Operational axioms for diagonalizing states

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- It gave rise to foundational puzzles, related to irreversibility.

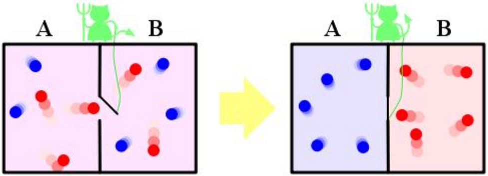

Figure: Maxwell's demon. Source:wikimedia commons

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Need for information-theoretic principles!

Method

Thermodynamics in GPTs!

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Let $\mathbf{p}, \mathbf{p}' \in \mathbb{R}^n$ be two probability distributions. We say that $\mathbf{p}$ is *majorized* by $\mathbf{p}'$ ($\mathbf{p} \preceq \mathbf{p}'$) if

$$\sum_{i=1}^{k} p_{[i]} \leq \sum_{i=1}^{k} p'_{[i]} \quad \text{for } i = 1, \ldots, n-1,$$

where $p_{[i]}$ is the i-th entry of the decreasing rearrangement of $\mathbf{p}$.

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It gives a preorder of quantum states based on their eigenvalues.

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Cf. also the next talk by Barnum et al.! (from a different angle)

Contents

Section 1

Framework and axioms

OPTs

We use a specific variant of GPTs, known as OPTs (operational-probabilistic theories).
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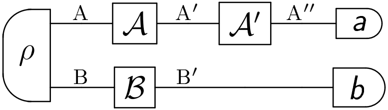
Circuits such as

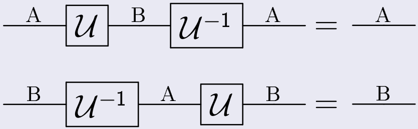


- A, B, etc. are systems
- $\mathcal{A}$, $\mathcal{B}$, etc. are transformations: they can be composed in sequence (e.g. $\mathcal{A}$ and $\mathcal{A}'$) or in parallel (e.g. $\mathcal{A}$ and $\mathcal{B}$)
- ρ is a state (a transformation with *no* input)
- a and b are effects (transformations with *no* output)

Reversible transformations

Reversible transformations

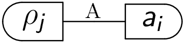
A transformation $\mathcal{U} : \mathrm{A} \to \mathrm{B}$ is called reversible if there exists a transformation $\mathcal{U}^{-1} : \mathrm{B} \to \mathrm{A}$ such that $\mathcal{U}^{-1}\mathcal{U} = \mathcal{I}_{\mathrm{A}}$, and $\mathcal{U}\mathcal{U}^{-1} = \mathcal{I}_{\mathrm{B}}$, where $\mathcal{I}_{\mathrm{S}}$ is the identity on system S.



Probabilistic structure & purity

- Circuits with no external wires represent probabilities

$$(a_i|\rho_j) \;:=$$



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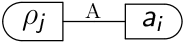

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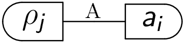

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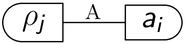
$$(a_i|\rho_j) :=$$ 

$$= p_{ij} \in [0,1] .$$

- This induces a sum for transformations.
- We define real vector spaces spanned by states and effects. We assume they are finite-dimensional.
- We can define coarse-graining and purity.

Purity

A transformation $\mathcal{T}$ is pure if $\mathcal{T} = \sum_i \mathcal{T}_i$ implies $\mathcal{T}_i = p_i \mathcal{T}$, where $\{p_i\}$ is a probability distribution.

Purity Preservation

Purity Preservation [Chiribella & Scandolo '15a]

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The sequential and parallel composition of pure transformations is a pure transformation.

- The product of two pure states is pure.
- Without Purity Preservation, we may have a "non-local" loss of information when composing transformations.

Causality

Causality [Chiribella et al. '10, Chiribella et al. '11]

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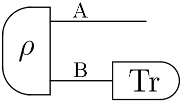
The outcome probabilities of present experiments are not affected by the choice of future measurements.

- Equivalently, for every system A there is a *unique* deterministic effect Tr_{A}.
- We can use Tr to define the marginals of bipartite states:

$$\rho_{\mathrm{A}} := \mathrm{Tr}_{\mathrm{B}} \rho_{\mathrm{AB}} =$$



Important in thermodynamics: we need to restrict ourselves to subsystems!

Purification [Chiribella et al. '10, Chiribella et al. '11]

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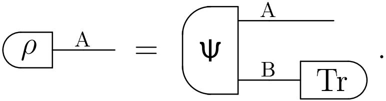
1. Every state ρ_A can be purified: there exists a pure state Ψ_{AB} such that



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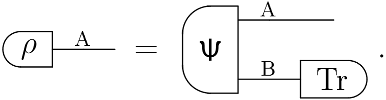
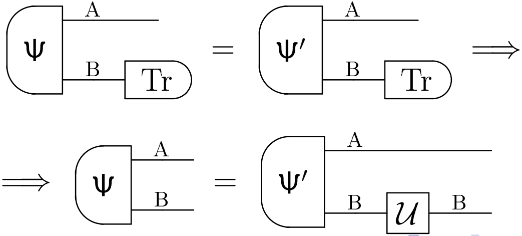


2. Different purifications of the same state differ by a reversible transformation on the purifying system:



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- It provides a formal justification of the thermodynamic procedure of enlarging a system to deal with an isolated system.

Purification is a good starting point for a theory of thermodynamics.

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- We can think of a as part of a yes/no test to check an elementary property of the system.
- Pure Sharpness ensures that every system has an elementary property.

Consequences of Pure Sharpness (+ Purification)

1. Duality pure states-pure effects: for every *pure* state α there is a unique *pure* effect $\alpha^\dagger$ such that $(\alpha^\dagger|\alpha) = 1$.

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2. Existence of perfectly distinguishable (pure) states.

Perfectly distinguishable states

The states $\{\rho_i\}_{i\in\mathsf{X}}$ are said *perfectly distinguishable* if there exists a measurement $\{a_j\}_{j\in\mathsf{X}}$ such that $(a_j|\rho_i) = \delta_{ij}$.

Section 2

Diagonalization

Diagonalizing states

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$$\rho = \sum_i p_i \alpha_i$$

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$$p_* := \max_{\alpha \text{ pure}} \left\{ p \in (0,1] : \rho = p\alpha + (1-p)\,\sigma \right\}.$$

- We have $\left(\alpha^\dagger | \rho\right) = p_*$, whence $\left(\alpha^\dagger | \sigma\right) = 0$, and $\left(\alpha^\dagger | \psi\right) = 0$ for any pure state ψ contained in σ.

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Preliminary result (from Purification) [Chiribella et al. '11]

We can associate a test $\{\mathcal{A}_i\}_{i\in\mathsf{X}}$ made of transformations with a measurement $\{a_i\}_{i\in\mathsf{X}}$ made of effects, where the $\mathcal{A}_i$'s occur with the same probability as the a_i's.
Moreover, if $(a|\rho) = 1$, then the associated transformation $\mathcal{A}$ doesn't disturb ρ.

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Effects destroy a system, but we can iterate the perfectly distinguishing test by using transformations!

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1. Consider the measurement $\left\{\alpha_1^{\dagger}, \mathrm{Tr} - \alpha_1^{\dagger}\right\}$. Apply the associated test $\left\{\mathcal{A}_1, \mathcal{A}_1^{\perp}\right\}$. If $\mathcal{A}_1$ occurs, the state is α_1. If not, the state is one of the others.

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2. Consider $\rho_1 = \frac{1}{n-1}\sum_{i=2}^{n} \alpha_i$. Since $\left(\mathrm{Tr} - \alpha_1^{\dagger}|\rho_1\right) = 1$, $\mathcal{A}_1^{\perp}$ does not disturb the states $\{\alpha_i\}_{i=2}^{n}$. Now repeat the procedure with the measurement $\left\{\alpha_2^{\dagger}, \mathrm{Tr} - \alpha_2^{\dagger}\right\}$ and the remaining states.

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In the end, we're able to identity the state with certainty! The α_i's are perfectly distinguishable!

Conclusions and further developments

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- All diagonalizations of a given state have the same eigenvalues (forthcoming paper with G. Chiribella).
- We can define majorization and Schur-concave functions (entropies!) [Scandolo '14]
- Adding the requirement that reversible transformations act transitively on maximal sets of perfectly distinguishable pure states (cf. [Barnum et al. '14]), the preorder of states given by majorization is equivalent to the one given by random reversible transformations in the GPT-version of the resource theory of purity. [Chiribella & Scandolo '15b]

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