Operational axioms for diagonalizing states

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The importance of thermodynamics

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- It gave rise to foundational puzzles, related to irreversibility.

**Figure**: Maxwell’s demon. Source: wikimedia commons
The need for an information-theoretic foundation of thermodynamics

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- Important for nanotechnology: systems at the nanoscale.
- Relationship between thermodynamics and information theory (Landauer, etc.)
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Need for information-theoretic principles!

Method

Thermodynamics in GPTs!
The tool of majorization

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Majorization

Let $\mathbf{p}, \mathbf{p}' \in \mathbb{R}^n$ be two probability distributions. We say that $\mathbf{p}$ is majorized by $\mathbf{p}'$ ($\mathbf{p} \preceq \mathbf{p}'$) if

$$
\sum_{i=1}^{k} p[i] \leq \sum_{i=1}^{k} p'[i] \quad \text{for } i = 1, \ldots, n-1,
$$

where $p[i]$ is the $i$-th entry of the decreasing rearrangement of $\mathbf{p}$. 

It gives a preorder of quantum states based on their eigenvalues.
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Cf. also the next talk by Barnum et al.! (from a different angle)
Section 1

Framework and axioms
OPTs

We use a specific variant of GPTs, known as OPTs (operational-probabilistic theories).
[Chiribella et al. ’10, Chiribella et al. ’11]
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Circuits such as

- $\rho$ is a state (a transformation with no input)
- $a$ and $b$ are effects (transformations with no output)
A transformation $\mathcal{U} : A \rightarrow B$ is called reversible if there exists a transformation $\mathcal{U}^{-1} : B \rightarrow A$ such that $\mathcal{U}^{-1}\mathcal{U} = \mathcal{I}_A$, and $\mathcal{U}\mathcal{U}^{-1} = \mathcal{I}_B$, where $\mathcal{I}_S$ is the identity on system $S$. 

\[
\begin{align*}
\begin{array}{ccc}
A & \xrightarrow{\mathcal{U}} & B \\
& \xrightarrow{\mathcal{U}^{-1}} & A \\
\end{array}
& = 
\begin{array}{c}
A
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{ccc}
B & \xrightarrow{\mathcal{U}^{-1}} & A \\
& \xrightarrow{\mathcal{U}} & B \\
\end{array}
& = 
\begin{array}{c}
B
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\end{align*}
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Circuits with no external wires represent probabilities

\[(a_i | \rho_j) := \rho_j A a_i = p_{ij} \in [0, 1].\]
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\[(a_i|\rho_j) := \rho_j \stackrel{A}{\longrightarrow} a_i = p_{ij} \in [0, 1].\]

This induces a sum for transformations.
Probabilistic structure & purity

- Circuits with no external wires represent probabilities

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Circuits with no external wires represent **probabilities**

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- This induces a sum for transformations.
- We define real vector spaces spanned by states and effects. We assume they are **finite-dimensional**.
- We can define coarse-graining and purity.

**Purity**

A transformation \(T\) is **pure** if \(T = \sum_i T_i\) implies \(T_i = p_i T\), where \(\{p_i\}\) is a probability distribution.
Purity Preservation

Purity Preservation [Chiribella & Scandolo ’15a]
The sequential and parallel composition of pure transformations is a pure transformation.
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- The product of two pure states is pure.
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- The product of two pure states is pure.
- Without Purity Preservation, we may have a “non-local” loss of information when composing transformations.
Causality [Chiribella et al. ’10, Chiribella et al. ’11]

The outcome probabilities of present experiments are not affected by the choice of future measurements.
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- Equivalently, for every system $A$ there is a unique deterministic effect $\text{Tr}_A$. 
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The outcome probabilities of present experiments are not affected by the choice of future measurements.

- Equivalently, for every system $A$ there is a *unique* deterministic effect $\text{Tr}_A$.
- We can use $\text{Tr}$ to define the *marginals* of bipartite states:

$$
\rho_A := \text{Tr}_B \rho_{AB} = \rho
$$

Important in thermodynamics: we need to restrict ourselves to subsystems!
Purification
[Chiribella et al. ’10, Chiribella et al. ’11]

Every state $\rho_A$ can be purified: there exists a pure state $\Psi_{AB}$ such that $\rho_A = \Psi_{A}^{T_B}$.

Different purifications of the same state differ by a reversible transformation on the purifying system: $\Psi_{AB}^{T_B} = \Psi_{AB}'^{T_B} \Rightarrow \rho_A = \rho_A'$. 
Every state $\rho_A$ can be purified: there exists a pure state $\Psi_{AB}$ such that

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Why Purification?

- It reconciles partial information and irreversibility with a picture where everything is **pure** and **reversible**.
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- It provides a **formal justification** of the thermodynamic procedure of enlarging a system to deal with an **isolated** system.
Why Purification?

- It reconciles partial information and irreversibility with a picture where everything is pure and reversible.
- Dilation and extension theorems can be reconstructed from it [Chiribella et al. ’10].
- It provides a formal justification of the thermodynamic procedure of enlarging a system to deal with an isolated system.

Purification is a good starting point for a theory of thermodynamics.
Pure Sharpness

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- We can think of \( a \) as part of a yes/no test to check an elementary property of the system.
- Pure Sharpness ensures that every system has an elementary property.
Duality pure states-pure effects: for every pure state $\alpha$ there is a unique pure effect $\alpha^\dagger$ such that $(\alpha^\dagger|\alpha) = 1$. 
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Existence of perfectly distinguishable (pure) states.

Perfectly distinguishable states

The states $\{\rho_i\}_{i \in X}$ are said perfectly distinguishable if there exists a measurement $\{a_j\}_{j \in X}$ such that $(a_j|\rho_i) = \delta_{ij}$. 
Section 2

Diagonalization
Diagonalizing states

Diagonalization

A diagonalization of a state $\rho$ is a convex decomposition of $\rho$ into perfectly distinguishable pure states.

$$\rho = \sum_i p_i \alpha_i$$

The $p_i$’s are called eigenvalues of the diagonalization.
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Define

$$p_* := \max_{\alpha \text{ pure}} \{ p \in (0, 1] : \rho = p\alpha + (1-p)\sigma \}.$$
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- Define

$$p_* := \max_{\alpha \text{ pure}} \{ p \in (0, 1] : \rho = p\alpha + (1 - p)\sigma \}.$$

- We have $(\alpha^\dagger|\rho) = p_*$, whence $(\alpha^\dagger|\sigma) = 0$, and $(\alpha^\dagger|\psi) = 0$ for any pure state $\psi$ contained in $\sigma$. 
The diagonalization algorithm

Consider a state $\rho$. 

1. Determine $p_1^* =: q_1$ and find $\alpha_1 =: \alpha_1^{\text{pure}}$, such that $\rho = q_1 \alpha_1 + (1 - q_1) \sigma_1$.

2. Repeat the same procedure for $\sigma_1$: find the maximum probability $q_2$ such that $\sigma_1 = q_2 \alpha_2 + (1 - q_2) \sigma_2$, with $\alpha_2 = \alpha_2^{\text{pure}}$.

3. Iterate the procedure. At the end, $\rho = \sum_{i=1}^{n} p_i \alpha_i$, where $p_1 = q_1$, and $p_i = q_i \prod_{j < i} (1 - q_j)$ for $i > 1$. 

$(\alpha_i | \alpha_j) = 0$ for $j > i$. 

G. Chiribella, C. M. Scandolo, Operational axioms for diagonalizing states
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2. Repeat the same procedure for $\sigma_1$: find the maximum probability $q_2$ such that $\sigma_1 = q_2 \alpha_2 + (1 - q_2) \sigma_2$, with $\alpha_2$ \textit{pure}.
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- $p_1 := q_1$, and $p_i := q_i \prod_{j<i} (1 - q_j)$ for $i > 1$.
- $(\alpha_i^\dagger | \alpha_j) = 0$ for $j > i$
We want to prove the $\alpha_i$’s are perfectly distinguishable.
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Preliminary result (from Purification) [Chiribella et al. ’11]

We can associate a test $\{A_i\}_{i \in X}$ made of transformations with a measurement $\{a_i\}_{i \in X}$ made of effects, where the $A_i$’s occur with the same probability as the $a_i$’s. Moreover, if $\langle a | \rho \rangle = 1$, then the associated transformation $A$ doesn’t disturb $\rho$. 
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Moreover, if $(a|\rho) = 1$, then the associated transformation $A$ doesn’t disturb $\rho$.

Effects destroy a system, but we can iterate the perfectly distinguishing test by using transformations!
Proving the $\alpha_i$'s are perfectly distinguishable

Consider the pure states $\{\alpha_i\}_{i=1}^n$, with $\left(\alpha_i^\dagger|\alpha_j\right) = 0$ for $j > i$. 
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Consider the pure states $\{\alpha_i\}_{i=1}^n$, with $(\alpha_i^\dagger|\alpha_j) = 0$ for $j > i$.

1. Consider the measurement $\{\alpha_1^\dagger, \text{Tr} - \alpha_1^\dagger\}$. Apply the associated test $\{A_1, A_1^\perp\}$. If $A_1$ occurs, the state is $\alpha_1$. If not, the state is one of the others.
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2. Consider $\rho_1 = \frac{1}{n-1} \sum_{i=2}^n \alpha_i$. Since $\left(\text{Tr} - \alpha_1^\dagger | \rho_1\right) = 1$, $A_1^\perp$ does not disturb the states $\{\alpha_i\}_{i=2}^n$. Now repeat the procedure with the measurement $\{\alpha_2^\dagger, \text{Tr} - \alpha_2^\dagger\}$ and the remaining states.
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In the end, we’re able to identity the state with certainty! The $\alpha_i$’s are perfectly distinguishable!
Conclusions and further developments

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- All diagonalizations of a given state have the same eigenvalues ([forthcoming paper with G. Chiribella]).
- We can define majorization and Schur-concave functions (entropies!) [Scandolo ’14]
- Adding the requirement that reversible transformations act transitively on maximal sets of perfectly distinguishable pure states (cf. [Barnum et al. ’14]), the preorder of states given by majorization is equivalent to the one given by random reversible transformations in the GPT-version of the resource theory of purity. [Chiribella & Scandolo ’15b]
References


G Chiribella, CMS, EPJ Web of Conferences 95, 03003 (2015).

