Contextuality, Cohomology, and Paradox
(arXiv:1502.03097)

Samson Abramsky, Rui Soares Barbosa,
Kohei Kishida, Ray Lal, and Shane Mansfield
(speaking)

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Outline

1. Topological model for contextuality.
2. Cohomology: Contextuality is like “impossible figures”.
3. Relation to QM no-go theorems.
Bell Non-Localyty

**Bell-type setup.** Input-output box for (2, 2, 2) scenario:

![Input-output box](image)

Distribution $p(o_A, o_B | a_i, b_j)$ for each context $\{a_i, b_j\}$.

<table>
<thead>
<tr>
<th>$(a_0, b_0)$</th>
<th>$(0, 0)$</th>
<th>$(0, 1)$</th>
<th>$(1, 0)$</th>
<th>$(1, 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(a_0, b_0)$</td>
<td>$1/2$</td>
<td>$0$</td>
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</tr>
<tr>
<td>$(a_0, b_1)$</td>
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</tr>
</tbody>
</table>
A table may be logically non-local / contextual. E.g. model by Hardy 1993:
\[
\begin{array}{cccc}
(a_0, b_0) & (0, 0) & (0, 1) & (1, 0) & (1, 1) \\
 & 1/2 & 0 & 0 & 1/2 \\
(a_0, b_1) & 3/8 & 1/8 & 1/8 & 3/8 \\
(a_1, b_0) & 3/8 & 1/8 & 1/8 & 3/8 \\
(a_1, b_1) & 1/8 & 3/8 & 3/8 & 1/8 \\
\end{array}
\]

No local probability table has this support. (Logical non-locality / contextuality implies probabilistic one.)
**Possiblility table**: non-zero $\mapsto 1$ ("possible")
0 $\mapsto 0$ ("impossible").

Support of a probability table is a possibility table.

\[
\begin{array}{c|cccc}
(a, b) & (0, 0) & (0, 1) & (1, 0) & (1, 1) \\
\hline
(a_0, b_0) & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\
(a_0, b_1) & \frac{3}{8} & \frac{1}{8} & \frac{1}{8} & \frac{3}{8} \\
(a_1, b_0) & \frac{3}{8} & \frac{1}{8} & \frac{1}{8} & \frac{3}{8} \\
(a_1, b_1) & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8}
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**Possibility table**: non-zero $\mapsto 1$ ("possible")
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Support of a probability table is a possibility table.

![Possibility Table](image)

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Support of a probability table is a possibility table.

Marginals, convex combination, no-signalling, locality, etc. all carry over to the possibilistic, logical versions.

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(Logical non-locality / contextuality implies probabilistic one.)
Theorem (Fine 1982 / Abramsky-Brandenburger 2011). A table $p(\cdot \mid a_i, b_j)_{i,j\in\{0,1\}}$ is local iff

There is a distribution $p(\cdot \mid a_i^0, a_1^i, b_0^i, b_1^i)$ that gives each $p(\cdot \mid a_i^0, b_0^i)$ as a marginal, e.g., $p(\cdot \mid A, O_B^0, a_0^0, b_0^0) = \sum_{O_A, O_B'} p(\cdot \mid A, O_A, b_0^0)$. i.e. a distribution over deterministic $(a_0^i, a_1^i, b_0^i, b_1^i)$.

i.e. the table is a convex combination of the deterministic tables for such $'s.$

Upshot. A no-signalling but non-local table is "Locally consistent": able to assign probabilities / possibilities consistently to the family of measurement contexts $f a_i^i, b_j^i; g$.

"Globally inconsistent": not able to to the set $f a_0, a_1; b_0, b_1 g$ of all measurements.

Topology on the set of measurements.
\textbf{Theorem} (Fine 1982 / Abramsky-Brandenburger 2011). A table $p(\cdot | a_i, b_j)_{i,j\in\{0,1\}}$ is local iff

- There is a distribution $p(\cdot | a_0, a_1, b_0, b_1)$ that gives each $p(\cdot | a_i, b_j)$ as a marginal, e.g.,

$$p(o_A, o_B | a_0, b_0) = \sum_{o,o'} p(o_A, o, o_B, o' | a_0, a_1, b_0, b_1);$$
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$$p(o_A, o_B \mid a_0, b_0) = \sum_{o,o'} p(o_A, o, o_B, o' \mid a_0, a_1, b_0, b_1);$$

- i.e. a distribution over deterministic

$$\lambda_{(a_0,a_1,b_0,b_1)} \mapsto (0,0,0,0),$$
$$\lambda_{(a_0,a_1,b_0,b_1)} \mapsto (0,0,0,1),$$
$$\vdots$$
$$\lambda_{(a_0,a_1,b_0,b_1)} \mapsto (1,1,1,1);$$

![Diagram of measurement contexts and distributions](image-url)
**Theorem** (Fine 1982 / Abramsky-Brandenburger 2011).

A table \( p(\cdot | a_i, b_j)_{i,j\in\{0,1\}} \) is local iff

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  \[
p(o_A, o_B | a_0, b_0) = \sum_{o,o'} p(o_A, o, o_B, o' | a_0, a_1, b_0, b_1);
  \]
  i.e. a distribution over deterministic
  \[
  \lambda(a_0,a_1,b_0,b_1) \mapsto (0,0,0,0),
  \]
  \[
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  \]
  \[
  \vdots
  \]
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**Upshot.** A no-signalling but non-local table is

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  able to assign probabilities / possibilities consistently to the family of measurement contexts \(\{a_i, b_j\}\);
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- “Locally consistent”:
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**Topology** on the set of measurements.
Topological Model for Contextuality

Topological spaces of variables and of their values.
Topological Model for Contextuality

Topological spaces of variables and of their values.

- measurements and outcomes
- sentences and truth values
- questions and answers
Topological spaces of **variables** and of their **values**.

- **measurements** and **outcomes**
- **sentences** and **truth values**
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For each **variable** $x$,
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For each variable $x$,

a dependent type $F(x)$ of values.
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For each variable $x$, a dependent type $F(x)$ of values.

"Bundle" $\sum_{x \in X} F(x)$
When we ask several questions, answers may obey constraints:
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- laws of physics, e.g., Charles’s law
- laws of logic

\[ \phi \quad \phi \quad \phi \]

\[ \begin{array}{c}
\text{tt} \\
\text{ff}
\end{array} \]

\[ \begin{array}{c}
\varphi \\
\neg \varphi \\
\neg \neg \varphi
\end{array} \]

\[ F(t) \]

\[ F(v) \]
When we ask several questions, answers may obey constraints:

- laws of physics, e.g., Charles’s law
- laws of logic

Distinguish good and bad ways of connecting dots in bundles ... just like “continuous sections”!
Hardy model:

<table>
<thead>
<tr>
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<th>00</th>
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$a_0 \bullet \bullet b_0$

$a_1 \bullet \bullet b_1$

Global section:

$\left( a_0; a_1; b_0; b_1 \right) \neq \left( 1; 0; 1; 0 \right)$.

Logical contextuality:

Not all sections extend to global ones.

Local consistency, global inconsistency
Hardy model:

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\begin{array}{c|ccccc}
 & 00 & 01 & 10 & 11 \\
\hline
a_0b_0 & 1 & 1 & 1 & 1 \\
a_0b_1 & 0 & 1 & 1 & 1 \\
a_1b_0 & 0 & 1 & 1 & 1 \\
a_1b_1 & 1 & 1 & 1 & 0 \\
\end{array}
\]

Global section:

\((a_0; a_1; b_0; b_1) \neq (1; 0; 1; 0)\).

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Logical contextuality: Not all sections extend to global ones. Local consistency, global inconsistency.
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Logical contextuality: Not all sections extend to global ones.
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Logical contextuality: Not all sections extend to global ones.

Local consistency, global inconsistency
Hardy:

Logical contextuality: Not all sections extend to global.
**PR box:**

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Strong contextuality: No global section at all.
**Logical contextuality:** Not all sections extend to global.

**Strong contextuality:** No global section at all.

Hierarhcy of contextuality:

**Probabilistic** $\supseteq$ **Logical** $\supseteq$ **Strong contextuality**
Logical contextuality: Not all sections extend to global.
Strong contextuality: No global section at all.
Hierarchies of contextuality:
- Probabilistic ⊈ Logical ⊈ Strong contextuality
Contextuality in Logical Paradoxes

Read bundles \( \pi : \sum_{x \in X} F(x) \to X \) in logic terms:
- \( x \in X \) are sentences,
- \( \ttt, \fff \in F(x) \) are truth values.
Contextuality in Logical Paradoxes

Read bundles $\pi : \sum_{x \in X} F(x) \rightarrow X$ in logic terms:

- $x \in X$ are sentences,
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“West is true”

“South is true”

“East is true”

“North is false”
Contextuality in Logical Paradoxes

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- $x \in X$ are sentences,
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This type of logical paradoxes (incl. the Liar Paradox) have the same topology as "paradoxes" of (strong) contextuality.
Contextuality in Logical Paradoxes

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Read bundles:

\[ \sum_{x \in X} F(x) \]

In logic terms, \( x \) are sentences, \( t, f \); \( F(x) \) are truth values.
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Contextuality in Logical Paradoxes

Read bundles: \[ \sum_{x \in X} \neg F(x) \]

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How to Formally Define …

Bundles that correspond to no-signalling possibility tables.
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Two equivalent formulations:
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1. Map of simplicial complexes

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(With some axioms, e.g. no-signalling.)
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2 makes it possible to apply cohomology.
Cohomology of Contextuality

Local consistency, global inconsistency…

Cohomological test for contextuality:

“Čech cohomology” gives a group homomorphism $\gamma$ that assigns to each section $s$ an “obstruction” $\gamma_s$ s.th. $s$ extends to a “cocycle” $s = 0$. (False positives, e.g. in Hardy model: Works for many cases; e.g. PR box: $a_0 b_0 a_1 b_1$).
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Cohomological test for contextuality:

“Čech cohomology” gives a group homomorphism $\gamma$ that assigns to each section $s$ an “obstruction” $\gamma_s$ s.th. $s$ extends to a “cocycle” $\iff \gamma_s = 0$. 
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\[ \uparrow \]

$s$ extends to global

\[ \begin{array}{ccc}
a_0 & b_0 & 0 \\
1 & 1 & 1 \\
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\end{array} \]
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- False positives,
e.g. in Hardy model.
- Works for many cases;
e.g. PR box:
“All vs Nothing” Argument

Joint outcomes may / may not satisfy parity equations:

\[
\begin{align*}
(0;0) & \quad x_2 = 0 \\
(0;1) & \quad x_2 = 1 \\
(1;0) & \quad x_2 = 1 \\
(1;1) & \quad x_2 = 0
\end{align*}
\]

\[a_0 \quad b_0 = 0 \]

\[a_1 \quad b_1 = 1 \]

LHS's \[\oplus\] RHS's

The equations are inconsistent, i.e. no global assignment to \(a_0, a_1, b_0, b_1\), i.e. strongly contextual!
“All vs Nothing” Argument

Joint outcomes may / may not satisfy parity equations:

\[(0, 0) \sim x \oplus y = 0\]
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\[ \bigoplus \text{LHS}'s = \bigoplus \text{RHS}'s \]
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“All vs nothing” arguments in QM can be formulated the same way.

- GHZ state: 
  \[ a_0 \oplus b_0 \oplus c_0 = 0 \]
  \[ a_0 \oplus b_1 \oplus c_1 = 1 \]
  \[ a_1 \oplus b_0 \oplus c_1 = 1 \]
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  \[ \bigoplus \text{LHS’s} = 0 \neq 1 = \bigoplus \text{RHS’s} \]

- Kochen-Specker-type:
  18 variables, each occurs twice, so \( \bigoplus \text{LHS’s} = 0 \);
  9 equations, all of parity 1, so \( \bigoplus \text{RHS’s} = 1 \).
Beyond QM, some NS tables suggest generalization.
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- “Box 25” of Pironio-Bancal-Scarani 2011 admits no parity argument, but satisfies

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\begin{align*}
    a_0 + 2b_0 &\equiv 0 \mod 3 & a_1 + 2c_0 &\equiv 0 \mod 3 \\
    a_0 + b_1 + c_0 &\equiv 2 \mod 3 & a_0 + b_1 + c_1 &\equiv 2 \mod 3 \\
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**Generalized all-vs-nothing argument** uses any commutative ring \( R \) (e.g. \( \mathbb{Z}_n \)) instead of \( \mathbb{Z}_2 \):
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- Linear equations $k_0 x_0 + \cdots + k_m x_m = p \quad (k_0, \ldots, k_m, p \in R)$. 
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- **Linear equations** $k_0x_0 + \cdots + k_mx_m = p \quad (k_0, \ldots, k_m, p \in R)$.
- Equations are inconsistent if a subset of them is s.th.
  - coefficients $k$ of each variable $x$ add up to 0,
  - parities $p$ do not.
“Strongly contextual by AvN argument” is explained by “strongly contextual by cohomology”:

**Theorem.**

Let $\mathcal{M}$ be a no-signalling bundle model. Then

- $\mathcal{M}$ admits a generalized AvN argument in a ring $R$

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**Heirarchy of strong contextuality:**

$$\text{AvN} \subsetneq \text{gen. AvN} \subsetneq \text{cohom. SC} \subseteq \text{SC}$$
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Conclusion

General, structural formalism independent of QM formalism. Uniform methods of detecting / showing contextuality.
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References


