A Graph Theoretic Perspective on CPM(Rel)

Dan Marsden

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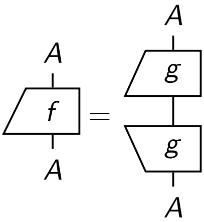
Selinger's CPM Construction

Category $\mathcal{C}$ a $\dagger$-compact closed monoidal category.

Positive Morphism

Endomorphism $f : A \to A$ is **positive** if there exists object B and morphism $g : A \to B$ such that:

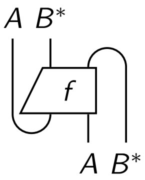


Selinger's CPM Construction

Category $\mathcal{C}$ a †-compact closed monoidal category.

The Category **CPM**($\mathcal{C}$)

- **Objects**: $\mathcal{C}$-objects.
- **Morphisms**: A morphism of type $A \to B$ is a $\mathcal{C}$-morphism $f : A^* \otimes A \to B^* \otimes B$ such that:



is positive.

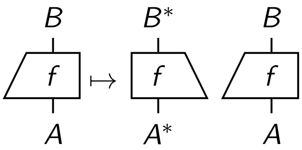
Selinger's CPM Construction

Category $\mathcal{C}$ a $\dagger$-compact closed monoidal category.

Relating $\mathcal{C}$ to $\mathbf{CPM}(\mathcal{C})$

There is a canonical functor:

$$\mathcal{C} \to \mathbf{CPM}(\mathcal{C})$$



A Linguistics Application

Compositional Distributional Semantics

- Non-commutative compact closed categories model grammar - pregroups (Lambek)
- Compact closed categories model semantics
- Functorial Semantics

$$P \to \mathbf{FdHilb}_{\mathbb{R}}$$

A Linguistics Application

Density Operators in Linguistics

- Ambiguity in language - "river **bank**" versus "financial **bank**" (Piedeleu)
- Hyponym / hypernym relationships - "dog" versus "mammal" (Balkir)
- Alternative models such as **Rel**

A Linguistics Application

Booleans

- Consider the two element set **Bool** $= \{\top, \bot\}$ as truth values
- In **Rel**, **Bool** has 4 states:

$$\emptyset,\ \{\top\},\ \{\bot\},\ \{\top,\bot\}$$

- In **CPM**(**Rel**), **Bool** has 5 states

What are the states in **CPM**(**Rel**)?

- (Selinger) States $I \to A$ in **CPM**(**Rel**) correspond to positive morphisms $A \to A$ in **Rel**, which are relations satisfying:

$$\begin{aligned} R(x,y) &\Rightarrow R(y,x) \\ R(x,y) &\Rightarrow R(x,x) \end{aligned}$$

- Can we count these?

States for small objects in **CPM(Rel)**

Elements	**Rel States**	**CPM(Rel) States**
0	1	1
1	2	2
2	4	5
3	8	18
4	16	113
5	32	1450

Another Perspective on States

Graphs

For each **CPM**(**Rel**) state with corresponding positive relation $R : A \to A$ we can construct a (simple labelled undirected) graph with:

- **Vertices** Elements $a \in A$ such that $R(a, a)$
- **Edges** Pairs $\{a, b\}$ with $R(a, b)$

Remark

For this talk, graphs are undirected, have no duplicate edges, but *always* have self loops.

Examples

Example

The relation $R : \{a, b\} \to \{a, b\}$:

$$R(a, a) = R(b, b) = \text{true} \qquad R(a, b) = R(b, a) = \text{false}$$

has graph:

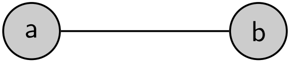
Examples

Example

The relation $R : \{a, b\} \to \{a, b\}$:

$$R(a, a) = R(a, b) = R(b, a) = R(b, b) = \text{true}$$

has graph:



States as Graphs

States are Graphs

In fact the states of a set A in $\mathbf{CPM}(\mathbf{Rel})$ bijectively correspond to the graphs on subsets of elements of A. A set of n elements then has:

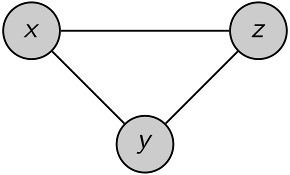
$$\sum_{0 \leq i \leq n} \binom{i}{n} 2^{n(n-1)/2}$$

states.

Pure States Graphically

Pure States are the Complete Graphs

The following is a pure state:



Pure States Graphically

Pure States are the Complete Graphs

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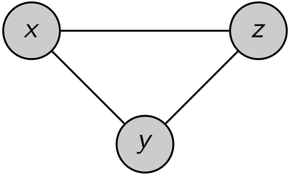
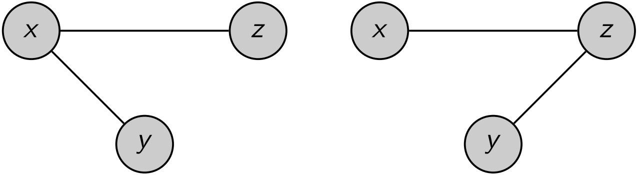


The following are not pure:



Graph State Duality

Morphisms as Graphs

As there is a bijective correspondence:

$$\frac{A \to B}{\overline{I \to A \otimes B}}$$

we can consider morphisms $A \to B$ as graphs on subsets of $A \times B$.

Composition and Identities Graphically

Identities and Composition

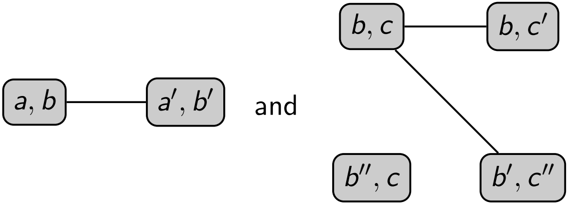
We can define a category $\mathcal{G}$ with objects sets and morphisms graphs on subsets of the cartesian products of the domain and codomain where:

- For each set A we define 1_A as the complete graph on the diagonal of $A \times A$.
- For the composition of two graphs $A \to B$ and $B \to C$
  - (a, c) is a vertex if there are vertices (a, b) and (b, c) in the original graphs
  - $\{(a, c), (a', c')\}$ is an edge if there are edges $\{(a, b), (a', b')\}$ and $\{(b, c), (b', c')$ in the original graphs

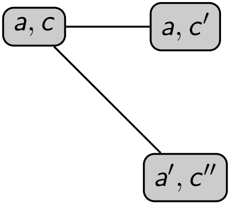
Composition and Identities Graphically

Example

The composition of the graphs:



is given by the graph:



An Isomorphism of Categories

We have an isomorphism of categories:

$$\mathbf{CPM}(\mathbf{Rel}) \cong \mathcal{G}$$

- $\mathbf{CPM}(\mathbf{Rel})$ is a †-compact monoidal category in which we can take unions of morphisms
- How do we describe this structure in terms of graphs?

Rel into $\mathcal{G}$

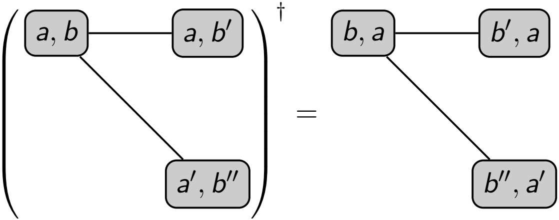
We have the canonical functor:

$$\mathbf{Rel} \to \mathcal{G}$$

sending a relation $R \subseteq A \times B$ to the complete graph on R.
In particular, pure states are complete graphs as claimed earlier.

The † Graphically

The dagger of a graph is the "same" graph with the elements of the vertex pairs swapped.



Monoidal Structure Graphically

Tensor Products

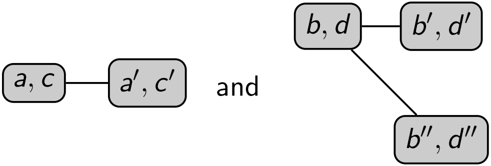
The tensor product of two graphs is the graph with:

- **Vertices**: Pairs of vertices from the component graphs
- **Edges**: There is an edge $\{(a, b, c, d), (a', b', c', d')\}$ if there is an edge $\{(a, b), (a', b')\}$ and an edge $\{(c, d), (c', d')\}$.

Monoidal Structure Graphically

Example

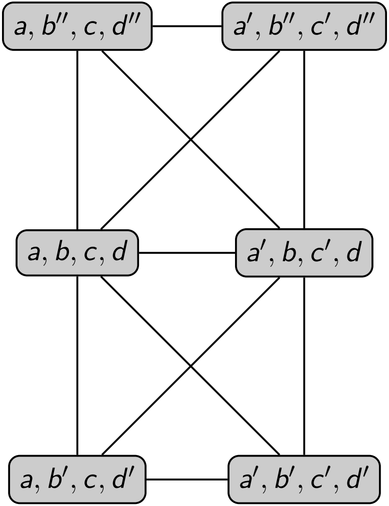
The tensor of the following pair of graphs:



Monoidal Structure Graphically

Example

is given by the graph:



Order Structure Graphically

For graphs $\gamma, \gamma' : A \to B$, we say that $\gamma \subseteq \gamma'$ if both the edges of γ are a subset of the edges of γ'. The union of a family of graphs $A \to B$ is given by taking the unions of the vertex and edge sets.

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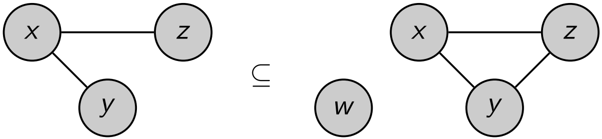
Ordering Example



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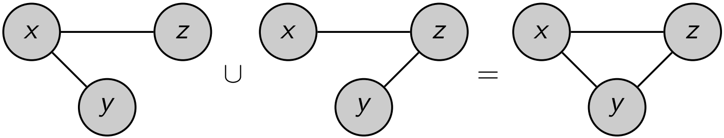
Union Example



Conclusion

- Simple visual reasoning about **CPM**(**Rel**)
- Applications - Stefano Gogioso talk...
- Further developments - Beautiful characterization of $\mathbf{CPM}^2(\mathbf{Rel})$ states by Oscar Cunningham
- Repeated iteration of the CPM construction (Daniela Ashoush)