Effect algebras, presheaves, non-locality and contextuality

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Non locality

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Alice

Bob

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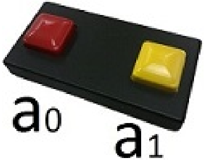

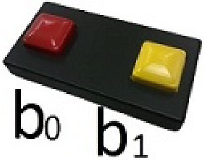

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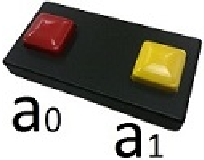

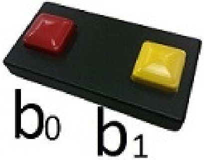

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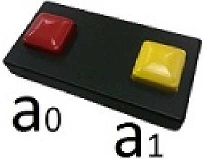

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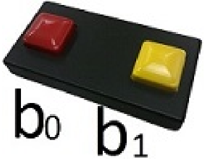

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- For example: $\mathrm{a}_0{:}1 \wedge \mathrm{b}_1{:}0$

Bell table

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Fact: this table cannot be obtained in a classical way, but can be obtained in QM.

No signaling probability tables

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- Probability: rows sum to 1
- No signaling (marginalization): Bob does not know what Alice chose as setting. e.g.:
  $p(a_0{:}0 \wedge b_0{:}0) + p(a_0{:}1 \wedge b_0{:}0) = p(a_1{:}0 \wedge b_0{:}0) + p(a_1{:}1 \wedge b_0{:}0)$

Classical finite probability theory

- Classically: consider state spaces

$$S_{\mathrm{A}} = \left\{ f : \{\mathrm{a}_0, \mathrm{a}_1\} \to \{0, 1\} \right\}$$

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- Probability distribution $p : \Omega \to [0,1]$ satisfying
  - $p(X) = 1$,
  - $p(\bigcup_i A_i) = \sum_i p(A_i) \quad \text{if} \quad A_i \cap A_j = \emptyset,\ i \neq j$

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$\to$ Replace Ω by something more general capturing the 'measure-only-once' phenomenon.

Effect algebras

Definition

An effect algebra $(E, \varovee, 0, 1)$ *comprises a partial commutative, associative monoid* $(E, \varovee, 0)$*, such that*

- $\forall e \in E \quad \exists$ *unique* $e^\perp$ *s.t.* $e \varovee e^\perp = 1 = 0^\perp$
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Note the partiality of $\owedge$.

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$\to$ Effect algebras are generalized probability spaces,
effect algebra morphisms to $[0,1]$ are probability distributions.

More generalization needed

- Only probabilities, no possibilities (Hardy).
- Relate to other work (Abramsky & Brandenburger).
- Any good good list has at least three points.

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- D extends to a functor $D : \mathbb{N} \to \mathbf{Set}$.
- Yoneda: $[Hom(N,-) \to D] \cong D(N)$.
- Functors $F : \mathbb{N} \to Set$ are "measure spaces", natural transformations $F \to D$ are "probability distributions".

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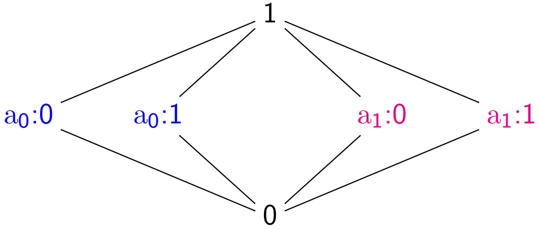
$$T \text{ extends to a functor.}$$

Theorem

Test functor is full, faithful and has a left adjoint.

Effect algebraic description

Define an effect algebra E_{A} for Alice



Similarly E_{B} for Bob.

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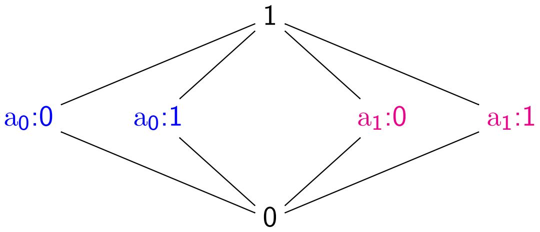

Similarly E_{B} for Bob.
Mix them together in the tensor product. $\mathrm{a}_1{:}1 \wedge \mathrm{b}_0{:}1$

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Define Boolean algebra B_{A} with atoms

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- Information of a_0 and a_1.
- "Deterministic hidden variables."
- Classical description.

B_{A} is the free completion of E_{A}.

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Theorem

The following structures are equivalent:

- *No-signaling probability table*
- *Bimorphism* $E_{\mathrm{A}}, E_{\mathrm{B}} \to [0, 1]$
- *Effect algebra morphism* $t : E_{\mathrm{A}} \otimes E_{\mathrm{B}} \to [0, 1]$

(Non) factorization

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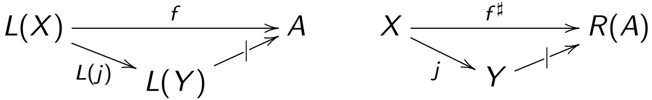
Paradox translates to

$$\begin{array}{ccccc} E_{\mathrm{A}} \otimes E_{\mathrm{B}} & & \xrightarrow{\quad t \quad} & & [0,1] \\ & \searrow & & \nearrow\!\!\!\!\!\!/ & \\ & & B_{\mathrm{A}} \otimes B_{\mathrm{B}} & & \end{array} \qquad \begin{array}{ccccc} E_{\mathrm{A}} \otimes E_{\mathrm{B}} & & \xrightarrow{\quad t \quad} & & [0,1] \\ & \searrow & & \nearrow & \\ & & Proj\,\mathcal{H} & & \end{array}$$

transporting non-factorization

For an adjunction $L \dashv R$ we have



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$$\begin{array}{ccccc} L(X) & \xrightarrow{\quad f \quad} & A & \qquad & X & \xrightarrow{\quad f^{\sharp} \quad} & R(A) \\ {\scriptstyle L(j)} \searrow & & \nearrow\!\!\!\!\!\!| & & {\scriptstyle j} \searrow & & \nearrow\!\!\!\!\!\!| \\ & L(Y) & & & & Y & \end{array}$$

Test functor has a left adjoint and $LT \cong Id$.

$$\begin{array}{ccccc} LT(E) \cong E & \xrightarrow{\quad t \quad} & [0,1] & \qquad & TE & \xrightarrow{\quad t^{\sharp} = Tt \quad} & D \\ {\scriptstyle i} \searrow & & \nearrow\!\!\!\!\!\!| & & {\scriptstyle i} \searrow & & \nearrow\!\!\!\!\!\!| \\ & LT(B) \cong B & & & & TB & \end{array}$$

- Transport from effect algebras to presheaves.

Slice category

Work relative to particular object.
Here: $T(B_{\mathrm{A}} \otimes B_{\mathrm{B}})$

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- Adjunction:

$$[\mathbb{N}, \mathbf{Set}] \underset{\pi}{\overset{(- \times T(B), \pi_2)}{\rightleftarrows}} [\mathbb{N}, \mathbf{Set}]/T(B) \quad (\top)$$

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Transport non-factoring to the slice category $[\mathbb{N}, \mathbf{Set}]/T(B_{\mathrm{A}} \otimes B_{\mathrm{B}})$

$$(T(E_{\mathrm{A}} \otimes E_{\mathrm{B}}), Ti) \xrightarrow{\langle Tt, Ti \rangle} (D \times T(B_{\mathrm{A}} \otimes B_{\mathrm{B}}), \pi_2)$$

$$(T(E_{\mathrm{A}} \otimes E_{\mathrm{B}}), Ti) \longrightarrow (T(B_{\mathrm{A}} \otimes B_{\mathrm{B}}), \mathrm{id}) \not\longrightarrow (D \times T(B_{\mathrm{A}} \otimes B_{\mathrm{B}}), \pi_2)$$

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- $(T(B_{\mathrm{A}} \otimes B_{\mathrm{B}}), id)$ is terminal.
- "The local section $\langle Tt, Ti \rangle$ has no global section."

Other work and other paradoxes

- Sequence of adjunctions linking to Abramsky & Brandenburger approach

$$\mathbf{EA} \underset{\top}{\overset{T}{\rightleftarrows}} \mathbf{Set}^{\mathbb{N}} \underset{\Sigma_{O^X}}{\overset{\Delta_{O^X}}{\rightleftarrows}} \mathbf{Set}^{\mathbb{N}}/\mathbb{N}(O^X, -) \simeq \mathbf{Set}^{(\mathbb{N}^{\mathrm{op}}/(O^X))^{\mathrm{op}}} \underset{I_!}{\overset{I^*}{\rightleftarrows}} \mathbf{Set}^{\mathcal{P}(X)^{\mathrm{op}}}$$

- By considering maps into $\{0,1\}$ where $1+1=1$ (not an effect algebra) we reconstruct the Hardy Paradox in a similar way.
- Looking at $[\mathbb{N}, Set]/T(Proj\mathcal{H})$ and maps into $\{0,1\}$ (as effect algebra) we reconstruct Kochen-Specker paradox.

Slogan:

Different contextuality scenarios arise from different slices of the presheaf category $[\mathbb{N}, \mathbf{Set}]$.