Categorical Semantics for Schrödinger's Equation

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Introduction

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- Quantum symmetries and their invariant observables.
- Schrödinger's Equation and Eilenberg-Moore morphisms.

Section 1

Representation Theory in CQM

†-Frobenius algebras

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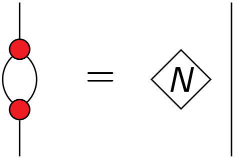

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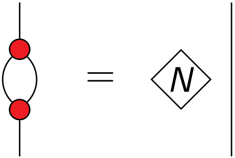

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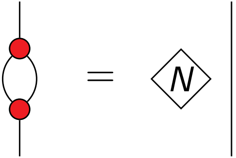
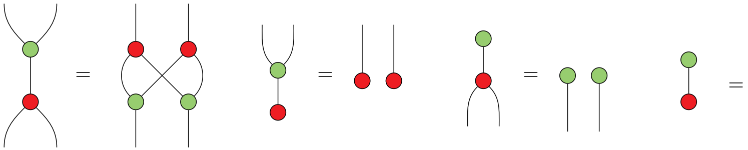


- †-qSFA $\equiv$ "quasi-special †-Frobenius algebra"
- †-qSCFA $\equiv$ "quasi-special commutative †-Frobenius algebra"

Strong Complementarity

We will say that a pair of †-qSFAs are **strongly complementary** if they satisfy the following (unscaled) bialgebra equations:



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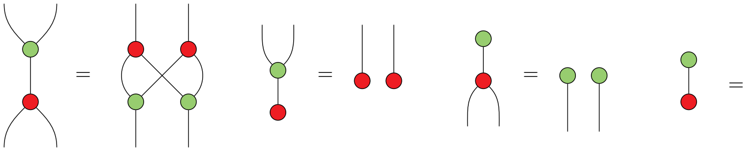


An **internal group** $(\mathcal{G}, \bullet, \circ)$ in a †-SMC consists of two strongly complementary †-qSFA $\bullet$ (the **group structure**) and †-SCFA $\circ$ (the **point structure**), with enough $\circ$-classical points. We say that an internal group is **abelian** if $\bullet$ is commutative.

Group of Classical Points

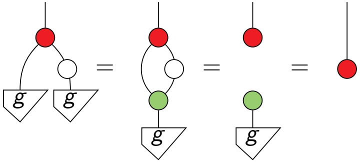
Lemma

Let $(\mathcal{G}, {\color{red}\bullet}, {\color{green}\bullet})$ be an internal group in a †-SMC. Then the monoid ( *,*  *) acts as a group ($K_{\color{green}\bullet}$,*  *,*  *) on the ${\color{green}\bullet}$-classical points (the* **group elements***), with the antipode*  *acting as group inverse.*

Group of Classical Points

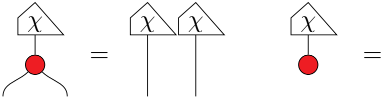
Lemma

Let $(\mathcal{G}, ●, ●)$ be an internal group in a †-SMC. Then the monoid $(⋏, ●)$ acts as a group $(K_{●}, ⋏, ●)$ on the ●-classical points (the **group elements***), with the antipode ϕ acting as group inverse.*



Multiplicative Characters

The **multiplicative characters** for an internal group $(\mathcal{G}, \bullet, \circ)$ in a †-SMC are the co-states $\langle\chi| : \mathcal{G} \to I$ such that:



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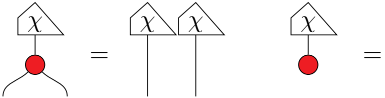

Lemma

The co-monoid (,) acts as a group on the multiplicative characters, with the antipode acting as group inverse.

Resolution of the Identity

- From now on, we work in †-SMCs enriched over finite commutative monoids, with appropriate distributivity laws, e.g. $a \otimes (b + c) = a \otimes b + a \otimes c$, or $a \cdot (b + c) = a \cdot b + a \cdot c$

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- When talking about a **resolution of the identity**, we mean a finite family $|x\rangle_{x \in X}$ of orthogonal, normalisable states s.t.:

$$\sum_{x \in X} \frac{1}{\langle x|x\rangle} |x\rangle\langle x| = id$$

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- When talking about a **partition of a state** $|\psi\rangle$, we mean a finite family $|x\rangle_{x \in X}$ of orthogonal, normalisable states s.t.:

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Let $(\mathcal{G}, \textcolor{red}{\bullet}, \textcolor{green}{\circ})$ *be an internal group. The following are equivalent:*

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$$\frac{1}{N} \sum_{\chi} \langle\chi| = \textcolor{red}{\bullet}$$

Fact: if either one holds, then $\textcolor{red}{\bullet}$ is necessarily commutative.

Multiplicative Characters in fdHilb

- Let $(\mathcal{G}, \bullet, \circ)$ be an abelian internal group in fdHilb

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- Let $(\mathcal{G}, \textcolor{red}{\bullet}, \textcolor{green}{\bullet})$ be an abelian internal group in fdHilb
- Then the multiplicative characters $\langle \chi | : \mathcal{G} \to \mathbb{C}$ are the linear extensions to $\mathcal{G}$ of the multiplicative characters $\chi \in G^{\wedge}$ of the finite abelian group $G = (K_{\textcolor{green}{\bullet}}, \textcolor{red}{\bullet}, \textcolor{red}{\bullet}) \cong \oplus_{x:X} \mathbb{Z}_{N_x}$:

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\begin{aligned}
\langle \chi | \; : \; \mathcal{G} \; &\longrightarrow \; \mathbb{C} \\
|g\rangle \; &\mapsto \; \chi(g)
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$$\begin{aligned} \langle\chi| \;:\; \mathcal{G} &\longrightarrow \mathbb{C} \\ |g\rangle &\mapsto \chi(g) \end{aligned}$$

- If $G \cong \mathbb{Z}_n$, they take the familiar (non-canonical) form:

$$\langle\chi_E|g\rangle = e^{-i\frac{2\pi}{N}E\,g} \quad \text{for } E \in \mathbb{Z}_N$$

Representations and Characters - no time today :(

- For non-abelian internal groups in compact-closed categories, there is a corresponding Lemma stating that representations  $: \mathcal{G} \to \mathcal{H} \otimes \mathcal{H}^{\star}$ form a resolution of the identity if and only if the characters $(\cap_{\mathcal{H}} \cdot$ $)$ form a partition of the co-unit .

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- The entire theory of symmetries and invariants can be developed in fdHilb for arbitrary finite symmetry groups.
- Today we focus on abelian internal groups, and in particular the abelian symmetry group $G = \mathbb{Z}_N$ of finite-dimensional cyclic time evolution

Dynamical Systems (1/2)

Consider an internal group $\mathbb{G} = (\mathcal{G}, \bullet, \bullet)$ in a †-SMC. We define a $\mathbb{G}$-**dynamical system** on a system $\mathcal{H}$ to be an Eilenberg-Moore algebra  $: \mathcal{H} \otimes \mathcal{G} \to \mathcal{H}$ for [the monad induced by] ($\mathcal{G}$,  , ):

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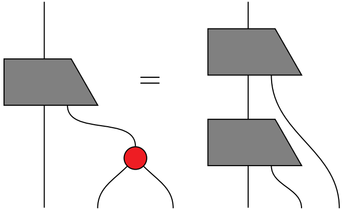

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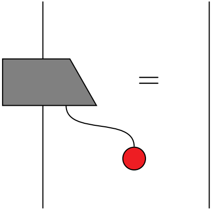

Unitary Dynamical Systems

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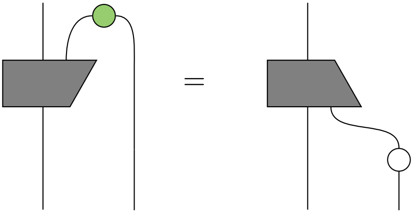

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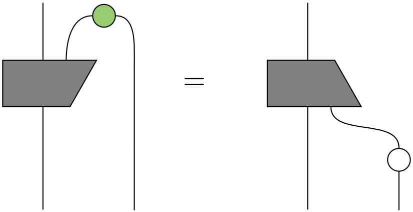

Equivalently, we could ask for it to be a ● -controlled unitary.

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$\Rightarrow$ **Categorical Quantum Dynamics ?**

Section 2

Symmetries and Invariants

Observables (1/3)

Consider a †-qSFA ● on a system $\mathcal{G}$ in a †-SMC. We define a ●-**classical observable** on a system $\mathcal{H}$ to be a self-adjoint Eilenberg-Moore co-algebra  : $\mathcal{H} \to \mathcal{H} \otimes \mathcal{G}$ for ($\mathcal{G}$,  , ):

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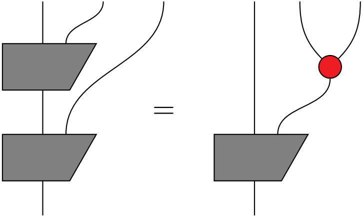

Observables (2/3)

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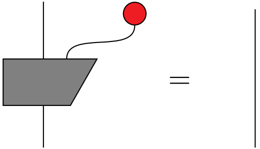

Observables (3/3)

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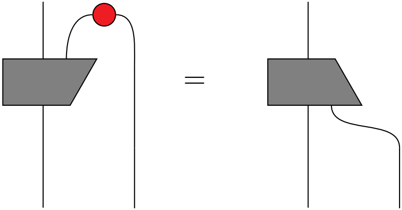

Projector-valued Spectra

Theorem

Let  $: \mathcal{H} \to \mathcal{H} \otimes \mathcal{G}$ *be a*  *-classical observable (in a †-SMC enriched over finite commutative monoids),*

Projector-valued Spectra

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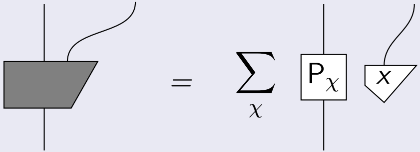
Let  $: \mathcal{H} \to \mathcal{H} \otimes \mathcal{G}$ *be a* ● *-classical observable (in a* †*-SMC enriched over finite commutative monoids), and assume there is a partition of the co-unit* $\frac{1}{N}\sum_{\chi}\langle\chi| =$  *into characters of* ● *.*

Projector-valued Spectra

Theorem

Let $: \mathcal{H} \to \mathcal{H} \otimes \mathcal{G}$ be a ●-classical observable (in a †-SMC enriched over finite commutative monoids), and assume there is a partition of the co-unit $\frac{1}{N}\sum_\chi \langle\chi| =$ ● into characters of ●. Then:



with $(P_\chi : \mathcal{H} \to \mathcal{H})_\chi$ a complete family of self-adjoint idempotents.

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- In particular,  is a (non-degenerate) $\color{red}\bullet$-classical observable.
- If $\mathbb{G} = (\mathcal{G}, \color{red}\bullet\color{black}, \color{green}\bullet\color{black})$ is an abelian internal group, the projectors are indexed by the multiplicative characters (which form a basis).
- If $\mathbb{G} = (\mathcal{G}, \color{red}\bullet\color{black}, \color{green}\bullet\color{black})$ is any internal group, the projectors are indexed by the characters (not a basis, but a matched family).

Dynamics-Observables Duality

Theorem

Let $\mathbb{G} = (\mathcal{G}, \bullet, \bullet)$ be an internal group in a †-SMC. Then a map  $: \mathcal{H} \otimes \mathcal{G} \to \mathcal{H}$ *is a unitary $\mathbb{G}$-dynamical system if and only if*  $:=$  $^\dagger : \mathcal{H} \to \mathcal{H} \otimes \mathcal{G}$ *is a $\bullet$-classical observable.*

We call  † the **Hamiltonian** of the unitary dynamical system .

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- In fdHilb and for $G \cong \mathbb{Z}_N$, a quantum dynamical system  is a family of unitaries $(U^t)_{t \in \mathbb{Z}_N}$, with U the generating unitary.
- The (multiplicative) characters $G^\wedge$ will label the allowed energy levels for the system as $\chi_E(t) = e^{-i\frac{2\pi}{N}Et}$.

Hamiltonians in fdHilb (2/2)

- The projector P_E on the E energy eigenspace is given by:

$$P_E = \frac{1}{N} \sum_{t \in \mathbb{Z}_N} e^{i \frac{2\pi}{N} E\, t} U^t$$

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- This is the same as the projector $(id_{\mathcal{H}} \otimes \langle \chi_E |) \cdot$  †.
- Therefore  † is indeed the CQM observable corresponding to the traditional Hamiltonian for the quantum system.

Symmetries and Invariants

Theorem

Let $\mathbb{G} = (\mathcal{G}, \bullet, \bullet)$ *be an internal group in a* $\dagger$*-SMC and consider a unitary* $\mathbb{G}$*-dynamical system*  $: \mathcal{H} \otimes \mathcal{G} \to \mathcal{H}$*. Then a* $\bullet$*-classical observable*  $: \mathcal{H} \to \mathcal{H} \otimes \mathcal{G}$ *commutes with*  *(it is an* **invariant***) if and only if it commutes with the Hamiltonian* $^\dagger : \mathcal{H} \to \mathcal{H} \otimes \mathcal{G}$

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Let $\mathbb{G} = (\mathcal{G}, \bullet, \bullet)$ be an internal group in a †-SMC and consider a unitary $\mathbb{G}$-dynamical system $: \mathcal{H} \otimes \mathcal{G} \to \mathcal{H}$. Then a $\bullet$-classical observable $: \mathcal{H} \to \mathcal{H} \otimes \mathcal{G}$ commutes with (it is an **invariant***) if and only if it commutes with the Hamiltonian $^\dagger : \mathcal{H} \to \mathcal{H} \otimes \mathcal{G}$*

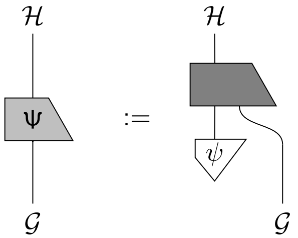
This makes the Hamiltonian † the most general invariant for .

Section 3

Schrödinger's Equation

Orbits

Let $\mathbb{G} = (\mathcal{G}, \bullet, \bullet)$ be an internal group in a $\dagger$-SMC and consider a unitary $\mathbb{G}$-dynamical system  : $\mathcal{H} \otimes \mathcal{G} \to \mathcal{H}$. The **orbit** of state $|\varphi\rangle : I \to \mathcal{H}$ under  is the following morphism $\Psi : \mathcal{G} \to \mathcal{H}$:



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Let $\mathbb{G} = (\mathcal{G}, \bullet, \circ)$ be an internal group in a †-SMC and consider a unitary $\mathbb{G}$-dynamical system  $: \mathcal{H} \otimes \mathcal{G} \to \mathcal{H}$. *The orbits of states are exactly the Eilenberg-Moore morphisms*  $\to$ .

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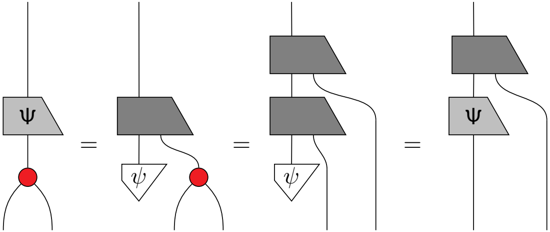
Let $\mathbb{G} = (\mathcal{G}, {\color{red}\bullet}, {\color{green}\bullet})$ *be an internal group in a* $\dagger$*-SMC and consider a unitary* $\mathbb{G}$*-dynamical system*  $: \mathcal{H} \otimes \mathcal{G} \to \mathcal{H}$*. The orbits of states are exactly the Eilenberg-Moore morphisms*  $\to$ *.*

Proof. [Orbit $\Rightarrow$ EM morphism]



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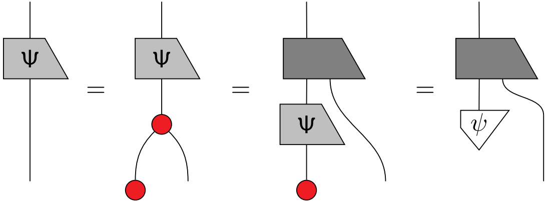
Proof. [EM morphism ⇒ orbit]



Hamiltonian Eigenstates

Theorem

Let $\mathbb{G} = (\mathcal{G}, \bullet, \bullet)$ be an internal group in a $\dagger$-SMC and consider a unitary $\mathbb{G}$-dynamical system  $: \mathcal{H} \otimes \mathcal{G} \to \mathcal{H}$*. A state $|\psi_\chi\rangle$ is an eigenstate of the Hamiltonian with eigenvalue χ, i.e.*

 $$^{\dagger} \cdot |\psi_\chi\rangle = |\psi_\chi\rangle \otimes |\chi\rangle$$

if and only if it is in the form $|\psi_\chi\rangle = \Psi \cdot |\chi\rangle$ for an orbit $\Psi : \mathcal{G} \to \mathcal{H}$

Schrödinger's Equation in fdHilb (1/2)

- The time-dependent Schrödinger's equation is written as:

$$i\hbar\partial_t|\Psi(t)\rangle = H|\Psi(t)\rangle$$

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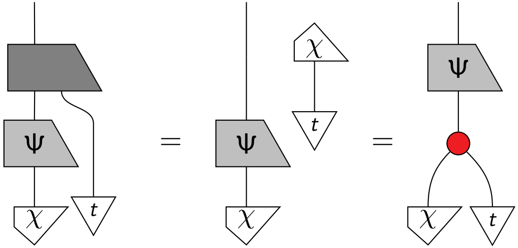
- This last form admits a finite-dimensional equivalent:

$$e^{-i\frac{2\pi}{N}Et}|\Psi_E(0)\rangle = U(t)|\Psi_E(0)\rangle$$

Schrödinger's Equation

In fdHilb, Schrödinger's equation can then be written as:



$$U(t)|\psi_E\rangle \quad = \quad e^{i\,2\pi/N\,E\cdot t}\;|\psi_E\rangle$$

Where we used the fact that the Hamiltonian eigenstates are exactly those in the form $\psi_\chi = \Psi \cdot |\chi\rangle$.

Schrödinger's Equation

Theorem

In fdHilb, *asking for a* $|\Psi(t)\rangle : \mathbb{Z}_N \to \mathcal{H}$ *to satisfy Schrödinger's equation is the same as asking for its linear extension* $\Psi_E : \mathcal{G} \to \mathcal{H}$ *to be an Eilenberg-Moore morphism*  $\to$ .

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We take this as our definition. We will say that a given morphism $\Psi : \mathcal{G} \to \mathcal{H}$ is a solution to **Schrödinger's equation** if and only if it is an Eilenberg-Moore morphism Ψ :  $\to$ .

Solutions to Schrödinger's Equation

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- The linear structure allows us to encode both orbit values and invariant components in the same map Ψ (which is cool).

Conclusions

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- Schrödinger's eq'n = definition of EM morphisms  $\rightarrow$ 
- Solutions to Schrödinger's eq'n = orbits of states under 
- + they encode the corresponding "energy" spectrum

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Thank You!

Thanks for Your Attention!

Any Questions?