Making the stabilizer ZX-calculus complete for scalars

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Outline

Background

Modifying the $\text{zx}$-calculus to keep account of scalars

The new completeness results

Conclusions
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Stabilizer quantum mechanics

- preparation of qubits in state $|0\rangle$
- Clifford unitaries, generated by

\[
S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad C_X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\]

- measurements in computational basis
Elements of stabilizer zx-calculus diagrams

- Green nodes with \( n \) inputs and \( m \) outputs,
  \( \alpha \in \{-\pi/2, 0, \pi/2, \pi\} \)

- Red nodes with \( k \) inputs and \( l \) outputs,
  \( \beta \in \{-\pi/2, 0, \pi/2, \pi\} \)

- Hadamard nodes with one input and one output
Elements of stabilizer ZX-calculus diagrams

- Green nodes with $n$ inputs and $m$ outputs,
  $\alpha \in \{-\pi/2, 0, \pi/2, \pi\}$

\[
\begin{bmatrix}
  m \\
  \vdots \\
  n
\end{bmatrix}
\alpha
\] := $|0\rangle \otimes^m \langle 0| \otimes^n + e^{i\alpha} |1\rangle \otimes^m \langle 1| \otimes^n$

- Red nodes with $k$ inputs and $l$ outputs,
  $\beta \in \{-\pi/2, 0, \pi/2, \pi\}$

\[
\begin{bmatrix}
  l \\
  \vdots \\
  k
\end{bmatrix}
\beta
\] := $|+\rangle \otimes^l \langle +| \otimes^k + e^{i\beta} |-\rangle \otimes^l \langle -| \otimes^k$

- Hadamard nodes with one input and one output

\[
\boxed{\text{H}} := |+\rangle \langle 0| + |-\rangle \langle 1|
\]
Scalar diagrams

Definition
A zx-calculus diagram is a scalar if it has no inputs or outputs.
Scalar diagrams

Definition
A $\text{zx}$-calculus diagram is a *scalar* if it has no inputs or outputs.

E.g.
![Diagram with nodes labeled $-\pi/2$, $\pi$, $H$, and $\pi/2$]
Scalar diagrams

Definition
A $\mathcal{ZX}$-calculus diagram is a scalar if it has no inputs or outputs.

E.g.

The empty diagram represents the identity scalar:

$$\left[ \begin{array}{c} \pi/2 \\ \pi/2 \end{array} \right] = 1$$
Zero diagrams

Definition
A \( \text{zx} \)-calculus diagram is a \textit{zero diagram} if it represents a zero matrix.
Zero diagrams

Definition
A \( \text{zx} \)-calculus diagram is a zero diagram if it represents a zero matrix.

E.g. \( \circ \pi \)

\[
\begin{align*}
\begin{bmatrix}
\circ & \pi
\end{bmatrix}
&= |0\rangle \otimes^0 \langle 0| \otimes^0 + e^{i\pi} |1\rangle \otimes^0 \langle 1| \otimes^0 = 1 - 1 = 0
\end{align*}
\]
Rules of the scalar-free $\text{ZX}$-calculus

- only the topology matters
- ignore non-zero scalar factors

Rules also hold upside-down and/or with the colours swapped.
Completeness

Definition
A graphical calculus for quantum theory is complete if any equality that can be derived using matrices can also be derived graphically, i.e. for any diagrams $D_1$ and $D_2$:

$$[D_1] = [D_2] \implies D_1 = D_2.$$
Completeness

Definition
A graphical calculus for quantum theory is complete if any equality that can be derived using matrices can also be derived graphically, i.e. for any diagrams \( D_1 \) and \( D_2 \):

\[
[D_1] = [D_2] \implies D_1 = D_2.
\]

Theorem (arXiv:1307.7025)
The scalar-free \(ZX\)-calculus is complete for stabilizer quantum mechanics.

Proof (sketch).
Any non-scalar stabilizer \(ZX\)-calculus diagram can be brought into a (non-unique) normal form called GS-LC form. If two GS-LC form diagrams represent the same operator up to scalar factor, then they are equal in the scalar-free \(ZX\)-calculus.
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Rules of the \( \text{zx-calculus without scalars} \)

- only the topology matters
- ignore non-zero scalar factors

\[
\begin{align*}
\alpha & = \alpha + \beta \\
\beta & = \cdots \\
\alpha & = \pi \\
\pi & = \alpha \\
\pi & = \pi/2 \\
\end{align*}
\]

Rules also hold upside-down and/or with the colours swapped.
Rules of the zx-calculus with scalars

- only the topology matters

\[ \alpha \cdots \beta = \alpha + \beta \]

\[ \pi = \pi \]

\[ \alpha = \alpha \]

\[ H = \frac{\pi}{2} \]

\[ -\alpha = -\alpha \]

Rules also hold upside-down and/or with the colours swapped.
Corollaries to the original stabilizer completeness proof

Assume every non-zero scalar diagram has an inverse.

E.g. \[ \begin{array}{c}
\text{\text{\includegraphics[width=2cm]{diagram1.png}}} \\
\text{\text{\includegraphics[width=2cm]{diagram2.png}}}
\end{array} \] but \[ \begin{array}{c}
\text{\text{\includegraphics[width=2cm]{diagram3.png}}} \\
\text{\text{\includegraphics[width=2cm]{diagram4.png}}}
\end{array} \]
Corollaries to the original stabilizer completeness proof

Assume every non-zero scalar diagram has an inverse.

E.g. \[ \begin{array}{c}
\text{Diagram 1}
\end{array} \]

Assume \[ \begin{array}{c}
\text{Diagram 2}
\end{array} \]

Then: \[ \begin{array}{c}
\text{Diagram 3}
\end{array} \]
Corollaries to the original stabilizer completeness proof

Assume every non-zero scalar diagram has an inverse.

Corollary

Any stabilizer scalar diagram can be decomposed into disconnected segments containing at most two nodes each.
Corollaries to the original stabilizer completeness proof

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Corollary

Any stabilizer scalar diagram can be decomposed into disconnected segments containing at most two nodes each.

Corollary

When a stabilizer zero diagram is brought into normal form and all scalar subdiagrams are decomposed as in the corollary above, the resulting diagram explicitly contains at least one of:

\[ \pi, \quad \frac{\pi}{2}, \quad -\frac{\pi}{2}, \quad \frac{\pi}{2} \]

Will see later that the above zero scalars can all be rewritten into each other, as

\[ \begin{bmatrix} \pi/2 \end{bmatrix} = e^{i\pi/4} \begin{bmatrix} \pi/2 \end{bmatrix}. \]
The star node and the star rule

Any non-zero scalar diagram built from disconnected segments containing at most two nodes each represents a number with absolute value greater than 1.

Introduce new node ★ – the *star node* – with $[\text{★}] = 1/2$, and a new rewrite rule – the *star rule*:

\[
\begin{align*}
\text{★} \odot & = \\
\text{★} \odot & =
\end{align*}
\]

Can then derive:

\[
\begin{align*}
\text{★} & = \\
\end{align*}
\]
The star node and the star rule

Any non-zero scalar diagram built from disconnected segments containing at most two nodes each represents a number with absolute value greater than 1.

Introduce new node ★ – the *star node* – with $[★] = 1/2$, and a new rewrite rule – the *star rule*:

\[
★ \circ =
\]

Can then derive:

\[
(★ \circ) \circ =
\]
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Completeness for non-zero stabilizer scalars

Theorem
The following is a unique normal form for non-zero stabilizer scalars: take one element of the set

\[
\left\{ \begin{array}{l}
\pi/2, \\
\pi/2, \\
\pi/2, \\
\pi/2, \\
-\pi/2, \\
-\pi/2, \\
\pi, \\
-\pi/2, \\
\pi, \\
\pi, \\
-\pi/2, \\
-\pi/2, \\
\pi, \\
-\pi/2, \\
\pi, \\
-\pi/2,
\end{array} \right. 
\]

and combine it with

- some number of copies of $\bullet$, or
- some number of copies of $\star$, or
- one copy of $\bullet$ and some number of copies of $\star$. 

Completeness for non-zero stabilizer scalars

**Theorem**

The following is a unique normal form for non-zero stabilizer scalars: take one element of the set

\[
\left\{ \right. \\
\left. \begin{array}{l}
\frac{\pi}{2}, \\
\frac{\pi}{2}, \\
\frac{\pi}{2}, \\
\frac{\pi}{2}, \\
\frac{-\pi}{2}, \\
\frac{-\pi}{2}, \\
\frac{-\pi}{2}, \\
\frac{-\pi}{2}
\end{array} \right
\}
\]

and combine it with

- some number of copies of \(\bigcirc\), or
- some number of copies of \(\star\), or
- one copy of \(\bigcirc\) and some number of copies of \(\star\).

Non-zero stabilizer scalar diagrams represent complex numbers \(\sqrt{2^r} e^{i s \pi / 4}\) for (possibly negative) integers \(r, s\).
Completeness for non-zero scaled stabilizer diagrams

Theorem

The scaled stabilizer $\text{zx}$-calculus with $\blacklozenge$ and the star rule is complete for non-zero scaled stabilizer diagrams.

Proof.

To derive equalities between non-zero scaled stabilizer diagrams:

▶ Deal with the non-scalar parts of the diagrams as in the scalar-free completeness proof [arXiv:1307.7025], but keep track of the scalars on the side.

▶ If the non-scalar parts are not equal up to scalar, the full diagrams cannot be equal.

▶ If the non-scalar parts are equal, bring the scalar parts into the normal form.

▶ The resulting diagrams are either identical or they do not represent the same matrix.
Completeness for non-zero scaled stabilizer diagrams

Theorem

The scaled stabilizer $zx$-calculus with $\star$ and the star rule is complete for non-zero scaled stabilizer diagrams.

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Completeness for non-zero scaled stabilizer diagrams

**Theorem**

*The scaled stabilizer $\text{ZX}$-calculus with $\dagger$ and the star rule is complete for non-zero scaled stabilizer diagrams.*

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To derive equalities between non-zero scaled stabilizer diagrams:

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Completeness for non-zero scaled stabilizer diagrams

Theorem

The scaled stabilizer $\text{zx}$-calculus with $\star$ and the star rule is complete for non-zero scaled stabilizer diagrams.

Proof.

To derive equalities between non-zero scaled stabilizer diagrams:

- Deal with the non-scalar parts of the diagrams as in the scalar-free completeness proof [arXiv:1307.7025], but keep track of the scalars on the side.
- If the non-scalar parts are not equal up to scalar, the full diagrams cannot be equal.
- If the non-scalar parts are equal, bring the scalar parts into the normal form.
Completeness for non-zero scaled stabilizer diagrams

Theorem
The scaled stabilizer \(ZX\)-calculus with ★ and the star rule is complete for non-zero scaled stabilizer diagrams.

Proof.
To derive equalities between non-zero scaled stabilizer diagrams:

▷ Deal with the non-scalar parts of the diagrams as in the scalar-free completeness proof [arXiv:1307.7025], but keep track of the scalars on the side.

▷ If the non-scalar parts are not equal up to scalar, the full diagrams cannot be equal.

▷ If the non-scalar parts are equal, bring the scalar parts into the normal form.

▷ The resulting diagrams are either identical or they do not represent the same matrix.
Stabilizer zero diagrams

New rules: the zero rule [suggested by Aleks Kissinger]:

\[ \pi = \pi \]

and the zero scalar rule:

\[ \pi \alpha = \pi \]
Stabilizer zero diagrams

New rules: the zero rule [suggested by Aleks Kissinger]:

\[ \pi = \pi \]

and the zero scalar rule:

\[ \pi \alpha = \pi \]

Theorem

*The scaled stabilizer ZX-calculus with the star rule, zero rule, and zero scalar rule is complete for zero diagrams.*

Proof.

This is a unique normal form for stabilizer zero diagrams:
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Conclusions and Outlook

- The $\textit{zx}$-calculus was known to be complete for stabilizer QM without scalars, i.e. equalities between operators could be derived up to scalar factor.

- We have modified the existing rewrite rules, added a new node, and added three new rewrite rules.

- With these, the $\textit{zx}$-calculus is complete for stabilizer QM with scalars, i.e. can now compute amplitudes and probabilities graphically.

- Can completeness be extended to larger fragment of QM, e.g. Clifford+T group?

Thank you!
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