DEMONIC Programming

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Outline

- Maxwell's Demon and Landauer's Hypothesis.
- Thermodynamics in 1 slide.
- A toy model system.
- Defining a programming language for the toy system.
- DEMONIC syntax and semantics.
- Allowed operations expressed in DEMONIC.
- Formal verification: a computational invariant.
- The Second Law and Landauer's Hypothesis proven.


KING LEAR
BY WILLIAM SHAKESPEARE
Oxford Theatre Guild presents KING LEAR
7-18 July, Merton College Gardens
7.30pm (2.30pm matinee, Saturday 11th)
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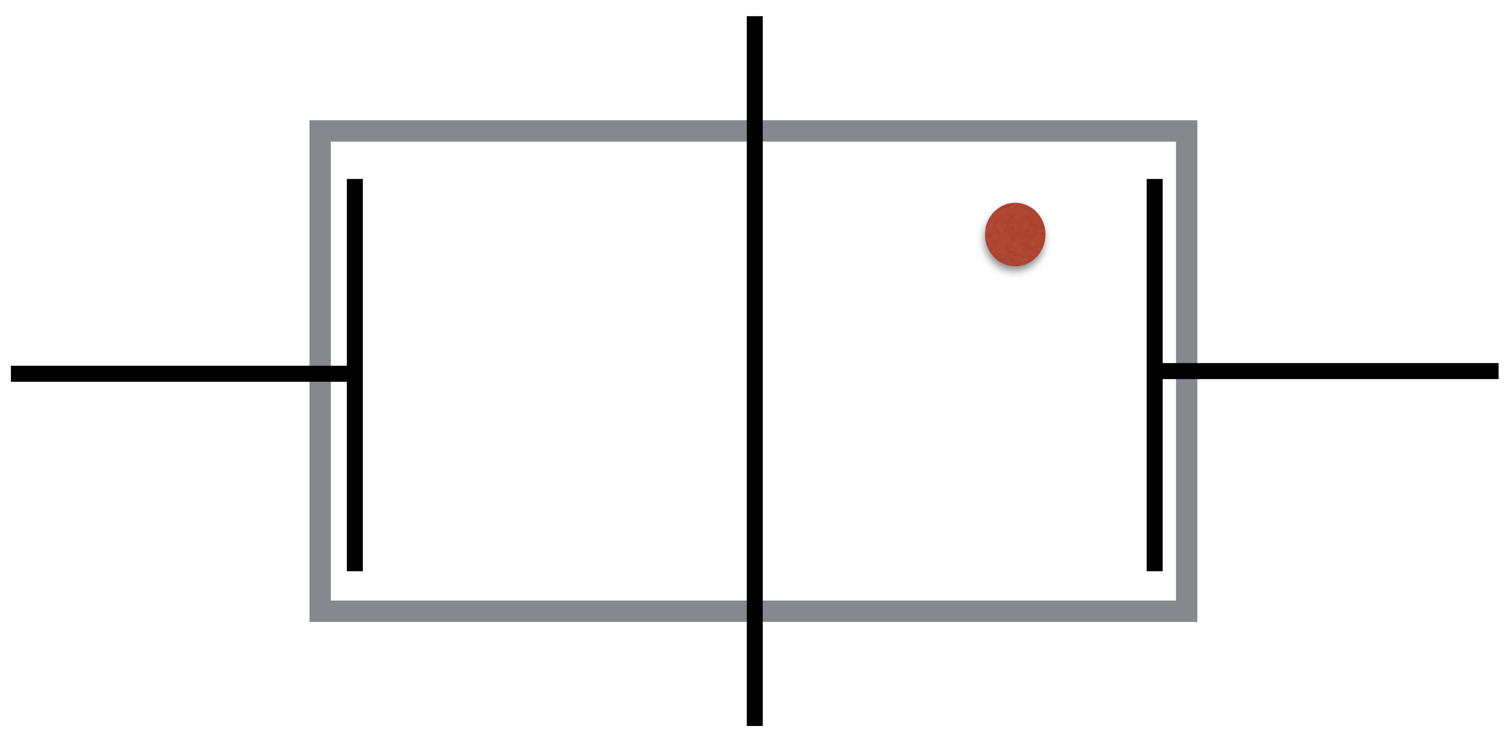
O for a muse of fire!

Henry V; opening words

O for a muse of ~~fire!~~

single-particle gas in equilibrium with a heat bath at temperature T!



The thermodynamics of computation

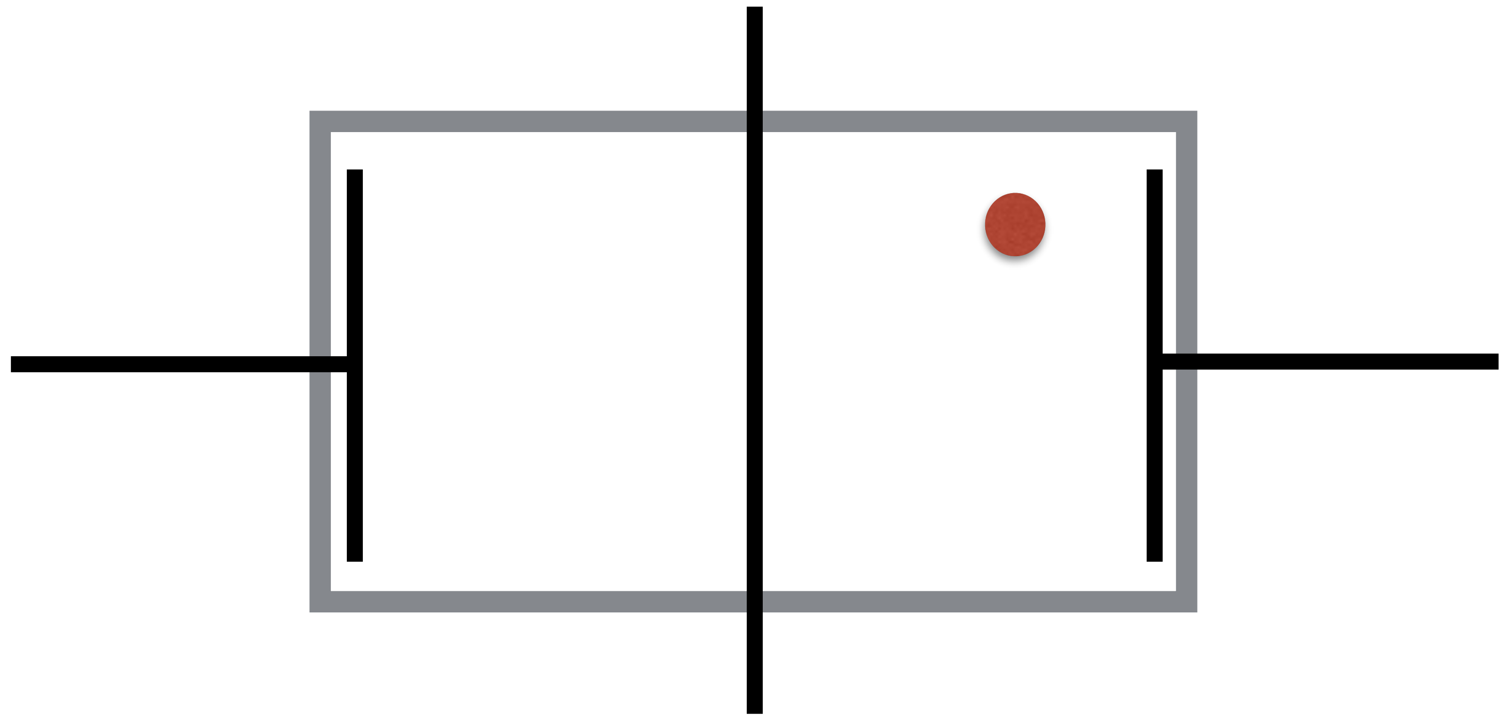
"Information is physical" — information processing is necessarily a physical process obeying laws of physics.

Thermodynamics: one-way entropy increase, therefore key constraints on what and how information can be processed.

The connection between information and entropy/thermodynamics: **Landauer's Hypothesis**: erasure of 1 bit requires kTln2 of work.

But . . . it has not been **proven**.

A toy system - thermodynamics 101



Single particle in a box, two pistons, one partition, heat bath T.

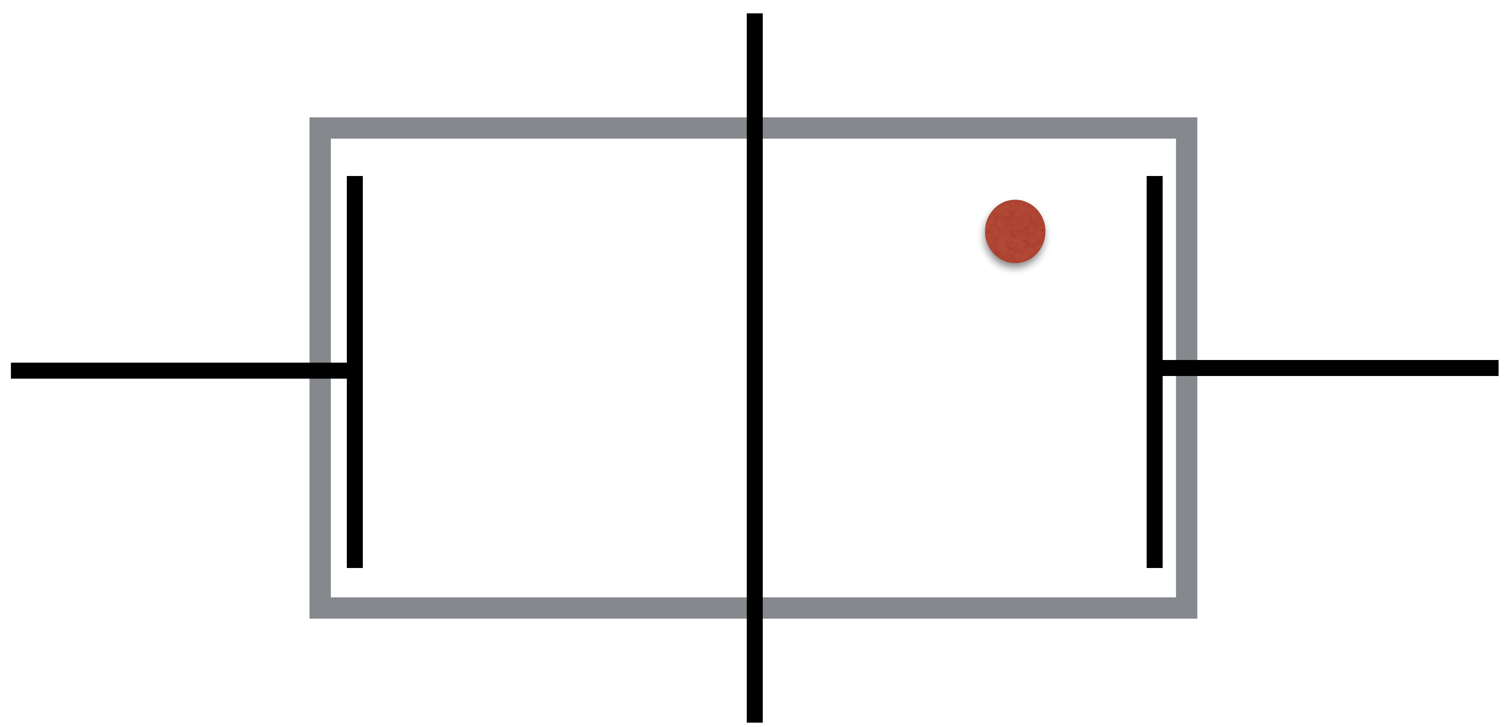
Variables: pressure p, volume V, entropy $H = -k(p_L \ln p_L + p_R \ln p_R)$

$$\frac{pV}{T} = \text{const} \qquad\qquad H_{\text{out}} - H_{\text{in}} = k \ln \frac{V_{\text{out}}}{V_{\text{in}}}$$

A toy system - thermodynamics 101

Allowed operations:

Insert/remove partition



A toy system - thermodynamics 101

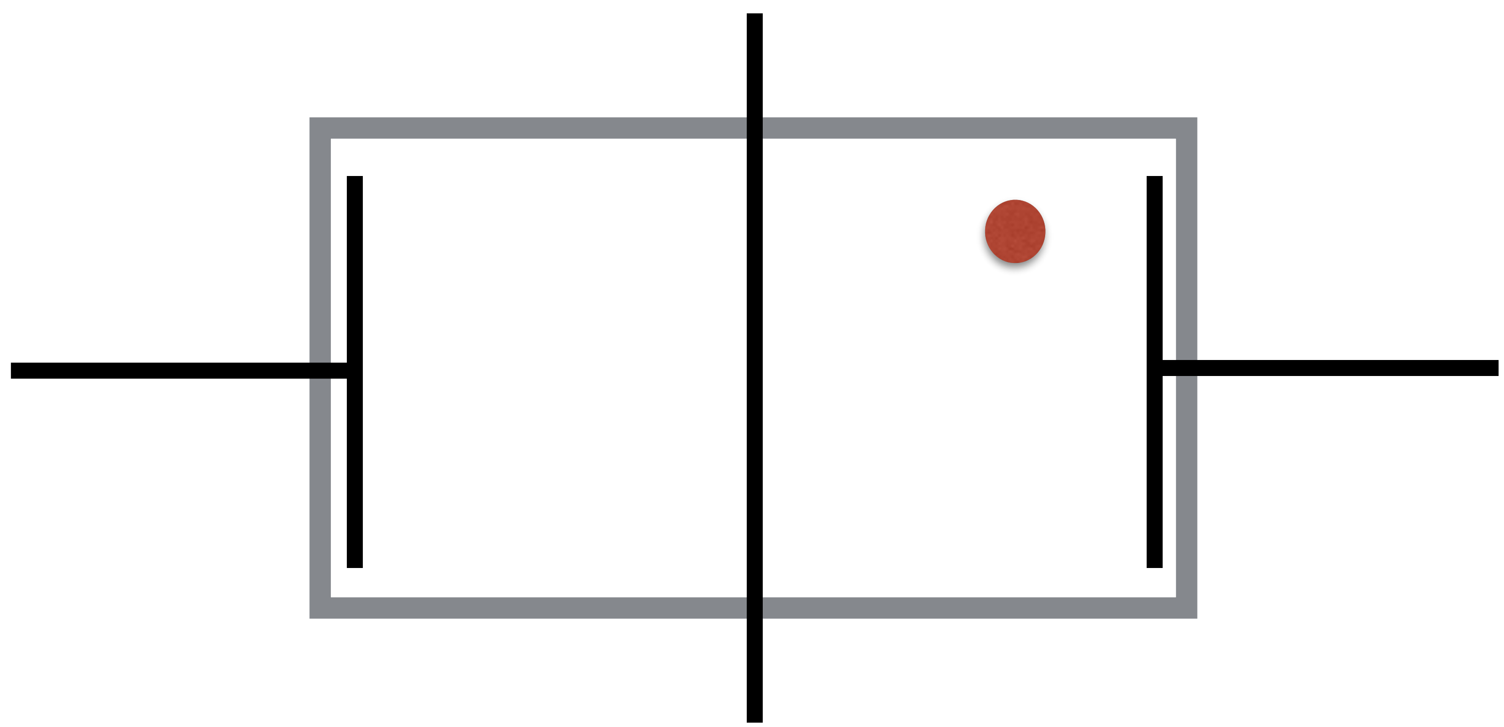
Allowed operations:

Insert/remove partition

A toy system - thermodynamics 101

Allowed operations:

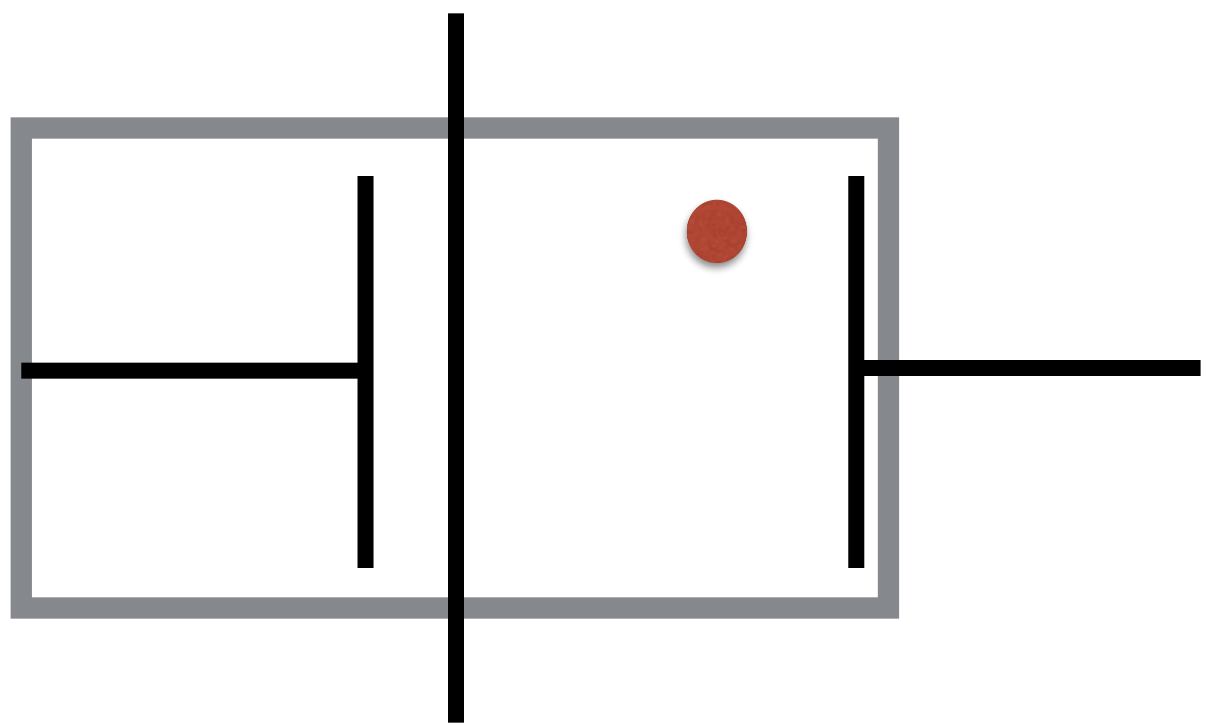
Insert/remove pistons left and right



A toy system - thermodynamics 101

Allowed operations:

Insert/remove pistons left and right



A toy system - thermodynamics 101

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Insert/remove pistons left and right

A toy system - thermodynamics 101

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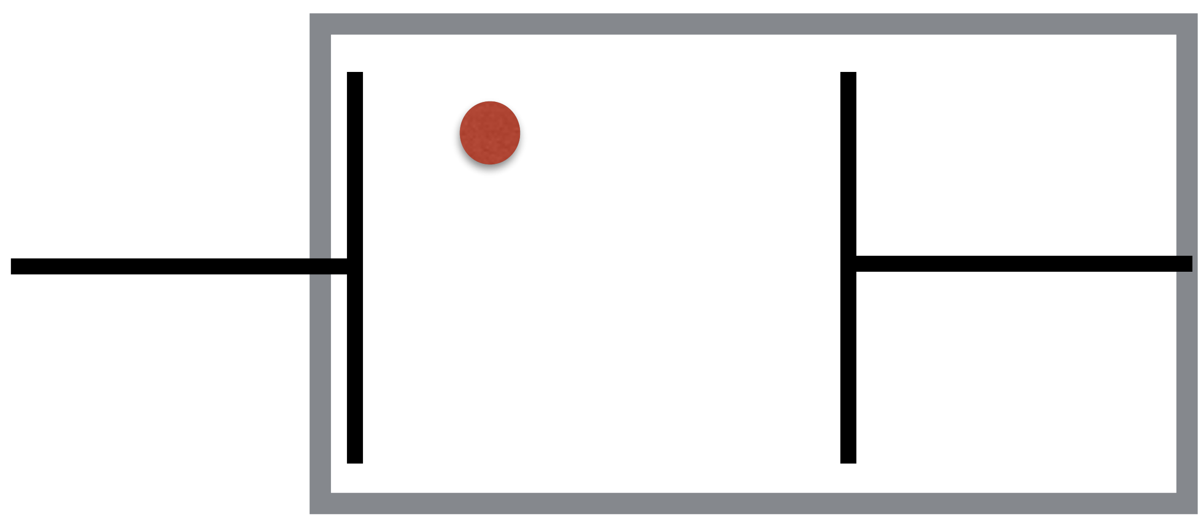


(NB isothermal compression: requires work kTln2)

Formalising the system

What we will do: extract out the **logical structure** of the state and allowed transitions into a **programming language**.

An allowed operation is a **function/basic statement** not a primitive. Allowed programs are built out of allowed statements.

state variable: $s = (X, A, I, w) \in \mathbb{T} \times \mathbb{B} \times \mathbb{B} \times \mathbb{Z}$

$X \in \mathbb{T} := \{0, \frac{1}{2}, 1\}$: prob of being on LHS.

A, *I*: Boolean flags for partition/a piston.

w: total work extracted from the system in unit of kTln2.

DEMONIC syntax

$$
\begin{aligned}
&\text{LProb} \ ::= \mathbb{T} := \ \mathtt{0} \mid \mathtt{1/2} \mid \mathtt{1} \\
&\text{Part} \ ::= \ \mathbb{B} := \ \mathtt{true} \mid \mathtt{false} \\
&\text{Pist} \ ::= \ \mathbb{B} \\
&\text{WUnit} \ ::= \ \mathbb{Z} \\
&\text{Field} \ ::= \ LProb \mid Part \mid Pist \mid WUnit \\
&\text{Fieldname} \ ::= \ \mathtt{X} \mid \mathtt{A} \mid \mathtt{I} \mid \mathtt{w} \quad (\text{where } W = wkT \ln 2) \\
&s \in \text{State} \ ::= \ (LProb, Part, Pist, WUnit) \\
&\text{BExp} \ ::= \ \mathbb{B} \mid State.A \mid State.I \\
&S \in \text{Statement} \ ::= \ S_1 ; S_2 \mid \ S_1 \oplus S_2 \mid State.Fieldname := Field \\
&\qquad\qquad\qquad\qquad\quad \mid \mathtt{if}\ BExp\ \mathtt{then}\ S_1\ \mathtt{else}\ S_2 \mid \mathtt{skip}
\end{aligned}
$$

DEMONIC operational semantics

$$\text{(assign)} \qquad \langle x := a,\ s\rangle \Rightarrow \langle \texttt{skip}, s[x \mapsto a]\rangle$$

$$\text{(comp1)} \qquad \frac{\langle S_1,\ s\rangle \Rightarrow_p \langle S_1', s'\rangle}{\langle S_1; S_2,\ s\rangle \Rightarrow_p \langle S_1'; S_2,\ s'\rangle}$$

$$\text{(comp2)} \qquad \langle \texttt{skip}; S,\ s\rangle \Rightarrow \langle S,\ s\rangle$$

$$\text{(if1)} \qquad \langle \texttt{if}\ B\ \texttt{then}\ S_1\ \texttt{else}\ S_2,\ s\rangle \Rightarrow \langle S_1,\ s\rangle \text{ if } [\![B]\!]s = \texttt{true}$$

$$\text{(if2)} \qquad \langle \texttt{if}\ B\ \texttt{then}\ S_1\ \texttt{else}\ S_2,\ s\rangle \Rightarrow \langle S_2,\ s\rangle \text{ if } [\![B]\!]s = \texttt{false}$$

$$\text{(prob1)} \qquad \langle S_1 \oplus S_2,\ s\rangle \Rightarrow_{1/2} \langle S_1,\ s\rangle$$

$$\text{(prob2)} \qquad \langle S_1 \oplus S_2,\ s\rangle \Rightarrow_{1/2} \langle S_2,\ s\rangle$$

Allowed thermodynamic operations

Inserting a partition:

$$\textbf{PartIn} \;=_{def}\; (s.A := \texttt{true})$$

Removing a partition:

$$\begin{aligned}\textbf{PartOut} \;=_{def}\; & \text{if } (s.A = \texttt{true}) \text{ then} \\ & \quad (\text{if } (s.I = \texttt{false}) \text{ then} \\ & \qquad (s.X := \tfrac{1}{2}) \text{ and } (s.A := \texttt{false}) \text{ else} \\ & \qquad\quad (s.A := \texttt{false})\ ) \\ & \qquad\qquad \text{else } \texttt{skip}\end{aligned}$$

Allowed thermodynamic operations

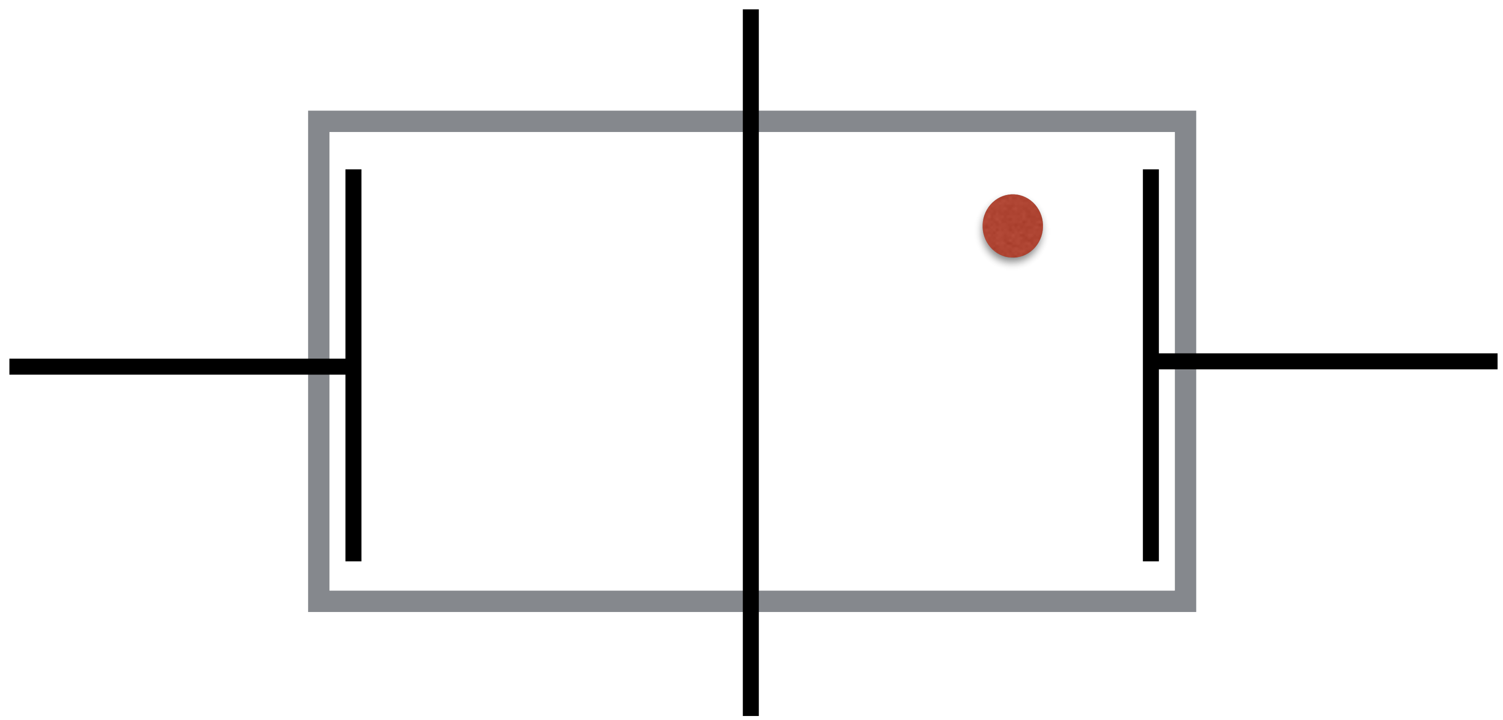
Removing a piston (left and right)

$$\textbf{LPistOut} \;=_{def}\; \text{if } (s.I = \texttt{false}) \text{ or } \neg(s.X = 0) \text{ then}$$
$$\texttt{skip} \text{ else}$$
$$(\text{ if } (s.A = \texttt{true}) \text{ then}$$
$$(s.I := \texttt{false}) \text{ else}$$
$$(s.I := \texttt{false}) \text{ and } (s.X := \tfrac{1}{2}) \text{ and } (s.w := w+1)\;)$$

$$\textbf{RPistOut} \;=_{def}\; \text{if } (s.I = \texttt{false}) \text{ or } \neg(s.X = 1) \text{ then}$$
$$\texttt{skip} \text{ else}$$
$$(\text{ if } (s.A = \texttt{true}) \text{ then}$$
$$(s.I := \texttt{false}) \text{ else}$$
$$(s.I := \texttt{false}) \text{ and } (s.X := \tfrac{1}{2}) \text{ and } (s.w := w+1)\;)$$

Allowed thermodynamic operations

Inserting a piston is more complicated…



Can we insert a piston to the right?
No! Would compress to zero volume, requiring infinite work.
But for a programming language we have to give the outcome if it were attempted.

Allowed thermodynamic operations

Consider this cycle

$$\begin{aligned}&\langle \mathit{PartIn};\mathit{LPistIn};\mathit{PartOut};\mathit{LPistOut},\ (\tfrac{1}{2},\ F,\ F,\ w_0)\rangle\\ &\Longrightarrow\ \langle \mathit{LPistIn};\mathit{PartOut};\mathit{LPistOut},\ (\tfrac{1}{2},\ T,\ F,\ w_0)\rangle\\ &\Longrightarrow_{1/2}\ \langle \mathit{PartOut};\mathit{LPistOut},\ (0,\ T,\ T,\ w_0)\rangle\\ &\Longrightarrow\ \langle \mathit{LPistOut},\ (0,\ F,\ T,\ w_0)\rangle\\ &\Longrightarrow\ \langle \mathit{skip},\ (\tfrac{1}{2},\ F,\ F,\ w_0+1)\rangle\end{aligned}$$

$$\begin{aligned}&\langle \mathit{PartIn};\mathit{LPistIn};\mathit{PartOut};\mathit{LPistOut},\ (\tfrac{1}{2},\ F,\ F,\ w_0)\rangle\\ &\Longrightarrow\ \langle \mathit{LPistIn};\mathit{PartOut};\mathit{LPistOut},\ (\tfrac{1}{2},\ T,\ F,\ w_0)\rangle\\ &\Longrightarrow_{1/2}\ \langle \mathit{PartOut};\mathit{LPistOut},\ (0,\ T,\ F,\ w_0-w_c)\rangle\\ &\Longrightarrow\ \langle \mathit{LPistOut},\ (\tfrac{1}{2},\ F,\ F,\ w_0-w_c)\rangle\\ &\Longrightarrow\ \langle \mathit{skip},\ (\tfrac{1}{2},\ F,\ F,\ w_0-w_c)\rangle\end{aligned}$$

Expected work extracted: $W_e = (w_0 + \frac{1}{2}(1-w_c))kT\ln 2.$

No perpetual motion implies $W_e \leq W_o$, i.e. $w_c \geq 1$

Allowed thermodynamic operations

Therefore...

$$
\begin{aligned}
\textbf{LPistIn} =_{def}\ & \text{if } (s.X = 1) \text{ then} \\
& (s.w := w - 1) \text{ else} \\
& \quad (\text{ if } (s.X = 0) \text{ then} \\
& \qquad (s.I := \texttt{true}) \text{ else} \\
& \qquad\quad (\text{ if } (s.A = \texttt{false}) \text{ then} \\
& \qquad\qquad (s.X := 1) \text{ and } (s.w := w - 1) \text{ and } (s.I := \texttt{true}) \text{ else} \\
& \qquad\qquad\quad [(s.X := 0) \text{ and } (s.I := \texttt{true})] \oplus [(s.X := 1) \text{ and } (s.w := w - 1)]))
\end{aligned}
$$

$$
\begin{aligned}
\textbf{RPistIn} =_{def}\ & \text{if } (s.X = 0) \text{ then} \\
& (s.w := w - 1) \text{ else} \\
& \quad (\text{ if } (s.X = 1) \text{ then} \\
& \qquad (s.I := \texttt{true}) \text{ else} \\
& \qquad\quad (\text{ if } (s.A = \texttt{false}) \text{ then} \\
& \qquad\qquad (s.X := 0) \text{ and } (s.w := w - 1) \text{ and } (s.I := \texttt{true}) \text{ else} \\
& \qquad\qquad\quad [(s.X := 1) \text{ and } (s.I := \texttt{true})] \oplus [(s.X := 0) \text{ and } (s.w := w - 1)]))
\end{aligned}
$$

Computational Invariant Statement

Probabilistic computational invariants are given over the set of probability distributions over states.

This is easy for a physicist: expectation values!

An **invariant statement** is a predicate that is true after a transition if it is true before, and **preserved under composition.**

What is the invariant statement for this single-particle system…?

Computational Invariant Statement

$$\langle wk\ln 2\rangle - \frac{1}{2}\left(\langle H(X)\rangle + H(\langle X\rangle)\right) \leq 0$$

Where $H(x) = -kT(x\ln x + (1-x)\ln(1-x))$

Every composition of the allowed thermodynamic operations satisfies this invariant afterwards if it satisfies it beforehand.

What is that entropic quantity???

The Second Law is a theorem of the system

Kelvin statement of the second law: $\nexists \gamma : \; (X_0,\, A_0,\, I_0,\, w_0) \xrightarrow{\gamma} (X_0,\, A_0,\, I_0,\, \langle w_f \rangle > w_0)$

Define the zero-point of the work counter as

$$w_0 = \frac{1}{2k\ln 2}\left(\langle H(X_0)\rangle + H(\langle X_0\rangle)\right)$$

then the invariant is satisfied initially. Final invariant gives

$$\langle w_f \rangle k \ln 2 - \tfrac{1}{2}\left(\langle H(X_0)\rangle + H(\langle X_0\rangle)\right) \leq 0$$

which straightforwardly implies

$$\langle w_f \rangle \leq w_0$$

for all allowed operations and compositions

Landauer Erasure

Two entropies make up the invariant entropy:

$\langle H(X) \rangle$: average entropy within a branch of the computation.

$H(\langle X \rangle)$: entropy of the probability distribution of the computation (across all its branches).

Consider $X=1/2$, partition=true. $\langle H(X) \rangle = H(\langle X \rangle) = k\ln 2$.

Measurement gives in two branches, $X=0$ and $X=1$.
$\langle H(X) \rangle = 0$ but $H(\langle X \rangle) = k\ln 2$ still.

Resetting the result to a known state gives one branch, eg. $X=0$.
$\langle H(X) \rangle = 0$ and $H(\langle X \rangle) = 0$.

Landauer Erasure

Given the invariant

$$\langle wk \ln 2 \rangle - \frac{1}{2}\left(\langle H(X) \rangle + H(\langle X \rangle)\right) \leq 0$$

Measurement of a bit of information requires at least $\frac{1}{2} kT \ln 2$

Resetting of a measured bit of information requires at least $\frac{1}{2} kT \ln 2$

Erasure (measure-then-reset) of an unknown bit of information requires a work cost of at least

$$\frac{1}{2} kT \ln 2 + \frac{1}{2} kT \ln 2 = kT \ln 2$$

Conclusions

We have used **formal semantics and verification** as a process logic for single-particle thermodynamics.

Basic transitions and operations are defined, as are their composition, and a new invariant statement found.

The Second Law is **provably** satisfied by **any combination of the basic operations.** This is not “up for debate”!

Landauer Erasure — work cost of measure then reset — is a formal consequence of the logical system.

Further work

Lots!

Extending to multi-particle states, extend to statistical mechanics, rederive partition function statements, extend definition of Landauer Erasure etc etc etc.

What is the new entropy? What's its connection to the Holevo quantity? What's the relationship to the Second Law?

And finally…

Where else in physics can we use these verification tools to prove formal statements about the possible states of a system??