Some Nearly Quantum Theories
(extended abstract)

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We consider possible non-signaling composites of probabilistic models based on euclidean Jordan algebras. Subject to some reasonable constraints, we show that no such composite exists having the exceptional Jordan algebra as a direct summand. We then construct several dagger compact categories of such Jordan-algebraic models. One of these neatly unifies real, complex and quaternionic mixed-state quantum mechanics, with the possible exception of the quaternionic "bit". Another is similar, except in that (i) it excludes the quaternionic bit, and (ii) the composite of two complex quantum systems comes with an extra classical bit. A no-go theorem forecloses any possibility of such a category including higher-dimensional spin factors.

1 Introduction

A series of recent papers [15, 11, 16, 14, 5] have shown that any of various packages of probabilistic or information-theoretic axioms force the state spaces of a finite-dimensional probabilistic theory to be those of formally real, or euclidean, Jordan algebras. Thus, euclidean Jordan algebras (hereafter, EJAs) form a natural class of probabilistic models. Moreover, it is one that keeps us in the general neighborhood of standard quantum mechanics, owing to the classification of simple EJAs as self-adjoint parts of real, complex and quaternionic matrix algebras (corresponding to real, complex and quaternionic quantum systems), the exceptional Jordan algebra of self-adjoint 3×3 matrices over the octonions, and one further class, the so-called *spin factors*. The latter are essentially "bits": their state-spaces are balls of arbitrary dimension, with antipodal points representing sharply distinguishable states.[1]

This raises the question of whether one can construct probabilistic *theories* (as opposed to a collection of models of individual systems) in which finite-dimensional complex quantum systems can be accommodated together with several — perhaps all — of the other basic types of EJAs listed above. Ideally, these would be symmetric monoidal categories; even better, we might hope to obtain compact closed, or still better, dagger-compact, categories of EJAs [1]. Also, one would like the resulting theory to embrace mixed states and CP mappings.

In this paper, we exhibit two dagger-compact categories of EJAs — called **URUE** and **URSE**, acronyms that will be explained below — that include all real, complex and quaternionic matrix algebras, with one conspicuous (and interesting) exception: the quaternionic bit, or "quabit", represented by $M_2(\mathbb{H})_{\text{sa}}$, the Jordan algebra of self-adjoint 2×2 quaternionic matrices. We are able to show that this cannot be added to **URUE** without destroying compact closure; whether **URSE** can be extended to include it remains open

[1] Where this ball has dimension 2, 3 or 5, these are just the state spaces of real, complex and quaternionic quantum bits.

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at present. **URSE** includes a faithful copy of finite-dimensional complex quantum mechanics, while in **URUE**, composites of complex quantum systems come with an extra classical bit — that is, a $\{0,1\}$ valued superselection rule.

We also show that there is scant hope of including more exotic Jordan algebras in a satisfactory categorical scheme. Even allowing for a very liberal definition of composite (our Definition 1 below), the exceptional Jordan algebra is ruled out altogether (Corollary 1), while non-quantum spin factors are ruled out if we want to regard states as morphisms — in particular, if we demand compact closure (see Example 1). Combined with the results of (any of) the papers cited above that derive a euclidean Jordan-algebraic structure from information-theoretic assumptions, these results provide a compelling motivation for a kind of unified quantum theory that accommodates real, complex and quaternionic quantum systems (possibly modulo the quabit) and permits the formation of composites of these.

A condition frequently invoked to rule out real and quaternionic QM is *local tomography*: the doctrine that the state of a composite of two systems should depend only on the joint probabilities it assigns to measurement outcomes on the component systems. Indeed, it can be shown [7] that standard complex QM with superselection rules is the only dagger-compact category of EJAs that includes the qubit. Accordingly, **URUE** and **URSE** are *not* locally tomographic. In our view, the very existence of these quite reasonable, well-behaved categories suggests that local tomography is not as well-motivated as is sometimes supposed.

Remark: A broadly similar proposal is advanced by Baez [3], who points out that one can view real and quaternionic quantum systems as pairs $(\mathbf{H},J)$, where $\mathbf{H}$ is a complex Hilbert space and J is an anti-unitary satisfying $J^2=\mathbf{1}$ (the real case) or $J^2=-\mathbf{1}$ (the quaternionic case). This yields a symmetric monoidal category in which objects are such pairs, morphisms $(\mathbf{H}_1,J_1)\to(\mathbf{H}_2,J_2)$ are linear mappings intertwining J_1 and J_2, and $(\mathbf{H}_1,J_1)\otimes(\mathbf{H}_2,J_2)=(\mathbf{H}_1\otimes\mathbf{H}_2,J_1\otimes J_2)$. The precise connection between this approach and ours is still under study.

2 Euclidean Jordan algebras

We begin with a concise review of some basic Jordan-algebraic background. References for this section are [2] and [8]. A *euclidean Jordan algebra* (hereafter: EJA) is a finite-dimensional commutative real algebra $(A,\cdot)$ with a multiplicative unit element u, satisfying the *Jordan identity*

$$a^2\cdot(a\cdot b)=a\cdot(a^2\cdot b)$$

for all $a,b\in A$, and equipped with an inner product satisfying

$$\langle a\cdot b|c\rangle=\langle b|a\cdot c\rangle$$

for all $a,b,c\in A$. The basic example is the self-adjoint part M_{sa} of a real, complex or quaternionic matrix algebra M, with $a\cdot b=(ab+ba)/2$ and with $\langle a|b\rangle=\text{tr}(ab)$. Any Jordan subalgebra of an EJA is also an EJA. So, too, is the *spin factor* $V_n=\mathbb{R}\times\mathbb{R}^n$, with the obvious inner product and with

$$(t,x)\cdot(s,y)=(ts+\langle x|y\rangle,ty+sx):$$

this can be embedded in $M_{2^n}(\mathbb{C})_{\text{sa}}$. Moreover, one can show that

$$V_2\simeq M_2(\mathbb{R})_{\text{sa}},\ V_3=M_2(\mathbb{C})_{\text{sa}},\ \text{and}\ V_5\simeq M_2(\mathbb{H})_{\text{sa}}.$$

Classification Direct sums of EJAs are also EJAs, so we can obtain more examples by forming direct sums of the EJAs of the types mentioned above. The *Jordan-von Neumann-Wigner Classification Theorem* (see [8] Chapter IV) provides a converse: every euclidean Jordan algebra is a direct sum of *simple* EJAs, each of which is isomorphic to a spin factor V_n, or to the self-adjoint part of a matrix algebra $M_n(\mathbb{K})$ where $\mathbb{K}$ is one of the classical division rings $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, *or*, if $n = 3$, to the *Octonions*, $\mathbb{O}$. This last example, which is not embeddable into the self-adjoint part of a complex matrix algebra, is called the *exceptional Jordan algebra*, or the *Albert algebra*. A Jordan algebra that *is* embeddable in $M_n(\mathbb{C})_{\text{sa}}$ for some n, is said to be *special*. It follows from the classification theorem that any EJA decomposes as a direct sum $A_{sp} \oplus A_{ex}$ where A_{sp} is special and A_{ex} is a direct sum of copies of the exceptional Jordan algebra.

Projections and the Spectral Theorem A *projection* in an EJA A is an element $p \in A$ with $p^2 = p$. If p, q are projections with $p \cdot q = 0$, we say that p and q are orthogonal. In this case, $p + q$ is another projection. A projection not representable as a sum of other projections is said to be *minimal* or *primitive*. A *Jordan frame* is a set $E \subseteq A$ of pairwise orthogonal minimal projections that sum to the Jordan unit. The *Spectral Theorem* (cf. e.g. [8], Theorem III.1.1) for EJAs asserts that every element $a \in A$ can be expanded as a linear combination $a = \sum_{x \in E} t_x x$ where E is some Jordan frame.

One can show that all Jordan frames for a given Euclidean Jordan algebra A have the same number of elements. This number is called the *rank* of A. By the Classification Theorem, all simple Jordan algebras having rank 4 or higher are special.

Order Structure Any EJA A is at the same time an ordered real vector space, with positive cone $A_+ = \{a^2 | a \in A\}$; for $a, b \in A$, $a \leq b$ iff $b - a \in A_+$. This allows us to interpret A as a probabilistic model: an *effect* (measurement-outcome) in A is an element $a \in A_+$ with $a \leq u$. A *state* on A is a positive linear mapping $\alpha : A \to \mathbb{R}$ with $\alpha(u) = 1$. If a is an effect, we interpret $\alpha(a)$ as the probability that a will be observed (if tested) in the state α.

The cone A_+ is *self-dual* with respect to the given inner product on A: an element $a \in V$ belongs to A_+ iff $\langle a|b \rangle \geq 0$ for all $b \in A_+$. Every state α then corresponds to a unique $b \in A_+$ with $\alpha(a) = \langle a|b \rangle$.

Remark: Besides being self-dual, the cone A_+ is *homogeneous*: any element of the *interior* of A_+ can be obtained from any other by an order-automorphism of A, that is, a linear automorphism $\phi : A \to A$ with $\phi(A_+) = A_+$. The *Koecher-Vinberg Theorem* ([10, 13]; see [8] for a modern proof) identifies EJAs as precisely the finite-dimensional ordered linear spaces having homogeneous, self-dual positive cones. This fact underwrites the derivations in several of the papers cited above [15, 16, 14].[2]

Reversible and universally reversible EJAs A Jordan subalgebra of M_{sa}, where M is a complex $*$-algebra, is *reversible* iff

$$a_1, ..., a_k \in A \;\Rightarrow\; a_1 a_2 \cdots a_k + a_k \cdots a_2 a_1 \in A,$$

where juxtaposition indicates multiplication in M. Note that with $k = 2$, this is just closure under the Jordan product on M_{sa}. An abstract EJA A is *reversible* iff it has a representation as a reversible Jordan subalgebra of some complex $*$-algebra. A reversible EJA is *universally reversible* (UR) iff it has *only* reversible representations.

Universal reversibility will play a large role in what follows. Of the four basic types of special Euclidean

[2] A different characterization of EJAs, in terms of projections associated with faces of the state space, is invoked in [5].

Jordan algebra considered above, the only ones that are not UR are the spin factors V_k with $k \geq 4$. For $k = 4$ and $k > 5$, V_k is not even reversible; V_5 — equivalently, $M_2(\mathbb{H})_{\mathrm{sa}}$ — has a reversible representation, but also non-reversible ones. Thus, if we adopt the shorthand

$$R_n = M_n(\mathbb{R})_{\mathrm{sa}},\ C_n = M_n(\mathbb{C})_{\mathrm{sa}},\ \text{and}\ Q_n = M_n(\mathbb{H})_{\mathrm{sa}},$$

we have R_n, C_n UR for all n, and Q_n UR for $n > 2$.

3 Composites of EJAs

A probabilistic theory must allow for some device for describing composite systems. Given EJAs A and B, understood as models for two physical systems, we'd like to construct an EJA AB that models the two systems considered together as a single entity. Is there any satisfactory way to do this? If so, how much latitude does one have?

The first question is answered affirmatively by a construction due to H. Hanche-Olsen [9], which we now review.

The universal tensor product A *representation* of a Jordan algebra A is a Jordan homomorphism $\pi : A \to M_{\mathrm{sa}}$, where $\mathbf{M}$ is a complex $*$-algebra. For any EJA A, there exists a (possibly trivial) $*$-algebra $C^*(A)$ and a representation $\psi_A : A \to C^*(A)_{\mathrm{sa}}$ with the universal property that any representation $\pi : A \to M_{\mathrm{sa}}$, where $\mathbf{M}$ is a C^*-algebra, decomposes uniquely as $\pi = \tilde{\pi} \circ \psi_A$, $\tilde{\pi} : C^*(A) \to \mathbf{M}$ a $*$-homomorphism. Evidently, $(C^*(A), \psi_A)$ is unique up to a canonical $*$-isomorphism. Since $\psi^{\mathrm{op}} : A \to C^*(A)^{\mathrm{op}}$ provides another solution to the same universal problem, there exists a canonical anti-automorphism Φ_A on $C^*(A)$, fixing every point of $\psi_A(A)$.

We refer to $(C^*(A), \psi_A)$ as the *universal representation* of A. A is exceptional iff $C^*(A) = \{0\}$. If A has no exceptional factors, then ψ_A is an injective. In this case, we will routinely identify A with its image $\psi_A(A) \leq C^*(A)$.

In [9], Hanche-Olsen defines the *universal* tensor product of two special EJAs A and B to be the Jordan subalgebra of $C^*(A) \otimes C^*(B)$ generated by $A \otimes B$. This is denoted $A \widetilde{\otimes} B$. It can be shown that

$$C^*(A \widetilde{\otimes} B) = C^*(A) \widetilde{\otimes} C^*(B) \ \text{ and } \ \Phi_{A \widetilde{\otimes} B} = \Phi_A \otimes \Phi_B.$$

Some further important facts about the universal tensor product are the following:

Proposition 1 Let A, B and C denote EJAs.

(a) If $\phi : A \to C$, $\psi : B \to C$ are unital Jordan homomorphisms with operator-commuting ranges[3], then there exists a unique Jordan homomorphism $A \widetilde{\otimes} B \to C$ taking $a \otimes b$ to $\phi(a) \cdot \psi(b)$ for all $a \in A$, $b \in B$.

(b) $A \widetilde{\otimes} B$ is UR unless one of the factors has a one-dimensional summand and the other has a representation onto a spin factor V_n with $n \geq 4$.

(c) If A is UR, then $A \widetilde{\otimes} M_n(\mathbb{C})_{\mathrm{sa}} = (C^*(A) \otimes M_n(\mathbb{C}))_{\mathrm{sa}}$.

These are Propositions 5.2, 5.3 and 5.4, respectively, in [9].

Note that part (b) implies that if A and B are irreducible and non-trivial, $A \widetilde{\otimes} B$ will always be UR, hence,

[3] Elements $x, y \in C$ *operator commute* iff $x \cdot (y \cdot z) = y \cdot (x \cdot z)$ for all $z \in C$.

the fixed-point set of $\Phi_A \otimes \Phi_B$. Using this one can compute $A\widetilde{\otimes}B$ for irreducible, universally reversible A and B [9]. Below, and for the balance of this paper, we use the shorthand $R_n := M_n(\mathbb{R})_{\text{sa}}$, $C_n = M_n(\mathbb{C})_{\text{sa}}$ and $Q_n = M_n(\mathbb{H})_{\text{sa}}$ (noting that Q_n is UR only for $n > 2$):

$\widetilde{\otimes}$	R_m	C_m	Q_m
R_n	R_{nm}	C_{nm}	Q_{nm}
C_n	C_{nm}	$C_{nm} \oplus C_{nm}$	C_{2nm}
Q_n	Q_{nm}	C_{2nm}	R_{4nm}

Figure 2

For $Q_2\widetilde{\otimes}Q_2$, a bit more work is required, but one can show that $Q_2\widetilde{\otimes}Q_2$ is the direct sum of four copies of $R_{16} = M_{16}(\mathbb{R})_{\text{sa}}$ [4].

General composites of EJAs The universal tensor product is an instance of the following (as it proves, only slightly) more general scheme. Recall that an order-automorphism of an EJA A is a linear bijection $\phi : A \to A$ taking A_+ onto itself. These form a Lie group, whose identity component we denote by $G(A)$.

Definition 1: A **composite** of EJAs A and B is a pair (AB, π) where AB is an EJA and $\pi : A \otimes B \to AB$ is a linear mapping such that

(a) If $a \in A_+$ and $b \in B_+$, then $\pi(a \otimes b) \in (AB)_+$, with $\pi(u \otimes u)$ the Jordan unit of AB;

(b) for all states α on A, β on B, there exists a state γ on AB such that $\gamma(\pi(a \otimes b)) = \alpha(a)\beta(b)$;

(c) for all automorphisms $\phi \in G(A)$ and $\psi \in G(B)$, there exists a preferred automorphism $\phi \otimes \psi \in G(AB)$ with $(\phi \otimes \psi)(\pi(a \otimes b)) = \pi(\phi(a) \otimes \psi(b))$. Moreover, we require that

$$(\phi_1 \otimes \psi_1) \circ (\phi_2 \otimes \psi_2) = (\phi_1 \circ \phi_2) \otimes (\psi_1 \circ \psi_2)$$

and

$$(\phi \otimes \psi)^\dagger = \phi^\dagger \otimes \psi^\dagger$$

for all $\phi, \phi_i \in G(A)$ and $\psi, \psi_i \in G(B)$.

It follows from (b) that π is injective (if $\pi(T) = 0$, then for any states α, β, there's a state γ of AB with $(\alpha \otimes \beta)(T) = \gamma(\pi(T)) = 0$; it follows that $T = 0$). Henceforth, we'll simply regard $A \otimes B$ as a subspace of AB.

Condition (c) calls for further comment. The dynamics of a physical system modeled by a Euclidean Jordan algebra A will naturally be represented by a one-parameter group $t \mapsto \phi_t$ of order automorphisms of A. As order-automorphisms in $G(A)$ are precisely the elements of such one-parameter groups, condition (c) is equivalent to the condition that, given dynamics $t \mapsto \phi_t$ and $t \mapsto \psi_t$ on A and B, respectively, there is a preferred dynamics on AB under which pure tensors $a \otimes b$ evolve according to $a \otimes b \mapsto \phi_t(a) \otimes \psi_t(b)$. In other words, there is a dynamics on AB under which A and B evolve independently.

Theorem 1: *If A and B are simple EJAs, then any composite AB is special, and an ideal in $A\widetilde{\otimes}B$.*

The basic idea of the proof is to show that if $p_1, ..., p_n$ is a Jordan frame in an irreducible summand of A, and $q_1, ..., q_m$ is a Jordan frame in an irreducible summand of B, then $\{p_i \otimes q_j | i = 1, ..., n, j = 1, ..., m\}$ is a pairwise orthogonal set of projections in AB, whence, the latter has rank at least four, and must therefore be special. For the details, we refer to [4].

Corollary 1: *If A is simple and B is exceptional, then no composite AB exists.*

In particular, if B is the exceptional factor, there exists no composite of B with itself.

Corollary 2: *If $A\widetilde{\otimes}B$ is simple, then $AB = A\widetilde{\otimes}B$ is the only possible composite of A and B.*

There are cases in which $A\widetilde{\otimes}B$ isn't simple, even where A and B are: namely, the cases in which A and B are both hermitian parts of complex matrix algebras. From table (2), we see that if $A = C_n$ and $B = C_m$, then $A\widetilde{\otimes}B = C_{nm} \oplus C_{nm}$. In this case, Proposition 1 gives us *two* choices for AB: either the entire direct sum above, or one of its isomorphic summands, i.e., the "obvious" composite $AB = C_{nm}$.

4 Embedded EJAs

Corollary 1 above justifies restricting our attention to special EJAs (often called Euclidean JC-algebras). In fact, it will be helpful to consider *embedded* EJAs, that is, Jordan subalgebras of specified (finite-dimensional) C^* algebras.

Definition 2: An *embedded JC algebra*, or EJC, is a pair (A, M_A) where A is a Jordan subalgebra of a finite-dimensional complex $*$-algebra M_A.

The notation $\mathbf{M}_A$ is intended to emphasise that the embedding $A \mapsto \mathbf{M}_A$ is part of the structure of interest. Given A, there is always a canonical choice for $\mathbf{M}_A$, namely the universal enveloping $*$-algebra $C^*(A)$ of A [9].

Definition 3: The *canonical product* of EJCs $(A, \mathbf{M}_A)$ and $(B, \mathbf{M}_B)$ is the pair $(A \odot B, \mathbf{M}_A \otimes \mathbf{M}_B)$ where $A \odot B$ is the Jordan subalgebra of $(\mathbf{M}_A \otimes \mathbf{M}_B)_{\mathrm{sa}}$ generated by the subspace $A \otimes B$.

Note that, as a matter of definition, $\mathbf{M}_{A\odot B} = \mathbf{M}_A \otimes \mathbf{M}_B$. If $\mathbf{M}_A = C^*(A)$ and $\mathbf{M}_B = C^*(B)$, then $A \odot B$ is the Hanche-Olsen tensor product.

One would like to know that $A \odot B$ is in fact a composite of A and B in the sense of Definition 1. Using a result of Upmeier [12], we can show that this is the case for *reversible* EJAs A and B. (that is, real, complex and quaternionic systems, and direct sums of these). Whether $A \odot B$ is a composite in the sense of Definition 1 when A or B is non-reversible spin factor remains an open question.

We can now form a category:

Definition 4: **EJC** is the category consisting of EJCs $(A, \mathbf{M}_A)$ and completely positive maps $\phi : \mathbf{M}_A \to \mathbf{M}_B$ with $\phi(A) \subseteq B$. We refer to such maps as *Jordan preserving*.

Proposition 2: *The canonical product $\odot$ is associative on* **EJC**. *More precisely, the associator mapping*

$$\alpha : \mathbf{M}_A \otimes (\mathbf{M}_B \otimes \mathbf{M}_C) \to (\mathbf{M}_A \otimes \mathbf{M}_B) \otimes \mathbf{M}_C$$

takes $A \odot (B \odot C)$ to $(A \odot B) \odot C$.

(Note that since the associator mapping is *CP*, this means that α is a morphism in **EJC**.) The proof is somewhat lengthy, so we refer the reader to the forthcoming paper [4].

Proposition 2 suggests that **EJC** might be symmetric monoidal under $\odot$. There is certainly a natural choice for the monoidal unit, namely $I = (\mathbb{R}, \mathbb{C})$. But the following example shows that tensor products of **EJC** morphisms are generally not morphisms:

Example 1: Let $(A, C^*(A))$ and $(B, C^*(B))$ be simple, universally embedded EJCs, and suppose that B is not UR (e.g., a spin factor V_n with $n > 3$). Let $\widehat{B}$ be the set of fixed points of the canonical involution Φ_B. Then by Corollary 2, $A \odot B = A\widetilde{\otimes}B$, the set of fixed points of $\Phi_A \otimes \Phi_B$. In particular, $u_A \otimes \widehat{B}$ is contained in $A \odot B$. Now let f be a state on $C^*(A)$: this is CP, and trivially Jordan-preserving, and so, a morphism in **EJC**. But

$$(f \otimes \mathrm{id}_B)(u_A \otimes \widehat{B}) = f(u_A)\widehat{B} = \widehat{B},$$

which isn't contained in B. So $f \otimes \mathrm{id}_B$ isn't Jordan-preserving.

5 Reversible and universally reversible EJCs

It seems that the category **EJC** is simply too large. We can try to obtain a better-behaved category by restricting the set of morphisms, or by restricting the set of objects, or both.

As a first pass, we might try this:

Definition 5: Let $(A, \mathbf{M}_A)$ and $(B, \mathbf{M}_B)$ be EJCs. A linear mapping $\phi : \mathbf{M}_A \to \mathbf{M}_B$ is *completely Jordan preserving* (CJP) iff $\phi \otimes 1_C$ takes $A \odot C$ to $B \odot C$ for all $(C, \mathbf{M}_C)$.

It is not hard to verify the following

Proposition 3: *If $\phi : \mathbf{M}_A \to \mathbf{M}_B$ and $\psi : \mathbf{M}_C \to \mathbf{M}_D$ are CJP, then so is*

$$\phi \otimes \psi : \mathbf{M}_{A \odot C} = \mathbf{M}_A \otimes \mathbf{M}_B \to \mathbf{M}_B \otimes \mathbf{M}_C = \mathbf{M}_{B \odot D}.$$

Thus, the category of EJC algebras and CJP mappings is symmetric monoidal.[4]

There are many examples: Jordan homomorphisms are CJP maps. If $a \in A$, the mapping

$$U_a : A \to A$$

given by $U_a = 2L_{a^2} - L_a^2$, where $L_a(b) = ab$, is also CJP. On the other hand, by Example 1 above, $\mathbf{CJP}(A, I)$ is *empty* for universally embedded simple A!

So not all CP maps are CJP; for instance, states are never CJP. More seriously, we can't interpret states as morphisms in this category. The problem is the non-UR spin factors in **CJP**. If we remove these, things are much better.

Definition 6: Let $\mathscr{C}$ be a subclass of embedded EJC algebras, closed under $\odot$ and containing I. A linear mapping $\phi : \mathbf{M}_A \to \mathbf{M}_B$ is CJP *relative to* $\mathscr{C}$ iff $\phi_A \otimes \mathrm{id}_C$ is Jordan preserving for all C *in* $\mathscr{C}$. $\mathbf{CJP}_{\mathscr{C}}$ is the category having objects elements of $\mathscr{C}$, mappings relatively CJP mappings.

Example 2: **URUE** is the class of universally reversible, universally embedded EJC algebras. **URSE** is the category of universally reversible, standardly embedded EJC algebras, and **RSE** is the category of reversible, standardly embedded EJC algebras. Equipped with relatively CJP mappings, both are symmetric monoidal categories.

Note that **RSE** consists of direct sums of real, complex and quaternionic quantum systems. **URSE** and **URUE** contain all real and complex quantum systems, and all quaternionic quantum systems *except* the "quabit", i.e., the quaternionic bit $Q_2 := M_2(\mathbb{H})_{\mathrm{sa}}$.

[4]Notice that scalars of this category are real numbers. It is sometimes suggested that quaternionic Hilbert spaces can't be accommodated in a symmetric monoidal category owing to the noncommutativity of $\mathbb{H}$, as the scalars in a symmetric monoidal category must always be commutative. As we are representing quaternionic quantum systems in terms of the associated real vector spaces of hermitian operators, this issue doesn't arise here.

In both of the categories **URUE** and **URSE**, states are morphisms. In fact, we are going to see that **URUE** and **URSE** inherit compact closure from the category $*$**-ALG** of finite-dimensional, complex $*$-algebras and CP maps, in which they are embedded.

It's worth taking a moment to review this compact structure. If $\mathbf{M}$ is a finite-dimensional complex $*$-algebra, let Tr denote the canonical trace on $\mathbf{M}$, regarded as acting on itself by left multiplication (so that $\mathrm{Tr}(a) = \mathrm{tr}(L_a)$, $L_a : \mathbf{M} \to \mathbf{M}$ being $L_a(b) = ab$ for all $b \in \mathbf{M}$). This induces an inner product on $\mathbf{M}$, given by $\langle a|b\rangle_{\mathbf{M}} = \mathrm{Tr}(ab^*)$[5]. Note that this inner product is self-dualizing, i.e,. $a \in \mathbf{M}_+$ iff $\langle a|b\rangle \geq 0$ for all $b \in \mathbf{M}_+$. Now let $\overline{\mathbf{M}}$ be the conjugate algebra, writing $\overline{a}$ for $a \in \mathbf{M}$ when regarded as belonging to $\overline{\mathbf{M}}$ (so that $\overline{ca} = \overline{c}\,\overline{a}$ for scalars $c \in \mathbb{C}$ and $\overline{a}\overline{b} = \overline{ab}$ for $a, b \in \mathbf{M}$). Note that $\langle \overline{a}|\overline{b}\rangle = \langle b|a\rangle$. Now define

$$f_{\mathbf{M}} = \sum_{e \in E} e \otimes \overline{e} \in \mathbf{M} \otimes \overline{\mathbf{M}}$$

where E is any orthonormal basis for $\mathbf{M}$ with respect to $\langle\ |\ \rangle_{\mathbf{M}}$. Then a computation shows that

$$\langle (a \otimes \overline{b}) f_{\mathbf{M}} | f_{\mathbf{M}} \rangle_{\mathbf{M} \otimes \overline{\mathbf{M}}} = \langle a|b\rangle_{\mathbf{M}}.$$

Since the left-hand side defines a positive linear functional on $\mathbf{M} \otimes \overline{\mathbf{M}}$, so does the right (remembering here that pure tensors generate $\mathbf{M} \otimes \overline{\mathbf{M}}$, as we're working in finite dimensions). Call this functional $\eta_{\mathbf{M}}$. That is,

$$\eta_{\mathbf{M}} : \mathbf{M} \otimes \overline{\mathbf{M}} \to \mathbb{C} \ \text{ is given by } \ \eta_{\mathbf{M}}(a \otimes \overline{b}) = \langle a|b\rangle = \mathrm{Tr}(ab^*)$$

and is, up to normalization, a state on $\mathbf{M} \otimes \overline{\mathbf{M}}$. A further computation now shows that

$$\langle a \otimes \overline{b} | f_{\mathbf{M}} \rangle_{\mathbf{M} \otimes \overline{\mathbf{M}}} = \eta(a \otimes \overline{b}).$$

It follows that $f_{\mathbf{M}}$ belongs to the positive cone of $\mathbf{M} \otimes \overline{\mathbf{M}}$, by self-duality of the latter. A final computation shows that, for any states α and $\overline{\alpha}$ on $\mathbf{M}$ and $\overline{\mathbf{M}}$, respectively, and any $a \in \mathbf{M}, \overline{a} \in \overline{\mathbf{M}}$, we have

$$(\eta_{\mathbf{M}} \otimes \alpha)(a \otimes f_{\overline{\mathbf{M}}}) = \alpha(a) \ \text{ and } \ (\overline{\alpha} \otimes \eta_{\mathbf{M}})(f_{\overline{M}} \otimes \overline{a}) = \overline{\alpha}(\overline{a}).$$

Thus, $\eta_{\mathbf{M}}$ and $f_{\overline{\mathbf{M}}}$ define a compact structure on $*$**-ALG**, for which the dual object of $\mathbf{M}$ is given by $\overline{\mathbf{M}}$.

Definition 7: The *conjugate* of a EJC algebra $(A, \mathbf{M}_A)$ is $(\overline{A}, \overline{\mathbf{M}}_A)$, where $\overline{A} = \{\overline{a} | a \in A\}$. We write η_A for $\eta_{\mathbf{M}_A}$ and f_A for $f_{\mathbf{M}_A}$.

5.1 Universally-embedded, universally reversible EJAs

Now consider the category **URUE** of universally reversible, universally embedded EJAs A, i.e., pairs $(A, \mathbf{M}_A)$ with A UR and $\mathbf{M}_A = C^*(A)$. Let Φ_A be the canonical involution on $C^*(A)$.

Lemma 1: *Let* $(A, \mathbf{M}_A)$ *belong to* **URUE**. *Then*

(a) $f_A \in A \odot \overline{A}$*;*

(b) $\eta_A \circ (\Phi_A \otimes \Phi_{\overline{A}}) = \eta_A$.

Proof: (a) follows from the fact that Φ_A is unitary, so that if E is an orthonormal basis, then so is $\{\Phi_A(e) | e \in E\}$. Since f_A is independent of the choice of orthonormal basis, it follows that f is invariant under $\Phi_A \otimes \Phi_{\overline{A}}$, hence, an element of $A \widetilde{\otimes} \overline{A}$. Now (b) follows from part (a) of the previous lemma. $\square$

[5]We are following the convention that complex inner products are conjugate linear in the *second* argument.

Define $\gamma_A : C^*(\overline{A}) \to C^*(A)$ by $\gamma(\overline{a}) = \Phi_A(a^*)$. This is a $*$-isomorphism, and intertwines Φ_A and Φ_A; hence, $\gamma_A \otimes \mathrm{id}_B : C^*(\overline{A}\widetilde{\otimes}B) \to C^*(A \otimes B)$ intertwines $\Phi_{\overline{A}} \otimes \Phi_B = \Phi_{\overline{A}\widetilde{\otimes}B}$ and $\Phi_A \otimes \Phi_B = \Phi_{A\widetilde{\otimes}B}$ — whence, takes $\overline{A}\widetilde{\otimes}B$ to $A\widetilde{\otimes}B$. In particular, γ_A is CJP relative to the class of UR, universally embedded EJCs.

Lemma 2: *Let A be a universally embedded UR EJC. Then for all $\alpha \in \mathbf{CJP}(A,I)$, there exists $a \in A$ with $\alpha(b) = \langle b|a\rangle$ for all $b \in A$.*

Proof: Since $\alpha \in C^*(A)^*$, there is certainly some $a \in C^*(A)$ with $\alpha = |a\rangle$. We need to show that $a \in A$. Since α is CJP,

$$\gamma_A \otimes \alpha : C^*(\overline{A}) \otimes C^*(A) = C^*(\overline{A} \otimes A) \to C^*(A)$$

is Jordan-preserving. In particular, $(\alpha \otimes \gamma_A)(f_A) \in A$. But

$$\begin{aligned} (\alpha \otimes \gamma_A)(f_A) &= \sum_{e\in E} (\alpha \otimes \gamma_A)(e \otimes \overline{e}) \\ &= \sum_{e\in E} \langle e|a\rangle \Phi(e^*) \\ &= \Phi(\sum_{e\in E} \langle e|a\rangle e^*) \\ &= \Phi((\sum_{e\in E} \langle a|e\rangle e)^*) = \Phi(a^*) = \gamma_A(\overline{a}). \end{aligned}$$

Hence, $\gamma_A(\overline{a}) \in A$, whence, $\overline{a} \in \overline{A}$, whence, $a \in A$. (Alternatively: $\Phi_A(a^*) \in A$ implies $a^* \in A$, whence, a is self-adjoint, whence, $a \in A$.) $\square$

It follows that η_A and f_A belong, as morphisms, to **URUE**. Hence, **URUE** inherits the compact structure from $*$-**ALG**, as promised.

The same holds for **URSE**. Specifically, we want to show that f_A belongs to $A \odot \overline{A}$ whenever A is a standardly embedded UR EJC.

Suppose that E is an orthonormal basis for $\mathbf{M}_A$: then so is $\{e^*|e \in E\}$; thus, since f_A is independent of the choice of basis, we have

$$f_A^* = \sum_{e\in E} e^* \otimes \overline{e}^* = \sum_{e^*\in E^*} e^* \otimes \overline{e^*} = f_A.$$

Thus, if $A \odot \widehat{A}$ is the self-adjoint part of $\mathbf{M}_A \otimes \overline{\mathbf{M}}_A$, then $f_A \in A \otimes \overline{A}$. This covers the case where $A = C_n$. We also have, by the results above, that $f_A \in A \odot \overline{A}$ whenever the latter equals $A\widetilde{\otimes}\overline{A}$. This covers $A = R_n$ and $A = Q_n$ for $n > 2$.

In fact, we can do a bit better. If $\mathbf{M}$ and $\mathbf{N}$ are finite-dimensional $*$-algebras and $\phi : \mathbf{M} \to \mathbf{N}$ is a linear mapping, let $\phi^\dagger$ denote the adjoint of ϕ with respect to the natural trace inner products on $\mathbf{M}$ and $\mathbf{N}$. It is not difficult to show that, for any $\mathbf{M}$ in $*$-**ALG**, $f_{\mathbf{M}}^\dagger = \eta_{\mathbf{M}}$ and vice versa; indeed, $*$-**ALG** is dagger-compact.

Definition 8: Let $(A,\mathbf{M}_A)$ and $(B,\mathbf{M}_B)$ be EJCs. A linear mapping $\phi : \mathbf{M}_A \to \mathbf{M}_B$ is †-*CJP* iff both ϕ and $\phi^\dagger$ are CJP. If $\mathscr{C}$ is a category of EJCs and CJP mappings, we write $\mathscr{C}^\dagger$ for the category having the same objects, but with morphisms restricted to †-CJP mappings in $\mathscr{C}$.

If A belongs to **URUE** or **URSE**, then f_A and η_A are both CJP and, hence, are both †-CJP with respect to the indicated category. Hence,

Theorem 2: *The categories* $\mathbf{URUE}^\dagger$ *and* $\mathbf{URSE}^\dagger$ *are dagger-compact.*

6 Conclusion

We have found two theories — the categories **URSE** and **URUE** — that, in slightly different ways, unify finite-dimensional real, complex and (almost all of) quaternionic quantum mechanics. By virtue of being compact closed, both theories continues to enjoy many of the information-processing properties of standard complex QM, e.g., the existence of conclusive teleportation and entanglement-swapping protocols [1].

It is worth pointing out that the composites in our categories are not "locally tomographic", i.e, a state ω on $A \odot B$ is not generally determined by the joint probability assignment $a, b \mapsto \omega(a \otimes b)$, where a and b are effects of A and B, respectively. Another way to put it is that $A \odot B$ is generally much larger than the vector-space tensor product $A \otimes B$. (As local tomography is well known to separate complex QM from its real and quaternionic variants, this is hardly surprising.)

Neither theory includes the quabit, Q_2. Example 1 shows that the Q_2 can't be added to **URUE** without a violation of compact closure. On the other hand, if f_{Q_2} belongs to the canonical composite $Q_2 \odot Q_2$, then the slightly larger category **RSE**, which consists of *all* finite-dimensional real, complex and quaternionic quantum systems, will be compact closed (indeed, dagger compact).[6]

The categories **URSE** and **URUE** contain interesting compact closed subcategories. In particular, real and quaternionic quantum systems (less the quabit), taken together, form a sub-theory, closed under composites. We *conjecture* that this is exactly what one gets by applying the CPM construction to Baez' (implicit) category of pairs $(\mathbf{H}, J)$, H a finite-dimensional Hilbert space and J an anti-unitary with $J^2 = \pm 1$ — and, again, excluding the quabit. Should **RSE** prove to be compact closed, we could entertain the stronger conjecture that this is exactly what one obtains by applying CPM to Baez' category.

Complex quantum systems also form a monoidal subcategory of **URSE**, which we might call $\mathbb{C}$**QM**: indeed, one that functions as an "ideal", in that if $A \in$ **URSE** and $B \in \mathbb{C}$**QM**, then $A \odot B \in \mathbb{C}$**QM** as well. This is provocative, as it suggests that a universe initially consisting of many systems of all three types, would eventually evolve into one in which complex systems greatly predominate.

The category **URUE** is somewhat mysterious. Like **URSE**, this encompasses real, complex and quaternionic quantum systems, excepting the quabit. In this theory, the composite of *complex* quantum systems comes with an extra classical bit — equivalently, a $\{0,1\}$-valued superselection rule. This functions to make the transpose operation — which is a Jordan automorphism of $\mathbf{M}_n(\mathbb{C})_{\text{sa}}$, but an antiautomorphism of $\mathbf{M}_n(\mathbb{C})$ — count as a morphism. The precise physical significance of this is a subject for further study.

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[6] Since this Abstract was first submitted, we believe we have settled this question in the affirmative. The details will appear elsewhere.

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