kappa_2 \circ !_{G1} \circ G!_{F1} = \kappa_2 \circ !_{GF1} = l_{GF,X} \circ GF\kappa_2 \quad , \end{aligned}$$

using (2) and (3). For $f \colon X \rightharpoonup Y$ in $\mathbf{A}_{+1}$, using (5),

$$(GF)_{+1} f = l_{GF,Y} \circ GFf = l_{G,FY} \circ G l_{F,Y} \circ GFf = G_{+1}(l_{F,Y} \circ Ff) = G_{+1}F_{+1}f \quad .$$

Hence $(GF)_{+1} = G_{+1}F_{+1}$. For $\alpha \colon F \Rightarrow F'$, one has

$$((G\alpha)_{+1})_X = \widehat{G\alpha_X} = G_{+1}\widehat{\alpha}_X = (G_{+1}\alpha_{+1})_X \quad ,$$

so that $(G\alpha)_{+1} = G_{+1}\alpha_{+1}$. For $\alpha \colon G \Rightarrow G'$, one has

$$((\beta F)_{+1})_X = \widehat{\beta}_{FX} = (\beta_{+1})_{F_{+1}X} = (\beta_{+1}F_{+1})_X \quad ,$$

and therefore $(\beta F)_{+1} = \beta_{+1}F_{+1}$. ■

Lemma B.4. *The mapping* $(\mathbf{C}, I) \mapsto \mathbf{C}_{\mathrm{t}}$ *for a FinPAC with effects* $(\mathbf{C}, I)$ *gives rise to a 2-functor* $(-)_{\mathrm{t}} \colon \mathbf{FPE} \to \mathbf{Eff}$.

Proof. Recall that $\mathbf{C}_{\mathrm{t}}$ is an effectus by Theorem 4.10. Let $F \colon (\mathbf{C}, I_{\mathbf{C}}) \to (\mathbf{D}, I_{\mathbf{D}})$ be an arrow in **FPE**. If $f \colon X \to Y$ is a total arrow in $\mathbf{C}$, then

$$\mathrm{Dp}(Ff) = 1_{FY} \circ Ff = 1_{FI} \circ F1_Y \circ Ff = 1_{FI} \circ F\,\mathrm{Dp}(f) = 1_{FI} \circ F1_X = 1_{FX} \quad ,$$

that is, Ff is total. Therefore F restricts to the functor $F_{\mathrm{t}} \colon \mathbf{C}_{\mathrm{t}} \to \mathbf{D}_{\mathrm{t}}$ in a commutative diagram:

$$\begin{array}{ccc} \mathbf{C}_{\mathrm{t}} & \xrightarrow{F_{\mathrm{t}}} & \mathbf{D}_{\mathrm{t}} \\ \downarrow & & \downarrow \\ \mathbf{C} & \xrightarrow{F} & \mathbf{D} \end{array}$$

Because $\mathbf{C}_{\mathrm{t}}$ inherits coproducts from $\mathbf{C}$, the functor F_{t} preserves finite coproducts as F does. Recall that $I_{\mathbf{C}}$ and $I_{\mathbf{D}}$ are the final objects in $\mathbf{C}_{\mathrm{t}}$ and $\mathbf{D}_{\mathrm{t}}$ respectively. By definition we have $FI_{\mathbf{C}} \cong I_{\mathbf{D}}$ in $\mathbf{D}$ and hence in $\mathbf{D}_{\mathrm{t}}$, so that F_{t} preserves the final object. Therefore F_{t} is an arrow in **Eff**.

Let $F, G \colon \mathbf{C} \to \mathbf{D}$ be arrows and $\alpha \colon F \Rightarrow G$ a 2-cell in **FPE**. Then

$$\begin{aligned} \mathrm{Dp}(\alpha_X) = 1_{GX} \circ \alpha_X &= 1_{GI_{\mathbf{C}}} \circ G1_X \circ \alpha_X && \text{since } G \text{ is an arrow in } \mathbf{FPE} \\ &= 1_{GI_{\mathbf{C}}} \circ \alpha_{I_{\mathbf{C}}} \circ F1_X && \text{by naturality of } \alpha \\ &= 1_{FI_{\mathbf{C}}} \circ F1_X && \text{since } \alpha \text{ is a 2-cell in } \mathbf{FPE} \\ &= 1_{FX} && \text{since } F \text{ is an arrow in } \mathbf{FPE} \quad , \end{aligned}$$

so that α_X is total. Hence we can restrict α to the natural transformation $\alpha_{\mathrm{t}} : F_{\mathrm{t}} \Rightarrow G_{\mathrm{t}}$ with $(\alpha_{\mathrm{t}})_X = \alpha_X$, which is obviously a 2-cell in **Eff**. Then it is easy to see that $(-)_{\mathrm{t}}$ gives a (1-)functor $\mathbf{FPE}(\mathbf{C},\mathbf{D}) \to \mathbf{Eff}(\mathbf{C}_{\mathrm{t}},\mathbf{D}_{\mathrm{t}})$.

Finally, it is also easy to check $(\mathrm{id}_{\mathbf{C}})_{\mathrm{t}} = \mathrm{id}_{\mathbf{C}_{\mathrm{t}}}$, $(GF)_{\mathrm{t}} = G_{\mathrm{t}}F_{\mathrm{t}}$, $(G\alpha)_{\mathrm{t}} = G_{\mathrm{t}}\alpha_{\mathrm{t}}$, $(\beta F)_{\mathrm{t}} = \beta_{\mathrm{t}}F_{\mathrm{t}}$ for arrows $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{E}$, and 2-cells $\alpha : F \Rightarrow F'$ and $\beta : G \Rightarrow G'$ in **FPE**. Therefore $(-)_{\mathrm{t}}$ gives a 2-functor $\mathbf{FPE} \to \mathbf{Eff}$. ■

Theorem B.5. *The 2-functors* $(-)_{+1} : \mathbf{Eff} \to \mathbf{FPE}$ *and* $(-)_{\mathrm{t}} : \mathbf{FPE} \to \mathbf{Eff}$ *form a 2-equivalence of 2-categories* $\mathbf{Eff} \simeq \mathbf{FPE}$*. Namely, there are 2-natural isomorphisms* $\mathrm{id}_{\mathbf{Eff}} \cong ((-)_{+1})_{\mathrm{t}}$ *and* $\mathrm{id}_{\mathbf{FPE}} \cong ((-)_{\mathrm{t}})_{+1}$.

Proof. We write $\Phi_{\mathbf{B}} : \mathbf{B} \to (\mathbf{B}_{+1})_{\mathrm{t}}$ for the isomorphism of categories in Proposition 4.6, which is given by $\Phi_{\mathbf{B}}X = X$ and $\Phi_{\mathbf{B}}f = \widehat{f}$. It preserves finite coproducts and the final object, so that $\Phi_{\mathbf{B}}$ is an arrow in **Eff**. Let $F : \mathbf{A} \to \mathbf{B}$ be an arrow in **Eff**. Because F_{+1} is a lifting of F, and $(F_{+1})_{\mathrm{t}}$ is a restriction of F_{+1}, the following diagram commutes.

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{\quad F \quad} & \mathbf{B} \\ {\scriptstyle \Phi_{\mathbf{A}}}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \Phi_{\mathbf{B}}} \\ (\mathbf{A}_{+1})_{\mathrm{t}} & \xrightarrow{(F_{+1})_{\mathrm{t}}} & (\mathbf{B}_{+1})_{\mathrm{t}} \end{array}$$

Let $\alpha : F \Rightarrow G$ be a 2-cell in **Eff**. Then

$$((\alpha_{+1})_{\mathrm{t}}\Phi_{\mathbf{A}})_X = ((\alpha_{+1})_{\mathrm{t}})_{\Phi_{\mathbf{A}}X} = (\alpha_{+1})_X = \widehat{\alpha}_X = \Phi_{\mathbf{B}}\alpha_X = (\Phi_{\mathbf{B}}\alpha)_X \quad ,$$

so that $(\alpha_{+1})_{\mathrm{t}}\Phi_{\mathbf{A}} = \Phi_{\mathbf{B}}\alpha$. Therefore Φ defines a 2-natural isomorphism $\mathrm{id}_{\mathbf{Eff}} \Rightarrow ((-)_{+1})_{\mathrm{t}}$.

Next, we write $\Psi_{\mathbf{C}} : (\mathbf{C}_{\mathrm{t}})_{+1} \to \mathbf{C}$ for the isomorphism of categories in Lemma 4.9, which is defined by $\Psi_{\mathbf{C}}X = X$ and $\Psi_{\mathbf{C}}f = \rhd_1 \circ f$. It preserves finite coproducts and the unit object I, since I is the final object of $\mathbf{C}_{\mathrm{t}}$. If we write 1_X for the top of $\mathbf{C}(X,I)$, then the top of $(\mathbf{C}_{\mathrm{t}})_{+1}(X,I)$ is $\widehat{1}_X = \kappa_1 \circ 1_X$, and therefore we have $\Psi_{\mathbf{C}}\widehat{1}_X = \rhd_1 \circ \kappa_1 \circ 1_X = 1_X$. Hence $\Psi_{\mathbf{C}}$ is an arrow in **FPE**. Let $F : \mathbf{C} \to \mathbf{D}$ be an arrow in **FPE**. Note that the following diagram commutes,

$$\begin{array}{ccc} F(X + I_{\mathbf{C}}) & \xrightarrow{l_{F_{\mathrm{t}},X}} & FX + I_{\mathbf{D}} \\ & {\scriptstyle F\rhd_1} \searrow & \downarrow{\scriptstyle \rhd_1} \\ & & FX \end{array}$$

since

$$\begin{aligned} \rhd_1 \circ l_{F_{\mathrm{t}},X} \circ F\kappa_1 &= \rhd_1 \circ \kappa_1 = \mathrm{id}_{FX} = F\mathrm{id}_X = F\rhd_1 \circ F\kappa_1 \\ \rhd_1 \circ l_{F_{\mathrm{t}},X} \circ F\kappa_2 &= \rhd_1 \circ \kappa_2 \circ 1_{FI_{\mathbf{C}}} = 0_{FI_{\mathbf{C}},FX} = F0_{I_{\mathbf{C}},X} = F\rhd_1 \circ F\kappa_2 \quad , \end{aligned}$$

where $F0_{I_{\mathbf{C}},X} = 0_{FI_{\mathbf{C}},FX}$ holds because F preserves the zero object. For $f : X \rightharpoonup Y$ in $(\mathbf{C}_{\mathrm{t}})_{+1}$, we have

$$\Psi_{\mathbf{D}}(F_{\mathrm{t}})_{+1}f = \rhd_1 \circ l_{F_{\mathrm{t}},Y} \circ Ff = F\rhd_1 \circ Ff = F(\rhd_1 \circ f) = F\Psi_{\mathbf{C}}f \quad ,$$

and hence $\Psi_{\mathbf{D}}(F_{\mathrm{t}})_{+1} = F\Psi_{\mathbf{C}}$. Let $\alpha : F \Rightarrow G$ be a 2-cell in **FPE**. Then

$$(\Psi_{\mathbf{D}}(\alpha_{\mathrm{t}})_{+1})_X = \Psi_{\mathbf{D}}((\alpha_{\mathrm{t}})_{+1})_X = \rhd_1 \circ \widehat{\alpha}_X = \alpha_X = \alpha_{\Psi_{\mathbf{C}}X} = (\alpha\Psi_{\mathbf{C}})_X \quad ,$$

so that $\Psi_{\mathbf{D}}(\alpha_{\mathrm{t}})_{+1} = \alpha\Psi_{\mathbf{C}}$. Therefore Ψ defines a 2-natural isomorphism $((-)_{\mathrm{t}})_{+1} \Rightarrow \mathrm{id}_{\mathbf{FPE}}$. ■

C Convex sets over a division effect monoid

Throughout this section, we let M be a division effect monoid (see Definition 6.3).

Lemma C.1. *For $r,s,t,u \in M$ with $r \le s$, $s \cdot t \le u$, $s \ne 0$ and $u \ne 0$, one has $(r/s)\cdot(st/u) = rt/u$. In particular $(r/s)\cdot(s/u) = r/u$, by setting $t = 1$.*

Proof. Since $(r/s)\cdot(st/u)\cdot u = (r/s)\cdot s \cdot t = r \cdot t$. ∎

Lemma C.2. *For each nonzero $t \in M$, the 'multiplication by t' map $(-)\cdot t \colon M \to {\downarrow}(t)$ is an effect module (over M) isomorphism, with the inverse $(-)/t \colon {\downarrow}(t) \to M$. In particular, $(-)/t$ is an effect module homomorphism: $0/t = 0$; $t/t = 1$; $(r \ovee s)/t = r/t \ovee s/t$; and $(rs/t) = r(s/t)$.*

Proof. The definition of division says that the map $(-)\cdot t \colon M \to {\downarrow}(t)$ is bijective. It is easy to see that $(-)\cdot t$ is an effect module homomorphism. Therefore, to prove it is an isomorphism, it suffices to show that it reflects the orthogonality: if $r \cdot t \perp s \cdot t$ and $r\cdot t \ovee s \cdot t \le t$, then $r \perp s$. Since the case $r = 0$ is trivial, we assume $r \ne 0$. Then $r \cdot t$ is nonzero too because $(-)\cdot t \colon M \to {\downarrow}(t)$ is bijective. Note that $s^\perp \cdot t = t \ominus s \cdot t \ge r \cdot t$ and hence $s^\perp \cdot t$ is nonzero as well. Then

$$r = (r\cdot t)/t = ((r\cdot t)/(s^\perp \cdot t))\cdot((s^\perp \cdot t)/t) = ((r\cdot t)/(s^\perp\cdot t))\cdot s^\perp \le s^\perp \ ,$$

so that $r \perp s$. ∎

The division allows us to construct coproducts in the category $\mathbf{Conv}_M$ explicitly, in the same way as the case $M = [0,1]$ done in [7]. First we construct a coproduct of the form $X+1$. For a convex set X over M, we define a "lifted" convex set $X_\bullet$ as follows.

$$X_\bullet = \{(x,r) \in (X \cup \{\bullet\}) \times M \mid x = \bullet \iff r = 0\} \ \ (= X \times (M \setminus \{0\}) \cup \{(\bullet,0)\})$$

For a formal convex sum $\sum_i |(x_i,r_i)\rangle s_i \in \mathcal{D}_M(X_\bullet)$, define the actual sum by

$$\textstyle\bigovee_i (x_i,r_i)s_i = \big(\bigovee_i x_i(r_is_i/t), t\big) \qquad \text{where: } t = \bigovee_i r_is_i \ . \tag{6}$$

Note that $\bigovee_i (r_is_i/t) = (\bigovee_i r_is_i)/t = t/t = 1$. The formula (6) is not completely rigorous in the case $t = 0$ or $x_i = \bullet$, but the meaning will be clear. For example, we often mean $(\bullet, 0)$ by $(e,0)$ even when e is an expression that does not make sense. Then, the diagram

$$X \xrightarrow{\kappa_1} X_\bullet \xleftarrow{\kappa_2} 1 \qquad \text{where: } \kappa_1(x) = (x,1)\ ;\ \kappa_2(\bullet) = (\bullet,0)$$

is a coproduct in $\mathbf{Conv}_M$, i.e. $X_\bullet$ is a coproduct $X+1$. For $f \colon X \to Y$ and $g \colon 1 \to Y$, define $[f,g] \colon X_\bullet \to Y$ by $[f,g](x,r) = f(x)r \ovee g(1)r^\perp$.

Let X and Y be convex sets over M. Define a (convex) subset $X + Y \subseteq X_\bullet \times Y_\bullet$ by: $((x,r),(y,s)) \in X+Y \iff r \perp s$ and $r \ovee s = 1$. Then the diagram

$$X \xrightarrow{\kappa_1} X+Y \xleftarrow{\kappa_2} Y \qquad \text{where: } \begin{array}{l} \kappa_1(x) = ((x,1),(\bullet,0)) \\ \kappa_2(y) = ((\bullet,0),(y,1)) \end{array}$$

is a coproduct in $\mathbf{Conv}_M$. For $f \colon X \to Z$ and $g \colon Y \to Z$, define $[f,g] \colon X+Y \to Z$ by $[f,g]((x,r),(y,s)) = f(x)r \ovee g(y)s$.

Finally, we show that $\mathbf{Conv}_M$ is an effectus. Note the difference of the joint monicity requirement between Definition 2.2 and [7, Definition 12].

Proposition C.3. *The category* $\mathbf{Conv}_M$ *of convex sets over a division effect monoid M is an effectus.*

Proof. The category $\mathbf{Conv}_M$ has binary coproducts as we described above. It also has the empty convex set $0 = \varnothing$ as an initial object (unless M is trivial, i.e. a singleton $\{0\}$; note that if M is trivial, then $\mathbf{Conv}_M$ is a trivial category), and the singleton convex set 1 as a final object. The pullback requirements are shown in the same way as the case $M = [0,1]$, see [7, Proposition 15].

We show the joint monicity requirement. Expanding the definition, we see the maps $[\rhd_i, \kappa_2]\colon (X+X)_\bullet \to X_\bullet$ are given by $[\rhd_1, \kappa_2](((x,r),(y,s)),t) = (x,rt)$ and $[\rhd_2, \kappa_2](((x,r),(y,s)),t) = (y,st)$. For $(z,t),(z',t') \in (X+X)_\bullet$, assume

$$[\rhd_1, \kappa_2](z,t) = [\rhd_1, \kappa_2](z',t') \;\; ; \quad [\rhd_2, \kappa_2](z,t) = [\rhd_2, \kappa_2](z',t') \;\; .$$

It is not hard to see that $(z,t) = (\bullet,0)$ if and only if $(z',t') = (\bullet,0)$, using the fact that a division effect monoid has no nontrivial zero divisors. Assume that $t \neq 0$, $t' \neq 0$, and let $z = ((x,r),(y,s))$ and $z' = ((x',r'),(y',s'))$. Then we have $(x,rt) = (x',r't')$ and $(y,st) = (y',s't')$. Then

$$t = 1 \cdot t = (r \ovee s)t = rt \ovee st = r't' \ovee s't' = (r' \ovee s')t' = 1 \cdot t' = t'$$

Hence $rt = r't' = r't$ implies $r = r'$ by the divisibility, and similarly we obtain $s = s'$. Because t is nonzero, $rt = 0$ if and only if $r = 0$. It follows that $(x,r) = (x',r')$. Similarly $(y,s) = (y',s')$. Therefore $(z,t) = (z',t')$. ■