A Graph Theoretic Perspective on CPM(Rel)

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Category $\mathcal{C}$ a $\dagger$-compact closed monoidal category.

**Positive Morphism**

Endomorphism $f : A \to A$ is **positive** if there exists object $B$ and morphism $g : A \to B$ such that:

\[
\begin{array}{c}
A \\
\downarrow f \\
A
\end{array}
= 
\begin{array}{c}
A \\
\downarrow g \\
A
\end{array}
\begin{array}{c}
\phantom{g} \\
\downarrow g \\
A
\end{array}
\]
Selinger’s CPM Construction

Category $\mathcal{C}$ a $\dagger$-compact closed monoidal category.

The Category $\text{CPM}(\mathcal{C})$

- **Objects**: $\mathcal{C}$-objects.
- **Morphisms**: A morphism of type $A \to B$ is a $\mathcal{C}$-morphism $f : A^* \otimes A \to B^* \otimes B$ such that:

\[
\begin{array}{ccc}
A & \otimes & B^* \\
\downarrow f & & \downarrow \\
A & \otimes & B^* \\
\end{array}
\]

is positive.
Category $\mathcal{C}$ a $\dagger$-compact closed monoidal category.

Relating $\mathcal{C}$ to $\text{CPM}(\mathcal{C})$

There is a canonical functor:

\[
\mathcal{C} \rightarrow \text{CPM}(\mathcal{C})
\]
A Linguistics Application

Compositional Distributional Semantics

- Non-commutative compact closed categories model grammar - pregroups (Lambek)
- Compact closed categories model semantics
- Functorial Semantics

\[ P \rightarrow \text{FdHilb}_\mathbb{R} \]
A Linguistics Application

Density Operators in Linguistics

- Ambiguity in language - “river bank” versus “financial bank” (Piedeleu)
- Hyponym / hypernym relationships - “dog” versus “mammal” (Balkir)
- Alternative models such as Rel
Booleans

- Consider the two element set \( \text{Bool} = \{\top, \bot\} \) as truth values
- In \( \text{Rel} \), \( \text{Bool} \) has 4 states:
  \[ \emptyset, \{\top\}, \{\bot\}, \{\top, \bot\} \]
- In \( \text{CPM(\text{Rel})} \), \( \text{Bool} \) has 5 states
What are the states in $\text{CPM}(\text{Rel})$?

- (Selinger) States $I \to A$ in $\text{CPM}(\text{Rel})$ correspond to positive morphisms $A \to A$ in $\text{Rel}$, which are relations satisfying:

  \[
  R(x, y) \Rightarrow R(y, x) \\
  R(x, y) \Rightarrow R(x, x)
  \]

- Can we count these?
States for small objects in $\text{CPM}(\text{Rel})$

<table>
<thead>
<tr>
<th>Elements</th>
<th>Rel States</th>
<th>$\text{CPM}(\text{Rel})$ States</th>
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Another Perspective on States

Graphs
For each $\text{CPM}(\text{Rel})$ state with corresponding positive relation $R : A \rightarrow A$ we can construct a (simple labelled undirected) graph with:

- **Vertices** Elements $a \in A$ such that $R(a, a)$
- **Edges** Pairs $\{a, b\}$ with $R(a, b)$

Remark
For this talk, graphs are undirected, have no duplicate edges, but *always* have self loops.
Example

The relation $R : \{a, b\} \rightarrow \{a, b\}$:

\[
R(a, a) = R(b, b) = \text{true} \quad \quad R(a, b) = R(b, a) = \text{false}
\]

has graph:

\[\text{a} \quad \text{b}\]
Example

The relation $R : \{a, b\} \rightarrow \{a, b\}$:

$$R(a, a) = R(a, b) = R(b, a) = R(b, b) = \text{true}$$

has graph:

```
  a --- b
```
States as Graphs

States are Graphs

In fact the states of a set $A$ in $\text{CPM}(\text{Rel})$ bijectively correspond to the graphs on subsets of elements of $A$. A set of $n$ elements then has:

$$\sum_{0 \leq i \leq n} \binom{i}{n} 2^{n(n-1)/2}$$

states.
Pure States Graphically

Pure States are the Complete Graphs

The following is a pure state:
Pure States Graphically

Pure States are the Complete Graphs

The following is a pure state:

The following are not pure:
Graph State Duality

Morphisms as Graphs

As there is a bijective correspondence:

\[
\begin{align*}
A &\rightarrow B \\
I &\rightarrow A \otimes B
\end{align*}
\]

we can consider morphisms \( A \rightarrow B \) as graphs on subsets of \( A \times B \).
Composition and Identities Graphically

Identities and Composition

We can define a category $\mathcal{G}$ with objects sets and morphisms graphs on subsets of the cartesian products of the domain and codomain where:

- For each set $A$ we define $1_A$ as the complete graph on the diagonal of $A \times A$.
- For the composition of two graphs $A \rightarrow B$ and $B \rightarrow C$
  - $(a, c)$ is a vertex if there are vertices $(a, b)$ and $(b, c)$ in the original graphs
  - $\{(a, c), (a', c')\}$ is an edge if there are edges $\{(a, b), (a', b')\}$ and $\{(b, c), (b', c')\}$ in the original graphs
Composition and Identities Graphically

Example

The composition of the graphs:

\[ (a, b), (a', b'), (b', c), (b'', c), (a, c), (a', c''), (b, c), (b, c'), (b', c'') \]

is given by the graph:
An Isomorphism of Categories

We have an isomorphism of categories:

$$\text{CPM}(\text{Rel}) \cong \mathcal{G}$$

- **CPM(Rel)** is a $\dagger$-compact monoidal category in which we can take unions of morphisms
- How do we describe this structure in terms of graphs?
We have the canonical functor:

\[
\text{Rel} \to \mathcal{G}
\]

sending a relation \( R \subseteq A \times B \) to the complete graph on \( R \). In particular, pure states are complete graphs as claimed earlier.
The dagger of a graph is the “same” graph with the elements of the vertex pairs swapped.

\[
\begin{pmatrix}
  a, b \\
  a', b''
\end{pmatrix}
\dagger
= 
\begin{pmatrix}
  b, a \\
  b'', a'
\end{pmatrix}
\]
Tensor Products
The tensor product of two graphs is the graph with:

- **Vertices**: Pairs of vertices from the component graphs
- **Edges**: There is an edge \{ (a, b, c, d), (a', b', c', d') \} if there is an edge \{ (a, b), (a', b') \} and an edge \{ (c, d), (c', d') \}.
Example

The tensor of the following pair of graphs:

\[
\begin{align*}
a, c & \longrightarrow a', c' \\
& \text{and} \\
& b, d \longrightarrow b', d' \longrightarrow b'', d''
\end{align*}
\]
Monoidal Structure Graphically

Example

is given by the graph:
For graphs $\gamma, \gamma': A \rightarrow B$, we say that $\gamma \subseteq \gamma'$ if both the edges of $\gamma$ are a subset of the edges of $\gamma'$. The union of a family of graphs $A \rightarrow B$ is given by taking the unions of the vertex and edge sets.
Order Structure Graphically

For graphs $\gamma, \gamma' : A \rightarrow B$, we say that $\gamma \subseteq \gamma'$ if both the edges of $\gamma$ are a subset of the edges of $\gamma'$. The union of a family of graphs $A \rightarrow B$ is given by taking the unions of the vertex and edge sets.

Ordering Example

\[ x \rightarrow z \subseteq x \rightarrow w \rightarrow y \]
Order Structure Graphically

For graphs $\gamma, \gamma' : A \to B$, we say that $\gamma \subseteq \gamma'$ if both the edges of $\gamma$ are a subset of the edges of $\gamma'$. The union of a family of graphs $A \to B$ is given by taking the unions of the vertex and edge sets.

Union Example

$$
\begin{array}{ccc}
\text{x} & \text{z} & \text{y} \\
\text{y} & \text{z} & \text{z} \\
\end{array}
\cup
\begin{array}{ccc}
\text{x} & \text{z} & \text{y} \\
\text{y} & \text{z} & \text{z} \\
\end{array}
= 
\begin{array}{ccc}
\text{x} & \text{z} & \text{y} \\
\text{y} & \text{z} & \text{z} \\
\end{array}
$$
Conclusion

- Simple visual reasoning about $\text{CPM}(\text{Rel})$
- Applications - Stefano Gogioso talk...
- Further developments - Beautiful characterization of $\text{CPM}^2(\text{Rel})$ states by Oscar Cunningham
- Repeated iteration of the CPM construction (Daniela Ashoush)