Some Nearly Quantum Theories
(extended abstract)

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We consider possible non-signaling composites of probabilistic models based on euclidean Jordan algebras. Subject to some reasonable constraints, we show that no such composite exists having the exceptional Jordan algebra as a direct summand. We then construct several dagger compact categories of such Jordan-algebraic models. One of these neatly unifies real, complex and quaternionic mixed-state quantum mechanics, with the possible exception of the quaternionic “bit”. Another is similar, except in that (i) it excludes the quaternionic bit, and (ii) the composite of two complex quantum systems comes with an extra classical bit. A no-go theorem forecloses any possibility of such a category including higher-dimensional spin factors.

1 Introduction

A series of recent papers \[15, 11, 16, 14, 5\] have shown that any of various packages of probabilistic or information-theoretic axioms force the state spaces of a finite-dimensional probabilistic theory to be those of formally real, or euclidean, Jordan algebras. Thus, euclidean Jordan algebras (hereafter, EJAs) form a natural class of probabilistic models. Moreover, it is one that keeps us in the general neighborhood of standard quantum mechanics, owing to the classification of simple EJAs as self-adjoint parts of real, complex and quaternionic matrix algebras (corresponding to real, complex and quaternionic quantum systems), the exceptional Jordan algebra of self-adjoint $3 \times 3$ matrices over the octonions, and one further class, the so-called spin factors. The latter are essentially “bits”: their state-spaces are balls of arbitrary dimension, with antipodal points representing sharply distinguishable states.

This raises the question of whether one can construct probabilistic theories (as opposed to a collection of models of individual systems) in which finite-dimensional complex quantum systems can be accommodated together with several — perhaps all — of the other basic types of EJAs listed above. Ideally, these would be symmetric monoidal categories: even better, we might hope to obtain compact closed, or still better, dagger-compact, categories of EJAs \[1\]. Also, one would like the resulting theory to embrace mixed states and CP mappings.

In this paper, we exhibit two dagger-compact categories of EJAs — called URUE and URSE, acronyms that will be explained below — that include all real, complex and quaternionic matrix algebras, with one conspicuous (and interesting) exception: the quaternionic bit, or “quabit”, represented by $M_2(\mathbb{H})_{\text{sa}}$, the Jordan algebra of self-adjoint $2 \times 2$ quaternionic matrices. We are able to show that this cannot be added to URUE without destroying compact closure; whether URSE can be extended to include it remains open.

\[1\]Where this ball has dimension 2, 3 or 5, these are just the state spaces of real, complex and quaternionic quantum bits.

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at present. **URSE** includes a faithful copy of finite-dimensional complex quantum mechanics, while in **URUE**, composites of complex quantum systems come with an extra classical bit — that is, a \{0,1\} valued superselection rule.

We also show that there is scant hope of including more exotic Jordan algebras in a satisfactory categorical scheme. Even allowing for a very liberal definition of composite (our Definition 1 below), the exceptional Jordan algebra is ruled out altogether (Corollary 1), while non-quantum spin factors are ruled out if we want to regard states as morphisms — in particular, if we demand compact closure (see Example 1). Combined with the results of (any of) the papers cited above that derive a euclidean Jordan-algebraic structure from information-theoretic assumptions, these results provide a compelling motivation for a kind of unified quantum theory that accommodates real, complex and quaternionic quantum systems (possibly modulo the quubit) and permits the formation of composites of these.

A condition frequently invoked to rule out real and quaternionic QM is *local tomography*: the doctrine that the state of a composite of two systems should depend only on the joint probabilities it assigns to measurement outcomes on the component systems. Indeed, it can be shown \[7\] that standard complex QM with superselection rules is the only dagger-compact category of EJAs that includes the qubit. Accordingly, **URUE** and **URSE** are not locally tomographic. In our view, the very existence of these quite reasonable, well-behaved categories suggests that local tomography is not as well-motivated as is sometimes supposed.

**Remark:** A broadly similar proposal is advanced by Baez \[3\], who points out that one can view real and quaternionic quantum systems as pairs \((H,J)\), where \(H\) is a complex Hilbert space and \(J\) is an anti-unitary satisfying \(J^2 = 1\) (the real case) or \(J^2 = -1\) (the quaternionic case). This yields a symmetric monoidal category in which objects are such pairs, morphisms \((H_1,J_1) \to (H_2,J_2)\) are linear mappings intertwining \(J_1\) and \(J_2\), and \((H_1,J_1) \otimes (H_2,J_2) = (H_1 \otimes H_2,J_1 \otimes J_2)\). The precise connection between this approach and ours is still under study.

## 2 Euclidean Jordan algebras

We begin with a concise review of some basic Jordan-algebraic background. References for this section are \[2\] and \[8\]. A *euclidean Jordan algebra* (hereafter: EJA) is a finite-dimensional commutative real algebra \((A,\cdot)\) with a multiplicative unit element \(u\), satisfying the *Jordan identity*

\[
a^2 \cdot (a \cdot b) = a \cdot (a^2 \cdot b)
\]

for all \(a,b \in A\), and equipped with an inner product satisfying

\[
\langle a \cdot b|c \rangle = \langle b|a \cdot c \rangle
\]

for all \(a,b,c \in A\). The basic example is the self-adjoint part \(M_{sa}\) of a real, complex or quaternionic matrix algebra \(M\), with \(a \cdot b = (ab + ba)/2\) and with \(\langle a|b \rangle = \text{tr}(ab)\). Any Jordan subalgebra of an EJA is also an EJA. So, too, is the *spin factor* \(V_n = \mathbb{R} \times \mathbb{R}^n\), with the obvious inner product and with

\[
(t,x) \cdot (s,y) = (ts + \langle x|y \rangle , ty + sx)\ :
\]

this can be embedded in \(M_{2^n} (\mathbb{C})_{sa}\). Moreover, one can show that

\[
V_2 \simeq M_2(\mathbb{R})_{sa}, \ V_3 = M_2(\mathbb{C})_{sa}, \text{ and } V_5 \simeq M_2(\mathbb{H})_{sa}.
\]
The cone will be observed (if tested) in the state $\alpha$. Order Structure Any EJA having rank 4 or higher are special. This number is called the $a$ expanded as a linear combination $\sum_{E} E$.

Spectral Theorem The Jordan frame $A$ is a set $E \subseteq A$ of pairwise orthogonal minimal projections that sum to the Jordan unit. The Spectral Theorem (cf. e.g. [8], Theorem III.1.1) for EJAs asserts that every element $a \in A$ can be expanded as a linear combination $a = \sum_{E} E$ where $E$ is some Jordan frame.

One can show that all Jordan frames for a given Euclidean Jordan algebra $A$ have the same number of elements. This number is called the rank of $A$. By the Classification Theorem, all simple Jordan algebras having rank 4 or higher are special.

Order Structure Any EJA $A$ is at the same time an ordered real vector space, with positive cone $A_+ = \{ a^2 | a \in A \}$; for $a, b \in A$, $a \leq b$ iff $b - a \in A_+$. This allows us to interpret $A$ as a probabilistic model: an effect (measurement-outcome) in $A$ is an element $a \in A_+$ with $a \leq u$. A state on $A$ is a positive linear mapping $\alpha : A \rightarrow \mathbb{R}$ with $\alpha(u) = 1$. If $a$ is an effect, we interpret $\alpha(a)$ as the probability that $a$ will be observed (if tested) in the state $\alpha$.

The cone $A_+$ is self-dual with respect to the given inner product on $A$: an element $a \in V$ belongs to $A_+$ iff $\langle a | b \rangle \geq 0$ for all $b \in A_+$. Every state $\alpha$ then corresponds to a unique $b \in A_+$ with $\alpha(a) = \langle a | b \rangle$.

Remark: Besides being self-dual, the cone $A_+$ is homogeneous: any element of the interior of $A_+$ can be obtained from any other by an order-automorphism of $A$, that is, a linear automorphism $\phi : A \rightarrow A$ with $\phi(A_+) = A_+$. The Koecher-Vinberg Theorem ([10] [13]; see [8] for a modern proof) identifies EJAs as precisely the finite-dimensional ordered linear spaces having homogeneous, self-dual positive cones. This fact underwrites the derivations in several of the papers cited above [15, 16, 14, 2].

Reversible and universally reversible EJAs A Jordan subalgebra of $M_{3\mathbb{A}}$, where $M$ is a complex $*$-algebra, is reversible iff

$$a_1, \ldots, a_k \in A \Rightarrow a_1 a_2 \cdots a_k + a_k \cdots a_2 a_1 \in A,$$

where juxtaposition indicates multiplication in $M$. Note that with $k = 2$, this is just closure under the Jordan product on $M_{3\mathbb{A}}$. An abstract EJA $A$ is reversible iff it has a representation as a reversible Jordan subalgebra of some complex $*$-algebra. A reversible EJA is universally reversible (UR) iff it has only reversible representations.

Universal reversibility will play a large role in what follows. Of the four basic types of special Euclidean

\[2\] A different characterization of EJAs, in terms of projections associated with faces of the state space, is invoked in [5].
Jordan algebra considered above, the only ones that are not UR are the spin factors $V_k$ with $k \geq 4$. For $k = 4$ and $k > 5$, $V_k$ is not even reversible; $V_5$ — equivalently, $M_2(\mathbb{H})_{\text{sa}}$ — has a reversible representation, but also non-reversible ones. Thus, if we adopt the shorthand

$$R_n = M_n(\mathbb{R})_{\text{sa}}, \quad C_n = M_n(\mathbb{C})_{\text{sa}}, \quad \text{and} \quad Q_n = M_n(\mathbb{H})_{\text{sa}},$$

we have $R_n, C_n$ UR for all $n$, and $Q_n$ UR for $n > 2$.

## 3 Composites of EJAs

A probabilistic theory must allow for some device for describing composite systems. Given EJAs $A$ and $B$, understood as models for two physical systems, we’d like to construct an EJA $AB$ that models the two systems considered together as a single entity. Is there any satisfactory way to do this? If so, how much latitude does one have?

The first question is answered affirmatively by a construction due to H. Hanche-Olsen [9], which we now review.

**The universal tensor product** A *representation* of a Jordan algebra $A$ is a Jordan homomorphism $\pi : A \rightarrow M_{\text{sa}}$, where $M$ is a complex $*$-algebra. For any EJA $A$, there exists a (possibly trivial) $*$-algebra $C^*(A)$ and a representation $\psi_A : A \rightarrow C^*(A)_{\text{sa}}$ with the universal property that any representation $\pi : A \rightarrow M_{\text{sa}}$, where $M$ is a $C^*$-algebra, decomposes uniquely as $\pi = \bar{\pi} \circ \psi_A$, $\bar{\pi} : C^*(A) \rightarrow M$ a $*$-homomorphism. Evidently, $(C^*(A), \psi_A)$ is unique up to a canonical $*$-isomorphism. Since $\psi_A : A \rightarrow C^*(A)_{\text{op}}$ provides another solution to the same universal problem, there exists a canonical anti-automorphism $\Phi_A$ on $C^*(A)$, fixing every point of $\psi_A(A)$.

We refer to $(C^*(A), \psi_A)$ as the *universal representation* of $A$. $A$ is exceptional iff $C^*(A) = \{0\}$. If $A$ has no exceptional factors, then $\psi_A$ is an injective. In this case, we will routinely identify $A$ with its image $\psi_A(A) \subseteq C^*(A)$.

In [9], Hanche-Olsen defines the *universal* tensor product of two special EJAs $A$ and $B$ to be the Jordan subalgebra of $C^*(A) \otimes C^*(B)$ generated by $A \otimes B$. This is denoted $A \hat{\otimes} B$. It can be shown that

$$C^*(A \hat{\otimes} B) = C^*(A) \hat{\otimes} C^*(B) \quad \text{and} \quad \Phi_{A \hat{\otimes} B} = \Phi_A \otimes \Phi_B.$$

Some further important facts about the universal tensor product are the following:

**Proposition 1** Let $A$, $B$, and $C$ denote EJAs.

(a) If $\phi : A \rightarrow C$, $\psi : B \rightarrow C$ are unital Jordan homomorphisms with operator-commuting ranges, then there exists a unique Jordan homomorphism $A \hat{\otimes} B \rightarrow C$ taking $a \otimes b$ to $\phi(a) \cdot \psi(b)$ for all $a \in A$, $b \in B$.

(b) $A \hat{\otimes} B$ is UR unless one of the factors has a one-dimensional summand and the other has a representation onto a spin factor $V_n$ with $n \geq 4$.

(c) If $A$ is UR, then $A \hat{\otimes} M_n(\mathbb{C})_{\text{sa}} = (C^*(A) \otimes M_n(\mathbb{C}))_{\text{sa}}$.

These are Propositions 5.2, 5.3 and 5.4, respectively, in [9].

Note that part (b) implies that if $A$ and $B$ are irreducible and non-trivial, $A \hat{\otimes} B$ will always be UR, hence,

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3Elements $x, y \in C$ *operator commute* iff $x \cdot (y \cdot z) = y \cdot (x \cdot z)$ for all $z \in C$. 

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the fixed-point set of $\Phi_A \otimes \Phi_B$. Using this one can compute $A \otimes B$ for irreducible, universally reversible $A$ and $B$ \cite{4}. Below, and for the balance of this paper, we use the shorthand $R_n := M_n(\mathbb{R})_{sa}$, $C_n = M_n(\mathbb{C})_{sa}$ and $Q_n = M_n(\mathbb{H})_{sa}$ (noting that $Q_n$ is UR only for $n > 2$):

$$
\begin{array}{c|ccc}
\otimes & R_m & C_m & Q_m \\
R_n & R_{nm} & C_{nm} & Q_{nm} \\
C_n & C_{2nm} & C_{nm} \oplus C_{nm} & C_{2nm} \\
Q_n & Q_{nm} & C_{2nm} & R_{4nm} \\
\end{array}
$$

Figure 2

For $Q_2 \tilde{\otimes} Q_2$, a bit more work is required, but one can show that $Q_2 \tilde{\otimes} Q_2$ is the direct sum of four copies of $R_{16} = M_{16}(\mathbb{R})_{sa}$ \cite{4}.

**General composites of EJAs** The universal tensor product is an instance of the following (as it proves, only slightly) more general scheme. Recall that an order-automorphism of an EJA $A$ is a linear bijection $\phi : A \to A$ taking $A_+$ onto itself. These form a Lie group, whose identity component we denote by $G(A)$.

**Definition 1**: A composite of EJAs $A$ and $B$ is a pair $(AB, \pi)$ where $AB$ is an EJA and $\pi : A \otimes B \to AB$ is a linear mapping such that

(a) If $a \in A_+$ and $b \in B_+$, then $\pi(a \otimes b) \in (AB)_+$, with $\pi(u \otimes u)$ the Jordan unit of $AB$;

(b) for all states $\alpha$ on $A$, $\beta$ on $B$, there exists a state $\gamma$ on $AB$ such that $\gamma(\pi(a \otimes b)) = \alpha(a) \beta(b)$;

(c) for all automorphisms $\phi \in G(A)$ and $\psi \in G(B)$, there exists a preferred automorphism $\phi \otimes \psi \in G(AB)$ with $(\phi \otimes \psi)(\pi(a \otimes b)) = \pi(\phi(a) \otimes \psi(b))$. Moreover, we require that

$$(\phi_1 \otimes \psi_1) \circ (\phi_2 \otimes \psi_2) = (\phi_1 \circ \phi_2) \otimes (\psi_1 \circ \psi_2)$$

and

$$(\phi \otimes \psi)^\dagger = \phi^\dagger \otimes \psi^\dagger$$

for all $\phi, \phi_i \in G(A)$ and $\psi, \psi_i \in G(B)$.

It follows from (b) that $\pi$ is injective (if $\pi(T) = 0$, then for any states $\alpha, \beta$, there’s a state $\gamma$ of $AB$ with $(\alpha \otimes \beta)(T) = \gamma(\pi(T)) = 0$; it follows that $T = 0$). Henceforth, we’ll simply regard $A \otimes B$ as a subspace of $AB$.

Condition (c) calls for further comment. The dynamics of a physical system modeled by a Euclidean Jordan algebra $A$ will naturally be represented by a one-parameter group $t \mapsto \phi_t$ of order automorphisms of $A$. As order-automorphisms in $G(A)$ are precisely the elements of such one-parameter groups, condition (c) is equivalent to the condition that, given dynamics $t \mapsto \phi_t$ on $A$ and $B$, respectively, there is a preferred dynamics on $AB$ under which pure tensors $a \otimes b$ evolve according to $a \otimes b \mapsto \phi_t(a) \otimes \psi_t(b)$. In other words, there is a dynamics on $AB$ under which $A$ and $B$ evolve independently.

**Theorem 1**: If $A$ and $B$ are simple EJAs, then any composite $AB$ is special, and an ideal in $A \tilde{\otimes} B$.

The basic idea of the proof is to show that if $p_1, \ldots, p_n$ is a Jordan frame in an irreducible summand of $A$, and $q_1, \ldots, q_m$ is a Jordan frame in an irreducible summand of $B$, then $\{p_i \otimes q_j | i = 1, \ldots, n, j = 1, \ldots, m\}$ is a pairwise orthogonal set of projections in $AB$, whence, the latter has rank at least four, and must therefore be special. For the details, we refer to \cite{4}.

**Corollary 1**: If $A$ is simple and $B$ is exceptional, then no composite $AB$ exists.
In particular, if $B$ is the exceptional factor, there exists no composite of $B$ with itself.

**Corollary 2:** If $A \tilde{\otimes} B$ is simple, then $AB = A \tilde{\otimes} B$ is the only possible composite of $A$ and $B$.

There are cases in which $A \tilde{\otimes} B$ isn’t simple, even where $A$ and $B$ are: namely, the cases in which $A$ and $B$ are both hermitian parts of complex matrix algebras. From table (2), we see that if $A = C_n$ and $B = C_m$, then $A \otimes B = C_{nm} \oplus C_{nm}$. In this case, Proposition 1 gives us two choices for $AB$: either the entire direct sum above, or one of its isomorphic summands, i.e., the “obvious” composite $AB = C_{nm}$.

### 4 Embedded EJAs

Corollary 1 above justifies restricting our attention to special EJAs (often called Euclidean JC-algebras). In fact, it will be helpful to consider embedded EJAs, that is, Jordan subalgebras of specified (finite-dimensional) $C^*$ algebras.

**Definition 2:** An embedded JC algebra, or EJC, is a pair $(A, M_A)$ where $A$ is a Jordan subalgebra of a finite-dimensional complex $*$-algebra $M_A$.

The notation $M_A$ is intended to emphasize that the embedding $A \hookrightarrow M_A$ is part of the structure of interest. Given $A$, there is always a canonical choice for $M_A$, namely the universal enveloping $*$-algebra $C^*(A)$ of $A$ [9].

**Definition 3:** The canonical product of EJCs $(A, M_A)$ and $(B, M_B)$ is the pair $(A \odot B, M_A \otimes M_B)$ where $A \odot B$ is the Jordan subalgebra of $(M_A \otimes M_B)_\text{sa}$ generated by the subspace $A \otimes B$.

Note that, as a matter of definition, $M_{A \odot B} = M_A \otimes M_B$. If $M_A = C^*(A)$ and $M_B = C^*(B)$, then $A \odot B$ is the Hanche-Olsen tensor product.

One would like to know that $A \odot B$ is in fact a composite of $A$ and $B$ in the sense of Definition 1. Using a result of Upmeier [12], we can show that this is the case for reversible EJAs $A$ and $B$. (that is, real, complex and quaternionic systems, and direct sums of these). Whether $A \odot B$ is a composite in the sense of Definition 1 when $A$ or $B$ is non-reversible spin factor remains an open question.

We can now form a category:

**Definition 4:** EJC is the category consisting of EJCs $(A, M_A)$ and completely positive maps $\phi : M_A \to M_B$ with $\phi(A) \subseteq B$. We refer to such maps as Jordan preserving.

**Proposition 2:** The canonical product $\odot$ is associative on EJC. More precisely, the associator mapping

$$\alpha : M_A \otimes (M_B \otimes M_C) \to (M_A \otimes M_B) \otimes M_C$$

takes $A \odot (B \odot C)$ to $(A \odot B) \odot C$.

(Note that since the associator mapping is CP, this means that $\alpha$ is a morphism in EJC.) The proof is somewhat lengthy, so we refer the reader to the forthcoming paper [4].

Proposition 2 suggests that EJC might be symmetric monoidal under $\odot$. There is certainly a natural choice for the monoidal unit, namely $I = (\mathbb{R}_+, \mathbb{C})$. But the following example shows that tensor products of EJC morphisms are generally not morphisms:

**Example 1:** Let $(A, C^*(A))$ and $(B, C^*(B))$ be simple, universally embedded EJCs, and suppose that $B$ is not UR (e.g., a spin factor $V_n$ with $n > 3$). Let $\tilde{B}$ be the set of fixed points of the canonical involution $\Phi_B$. Then by Corollary 2, $A \odot B = A \tilde{\otimes} B$, the set of fixed points of $\Phi_A \otimes \Phi_B$. In particular, $u_A \otimes \tilde{B}$ is contained
in $A \odot B$. Now let $f$ be a state on $C^*(A)$: this is CP, and trivially Jordan-preserving, and so, a morphism in $\mathbf{EJC}$. But

$$(f \otimes \text{id}_B)(u_A \otimes \hat{B}) = f(u_A)\hat{B} = \hat{B},$$

which isn’t contained in $B$. So $f \otimes \text{id}_B$ isn’t Jordan-preserving.

## 5 Reversible and universally reversible $\mathbf{EJC}$s

It seems that the category $\mathbf{EJC}$ is simply too large. We can try to obtain a better-behaved category by restricting the set of morphisms, or by restricting the set of objects, or both.

As a first pass, we might try this:

**Definition 5:** Let $(A,M_A)$ and $(B,M_B)$ be EJCs. A linear mapping $\phi : M_A \to M_B$ is completely Jordan preserving (CJP) iff $\phi \otimes 1_C$ takes $A \odot C$ to $B \odot C$ for all $(C,M_C)$.

It is not hard to verify the following

**Proposition 3:** If $\phi : M_A \to M_B$ and $\psi : M_C \to M_D$ are CJP, then so is $\phi \otimes \psi : M_A \odot C = M_A \otimes M_B \to M_B \otimes M_C = M_B \odot D$.

Thus, the category of EJC algebras and CJP mappings is symmetric monoidal.\[4\]

There are many examples: Jordan homomorphisms are CJP maps. If $a \in A$, the mapping

$$U_a : A \to A$$

given by $U_a = 2L_a^2 - L_a^2$, where $L_a(b) = ab$, is also CJP. On the other hand, by Example 1 above, $\text{CJP}(A,I)$ is empty for universally embedded simple $A$!

So not all CP maps are CJP; for instance, states are never CJP. More seriously, we can’t interpret states as morphisms in this category. The problem is the non-UR spin factors in $\text{CJP}$. If we remove these, things are much better.

**Definition 6:** Let $\mathscr{C}$ be a subclass of embedded EJC algebras, closed under $\odot$ and containing $I$. A linear mapping $\phi : M_A \to M_B$ is CJP relative to $\mathscr{C}$ iff $\phi \otimes \text{id}_C$ is Jordan preserving for all $C$ in $\mathscr{C}$. $\text{CJP}_\mathscr{C}$ is the category having objects elements of $\mathscr{C}$, mappings relatively CJP mappings.

**Example 2:** $\mathbf{URUE}$ is the class of universally reversible, universally embedded EJC algebras. $\mathbf{URSE}$ is the category of universally reversible, standardly embedded EJC algebras, and $\mathbf{RSE}$ is the category of reversible, standardly embedded EJC algebras. Equipped with relatively CJP mappings, both are symmetric monoidal categories.

Note that $\mathbf{RSE}$ consists of direct sums of real, complex and quaternionic quantum systems. $\mathbf{URSE}$ and $\mathbf{URUE}$ contain all real and complex quantum systems, and all quaternionic quantum systems except the “quabit”, i.e., the quaternionic bit $Q_2 := M_2(\mathbb{H})_{sa}$.

\[4\] Notice that scalars of this category are real numbers. It is sometimes suggested that quaternionic Hilbert spaces can’t be accommodated in a symmetric monoidal category owing to the noncommutativity of $\mathbb{H}$, as the scalars in a symmetric monoidal category must always be commutative. As we are representing quaternionic quantum systems in terms of the associated real vector spaces of hermitian operators, this issue doesn’t arise here.
In both of the categories $\text{URUE}$ and $\text{URSE}$, states are morphisms. In fact, we are going to see that $\text{URUE}$ and $\text{URSE}$ inherit compact closure from the category $\text{*-ALG}$ of finite-dimensional, complex $\text{*}$-algebras and CP maps, in which they are embedded.

It’s worth taking a moment to review this compact structure. If $M$ is a finite-dimensional complex $\text{*}$-algebra, let $\text{Tr}$ denote the canonical trace on $M$, regarded as acting on itself by left multiplication (so that $\text{Tr}(a) = \text{tr}(L_a)$, $L_a : M \rightarrow M$ being $L_a(b) = ab$ for all $b \in M$). This induces an inner product on $M$, given by $\langle a| b \rangle_M = \text{Tr}(ab^*)$.

Note that this inner product is self-dualizing, i.e., $a \in M_+$ iff $\langle a| b \rangle \geq 0$ for all $b \in M_+$.

Now let $\overline{M}$ be the conjugate algebra, writing $\overline{a}$ for $a \in M$ when regarded as belonging to $\overline{M}$ (so that $\overline{ca} = \overline{c} \overline{a}$ for scalars $c \in \mathbb{C}$ and $\overline{ab} = \overline{a} \overline{b}$ for $a, b \in M$). Note that $\langle \overline{a}| \overline{b} \rangle = \langle b| a \rangle$. Now define

$$f_M = \sum_{e \in E} e \otimes \overline{e} \in M \otimes \overline{M}$$

where $E$ is any orthonormal basis for $M$ with respect to $\langle \cdot | \cdot \rangle_M$. Then a computation shows that

$$\langle (a \otimes \overline{b}) f_M | f_M \rangle_{M \otimes \overline{M}} = \langle a| b \rangle_M.$$

Since the left-hand side defines a positive linear functional on $M \otimes \overline{M}$, so does the right (remembering here that pure tensors generate $M \otimes \overline{M}$, as we’re working in finite dimensions). Call this functional $\eta_M$. That is,

$$\eta_M : M \otimes \overline{M} \rightarrow \mathbb{C}$$

is given by $\eta_M(a \otimes \overline{b}) = \langle a| b \rangle = \text{Tr}(ab^*)$

and is, up to normalization, a state on $M \otimes \overline{M}$. A further computation now shows that

$$\langle a \otimes \overline{b} | f_M \rangle_{M \otimes \overline{M}} = \eta(a \otimes \overline{b}).$$

It follows that $f_M$ belongs to the positive cone of $M \otimes \overline{M}$, by self-duality of the latter. A final computation shows that, for any states $\alpha$ and $\overline{\alpha}$ on $M$ and $\overline{M}$, respectively, and any $a \in M, \overline{a} \in \overline{M}$, we have

$$(\eta_M \otimes \alpha)(a \otimes f_M) = \alpha(a) \quad \text{and} \quad (\overline{\alpha} \otimes \eta_M)(f_M \otimes \overline{a}) = \overline{\alpha}(\overline{a}).$$

Thus, $\eta_M$ and $f_M$ define a compact structure on $\text{*-ALG}$, for which the dual object of $M$ is given by $\overline{M}$.

**Definition 7:** The conjugate of a EJC algebra $(A,M_A)$ is $(\overline{A},\overline{M_A})$, where $\overline{A} = \{\overline{a} : a \in A\}$. We write $\eta_A$ for $\eta_{M_A}$ and $f_A$ for $f_{M_A}$.

### 5.1 Universally-embedded, universally reversible EJAs

Now consider the category $\text{URUE}$ of universally reversible, universally embedded EJAs $A$, i.e., pairs $(A,M_A)$ with $A$ UR and $M_A = C^*(A)$. Let $\Phi_A$ be the canonical involution on $C^*(A)$.

**Lemma 1:** Let $(A,M_A)$ belong to $\text{URUE}$. Then

$$(a) \quad f_A \in A \otimes \overline{A};$$

$$(b) \quad \eta_A \circ (\Phi_A \otimes \Phi_{\overline{A}}) = \eta_A.$$

**Proof:** (a) follows from the fact that $\Phi_A$ is unitary, so that if $E$ is an orthonormal basis, then so is $\{\Phi_A(e) : e \in E\}$. Since $f_A$ is independent of the choice of orthonormal basis, it follows that $f$ is invariant under $\Phi_A \otimes \Phi_{\overline{A}}$, hence, an element of $A \otimes \overline{A}$. Now (b) follows from part (a) of the previous lemma. □
Define $\gamma_A : C^*(\overline{A}) \to C^*(A)$ by $\gamma_A(a) = \Phi_A(a^*)$. This is a $*$-isomorphism, and intertwines $\Phi_A$ and $\Phi_A$; hence, $\gamma_A \otimes \text{id}_B : C^*(\overline{A} \otimes B) \to C^*(A \otimes B)$ intertwines $\Phi_{A \otimes B} = \Phi_A \otimes \text{id}_B$ and $\Phi_A \otimes \Phi_B = \Phi_{A \otimes B}$ — whence, takes $\overline{A} \otimes B$ to $A \otimes B$. In particular, $\gamma_A$ is CJP relative to the class of UR, universally embedded EJCs.

**Lemma 2:** Let $A$ be a universally embedded UR EJC. Then for all $\alpha \in \text{CJP}(A, I)$, there exists $a \in A$ with $\alpha(b) = \langle b|a \rangle$ for all $b \in A$.

**Proof:** Since $\alpha \in C^*(A)^*$, there is certainly some $a \in C^*(A)$ with $\alpha = |a\rangle$. We need to show that $a \in A$. Since $\alpha$ is CJP,

$$\gamma_A \otimes \alpha : C^*(A) \otimes C^*(A) = C^*(A \otimes A) \to C^*(A)$$

is Jordan-preserving. In particular, $(\alpha \otimes \gamma_A)(f_A) \in A$. But

$$\begin{align*}
(\alpha \otimes \gamma_A)(f_A) &= \sum_{e \in E} (\alpha \otimes \gamma_A)(e \otimes \overline{e}) \\
&= \sum_{e \in E} \langle e|a \rangle \Phi(e^*) \\
&= \Phi(\sum_{e \in E} \langle e|a \rangle e^*) \\
&= \Phi(\langle \sum_{e \in E} \langle a|e \rangle e^* \rangle) = \Phi(a^*) = \gamma_A(a).
\end{align*}$$

Hence, $\gamma_A(a) \in A$, whence, $\overline{a} \in \overline{A}$, whence, $a \in A$. (Alternatively: $\Phi_A(a^*) \in A$ implies $a^* \in A$, whence, $a$ is self-adjoint, whence, $a \in A$.) $\square$

It follows that $\eta_A$ and $f_A$ belong, as morphisms, to URUE. Hence, URUE inherits the compact structure from $\text{*-ALG}$, as promised.

The same holds for URSE. Specifically, we want to show that $f_A$ belongs to $A \odot \overline{A}$ whenever $A$ is a standardly embedded UR EJC.

Suppose that $E$ is an orthonormal basis for $M_A$: then so is $\{e^*|e \in E\}$; thus, since $f_A$ is independent of the choice of basis, we have

$$f_A^* = \sum_{e \in E} e^* \otimes \overline{e}^* = \sum_{e^* \in E^*} e^* \otimes \overline{e^*} = f_A.$$

Thus, if $A \odot \overline{A}$ is the self-adjoint part of $M_A \otimes \overline{M_A}$, then $f_A \in A \odot \overline{A}$. This covers the case where $A = C_n$. We also have, by the results above, that $f_A \in A \odot \overline{A}$ whenever the latter equals $A \odot \overline{A}$. This covers $A = R_n$ and $A = Q_n$ for $n > 2$.

In fact, we can do a bit better. If $M$ and $N$ are finite-dimensional $\ast$-algebras and $\phi : M \to N$ is a linear mapping, let $\phi^\dagger$ denote the adjoint of $\phi$ with respect to the natural trace inner products on $M$ and $N$. It is not difficult to show that, for any $M$ in $\text{-ALG}$, $f_M^\dagger = \eta_M$ and vice versa; indeed, $\text{-ALG}$ is dagger-compact.

**Definition 8:** Let $(A, M_A)$ and $(B, M_B)$ be EJCs. A linear mapping $\phi : M_A \to M_B$ is $\dagger$-CJP if both $\phi$ and $\phi^\dagger$ are CJP. If $\mathcal{C}$ is a category of EJCs and CJP mappings, we write $\mathcal{C}^\dagger$ for the category having the same objects, but with morphisms restricted to $\dagger$-CJP mappings in $\mathcal{C}$.

If $A$ belongs to URUE or URSE, then $f_A$ and $\eta_A$ are both CJP and, hence, are both $\dagger$-CJP with respect to the indicated category. Hence,

**Theorem 2:** The categories URUE$^\dagger$ and URSE$^\dagger$ are dagger-compact.
6 Conclusion

We have found two theories — the categories URSE and URUE — that, in slightly different ways, unify finite-dimensional real, complex and (almost all of) quaternionic quantum mechanics. By virtue of being compact closed, both theories continues to enjoy many of the information-processing properties of standard complex QM, e.g., the existence of conclusive teleportation and entanglement-swapping protocols [1].

It is worth pointing out that the composites in our categories are not “locally tomographic”, i.e, a state $\omega$ on $A \otimes B$ is not generally determined by the joint probability assignment $a, b \mapsto \omega(a \otimes b)$, where $a$ and $b$ are effects of $A$ and $B$, respectively. Another way to put it is that $A \otimes B$ is generally much larger than the vector-space tensor product $A \otimes B$. (As local tomography is well known to separate complex QM from its real and quaternionic variants, this is hardly surprising.)

Neither theory includes the quabit, $Q_2$. Example 1 shows that the $Q_2$ can’t be added to URUE without a violation of compact closure. On the other hand, if $f_{Q_2}$ belongs to the canonical composite $Q_2 \otimes Q_2$, then the slightly larger category RSE, which consists of all finite-dimensional real, complex and quaternionic quantum systems, will be compact closed (indeed, dagger compact).[6]

The categories URSE and URUE contain interesting compact closed subcategories. In particular, real and quaternionic quantum systems (less the quabit), taken together, form a sub-theory, closed under composites. We conjecture that this is exactly what one gets by applying the CPM construction to Baez’ (implicit) category of pairs $(H, J)$, $H$ a finite-dimensional Hilbert space and $J$ an anti-unitary with $J^2 = \pm 1$ — and, again, excluding the quabit. Should RSE prove to be compact closed, we could entertain the stronger conjecture that this is exactly what one obtains by applying CPM to Baez’ category.

Complex quantum systems also form a monoidal subcategory of URSE, which we might call CQM: indeed, one that functions as an “ideal”, in that if $A \in$ URSE and $B \in$ CQM, then $A \otimes B \in$ CQM as well. This is provocative, as it suggests that a universe initially consisting of many systems of all three types, would eventually evolve into one in which complex systems greatly predominate.

The category URUE is somewhat mysterious. Like URSE, this encompasses real, complex and quaternionic quantum systems, excepting the quabit. In this theory, the composite of complex quantum systems comes with an extra classical bit — equivalently, a $\{0, 1\}$-valued superselection rule. This functions to make the transpose operation — which is a Jordan automorphism of $M_n(\mathbb{C})_{sa}$, but an antiautomorphism of $M_n(\mathbb{C})$ — count as a morphism. The precise physical significance of this is a subject for further study.

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6Since this Abstract was first submitted, we believe we have settled this question in the affirmative. The details will appear elsewhere.
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