Making the stabilizer ZX-calculus complete for scalars

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Outline

Background

Modifying the ZX-calculus to keep account of scalars

The new completeness results

Conclusions

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Stabilizer quantum mechanics

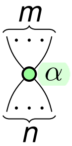
- preparation of qubits in state $|0\rangle$
- Clifford unitaries, generated by

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \; H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \; C_X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

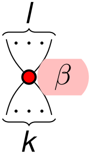
- measurements in computational basis

Elements of stabilizer ZX-calculus diagrams

- green nodes with n inputs and m outputs,
  $\alpha \in \{-\pi/2, 0, \pi/2, \pi\}$



- red nodes with k inputs and l outputs,
  $\beta \in \{-\pi/2, 0, \pi/2, \pi\}$



- Hadamard nodes with one input and one output

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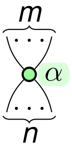

$$[\![\cdot]\!] := |0\rangle^{\otimes m} \langle 0|^{\otimes n} + e^{i\alpha} |1\rangle^{\otimes m} \langle 1|^{\otimes n},$$

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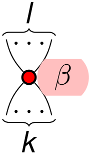

$$[\![\cdot]\!] := |+\rangle^{\otimes l} \langle +|^{\otimes k} + e^{i\beta} |-\rangle^{\otimes l} \langle -|^{\otimes k},$$

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$$[\![\cdot]\!] := |+\rangle \langle 0| + |-\rangle \langle 1|$$

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Definition

A ZX-calculus diagram is a *scalar* if it has no inputs or outputs.

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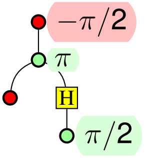

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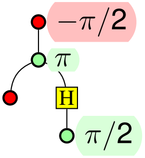
E.g. 

The empty diagram represents the identity scalar:

$$[\![\quad]\!] = 1$$

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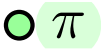
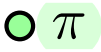
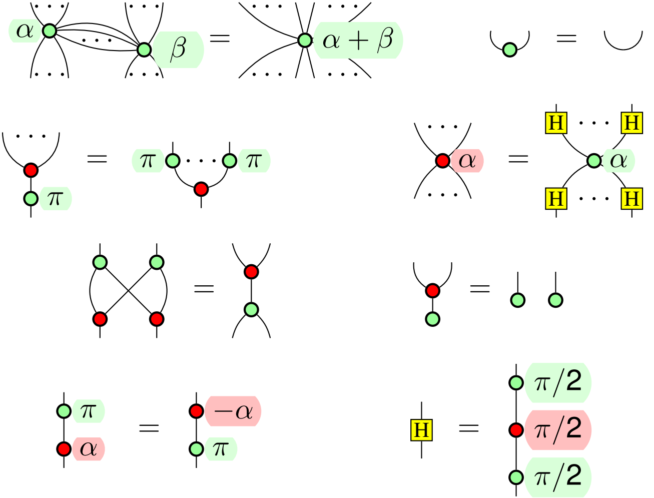
E.g. 

$$[\![\text{ }]\!] = |0\rangle^{\otimes 0} \langle 0|^{\otimes 0} + e^{i\pi} |1\rangle^{\otimes 0} \langle 1|^{\otimes 0} = 1 - 1 = 0$$



Rules of the scalar-free ZX-calculus

- only the topology matters
- ignore non-zero scalar factors



Rules also hold upside-down and/or with the colours swapped.

Completeness

Definition

A graphical calculus for quantum theory is *complete* if any equality that can be derived using matrices can also be derived graphically, i.e. for any diagrams D_1 and D_2:

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Theorem (arXiv:1307.7025)

The scalar-free ZX*-calculus is complete for stabilizer quantum mechanics.*

Proof (sketch).

Any non-scalar stabilizer ZX-calculus diagram can be brought into a (non-unique) normal form called GS-LC form.
If two GS-LC form diagrams represent the same operator up to scalar factor, then they are equal in the scalar-free ZX-calculus. □

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Rules of the ZX-calculus without scalars

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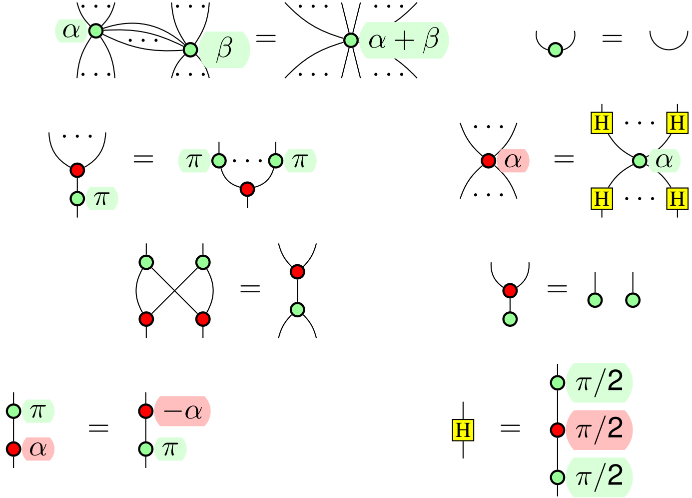

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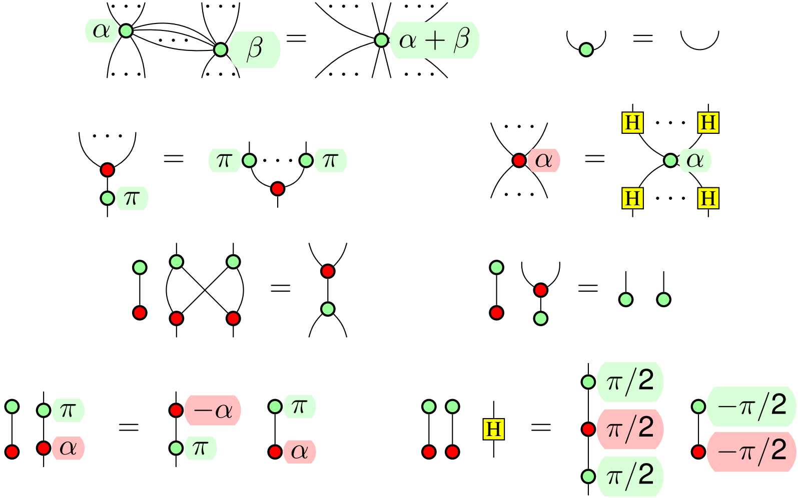

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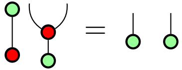
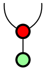
Corollaries to the original stabilizer completeness proof

Assume every non-zero scalar diagram has an inverse.

E.g.  but  = ??

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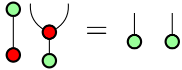
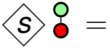
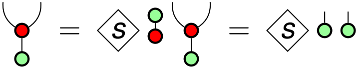
E.g. 

Assume  . Then:



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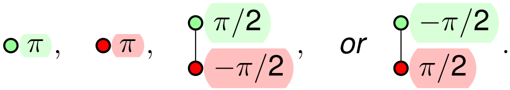
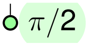
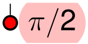
When a stabilizer zero diagram is brought into normal form and all scalar subdiagrams are decomposed as in the corollary above, the resulting diagram explicitly contains at least one of:



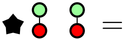
Will see later that the above zero scalars can all be rewritten into each other, as ⟦⟧ $= e^{i\pi/4}$ ⟦⟧.

The star node and the star rule

Any non-zero scalar diagram built from disconnected segments containing at most two nodes each represents a number with absolute value greater than 1.

Introduce new node ⬟ – the *star node* – with $[\![⬟]\!] = 1/2$, and a new rewrite rule – the *star rule*:

Can then derive:



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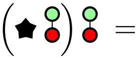

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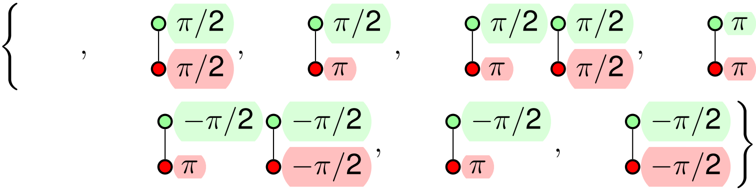
The following is a unique normal form for non-zero stabilizer scalars: take one element of the set



and combine it with

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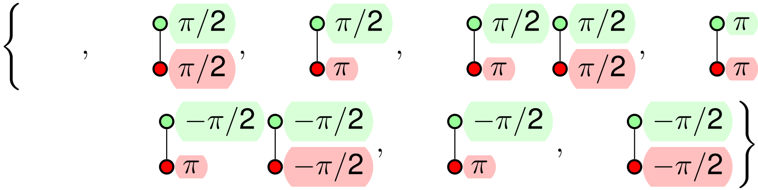

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Non-zero stabilizer scalar diagrams represent complex numbers $\sqrt{2^r}e^{is\pi/4}$ for (possibly negative) integers r, s.

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- If the non-scalar parts are equal, bring the scalar parts into the normal form.
- The resulting diagrams are either identical or they do not represent the same matrix.

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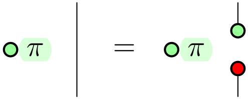
Stabilizer zero diagrams

New rules: the *zero rule* [suggested by Aleks Kissinger]:



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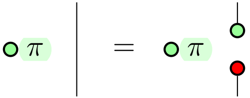

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Theorem

The scaled stabilizer ZX-calculus with the star rule, zero rule, and zero scalar rule is complete for zero diagrams.

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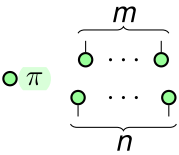
This is a unique normal form for stabilizer zero diagrams:



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