A diagrammatic axiomatisation of the GHZ and W quantum states

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Oxford, 17 July 2015

The unhelpful third party

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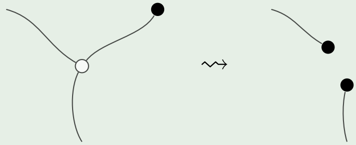
GHZ: $|000\rangle + |111\rangle$



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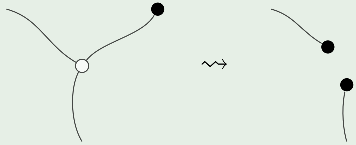
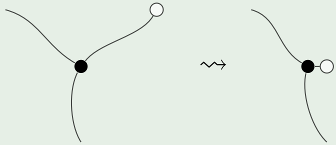
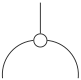


W: $|001\rangle + |010\rangle + |100\rangle$



Only the arity counts

By map-state duality, a *tripartite state* is the same as a binary operation

$|0\rangle\langle 00| + |1\rangle\langle 11|$  $\mapsto \; 000 + 111 \; \leftarrow\!\!\shortmid$  $|000\rangle + |111\rangle$

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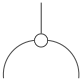
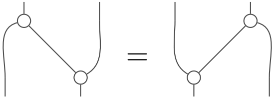
$|0\rangle\langle 00| + |1\rangle\langle 11|$  $\mapsto \quad 000 + 111 \quad \leftarrow\!\shortmid$  $|000\rangle + |111\rangle$

- Bob & Aleks, 2010: we can associate *commutative Frobenius algebras* (with different properties) to the GHZ and W states



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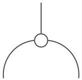
$|0\rangle\langle 00| + |1\rangle\langle 11|$  $\mapsto \; 000 + 111 \; \leftarrow\!\mapsto$  $|000\rangle + |111\rangle$

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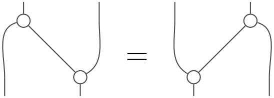

GHZ and W as **building blocks** for higher SLOCC classes?

Axiomatise

Goal: An as-complete-as-possible diagrammatic axiomatisation of the relations between GHZ and W

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Desiderata (the basic ZX calculus meets these!):

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Desiderata (the basic ZX calculus meets these!):

- *a faithful graphical representation of symmetries* (if something looks symmetrical, it better be)
- the axioms should look familiar to algebraists and/or topologists

The ZW calculus

Result: the ZW calculus is complete for the category of abelian groups generated by $\mathbb{Z} \oplus \mathbb{Z}$ through tensoring†

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Warning

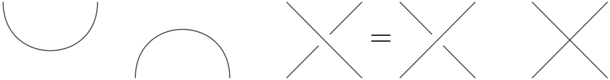
I'll show you a different (but equivalent) version from the one in the paper

The construction of ZW

1. Layer one: Cross
2. Layer two: Even
3. Layer three: Odd
4. Layer four: Copy

A matter of space

- **The new generators:** cup, cap, symmetric braiding, crossing



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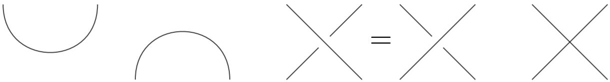
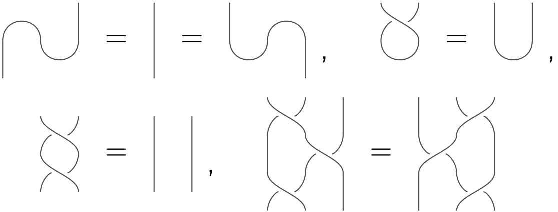


- **What they satisfy:**
  Cup + cap + braiding: zigzag equations + symmetric Reidemeister I, II, III



Who framed Reidemeister?

Cup + cap + crossing: symmetric Reidemeister II, III;
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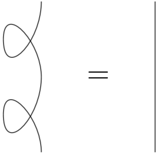

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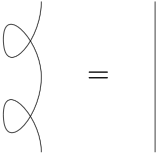

(logic of **blackboard-framed links**, but with a symmetric braiding)

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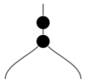
Black dots

- **The new generator:** W algebra



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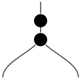

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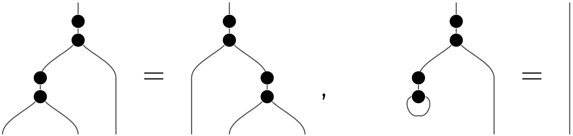

The W bialgebra...

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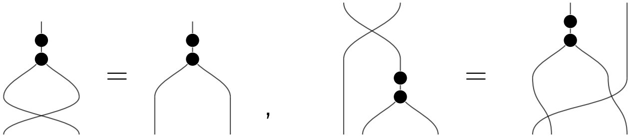

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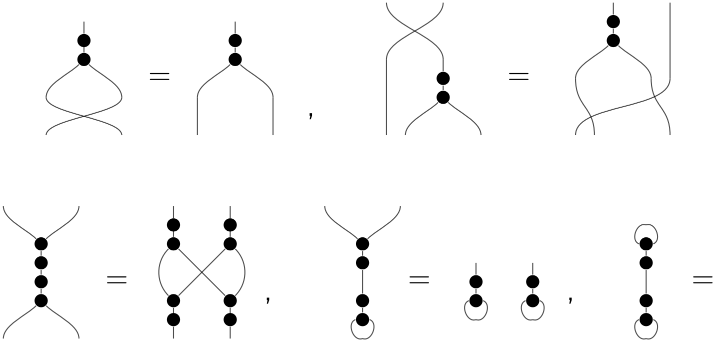

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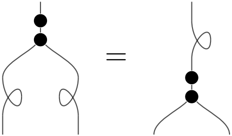

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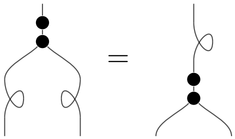
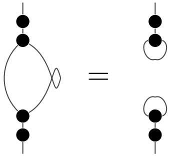
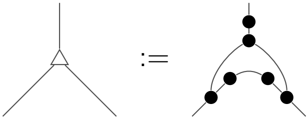


- **Will be provable:**



From ZW to ZX

One can build a gate



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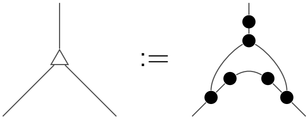
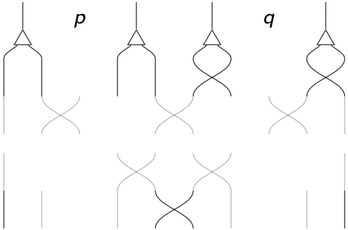


This is actually the **ternary red gate** of the ZX calculus, aka $\mathbb{Z}_2$ on the computational basis

(SLOCC-equivalent to GHZ)

Fun fact

Then, interpreted in $\mathbf{Vec}_{\mathbb{R}}$,



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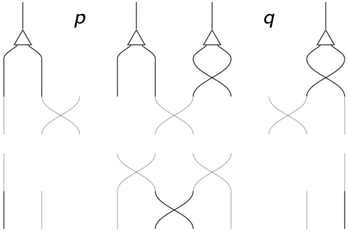

represents multiplication in $\mathrm{Cl}_{p,q}(\mathbb{R})$, the real Clifford algebra with signature (p, q) $\rightsquigarrow$

braiding : crossing = commutation : anticommutation

The construction of ZW

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The X gate

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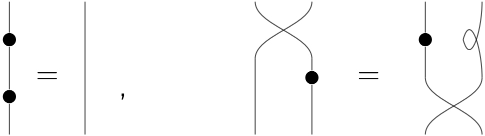

Purity

So far: only purely even/purely odd maps

$\rightsquigarrow$ works for fermions

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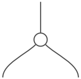

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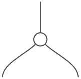

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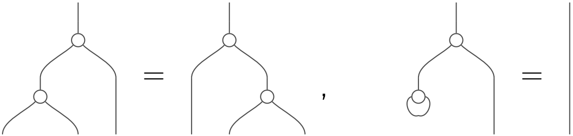

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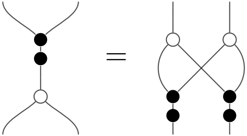

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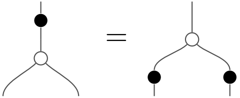

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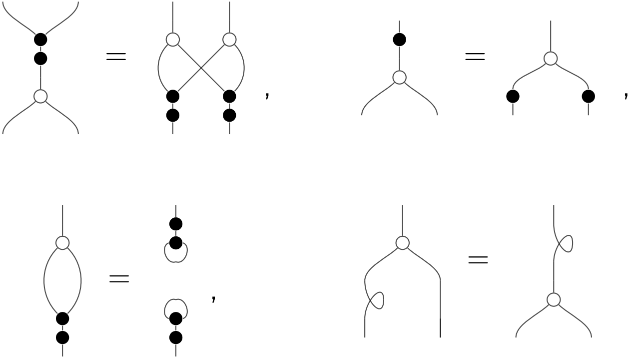
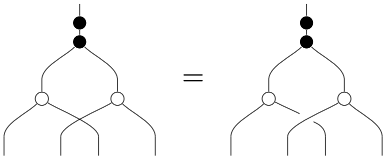


Detach

- **What it satisfies** (finally)**:**



(*crossing elimination rule*)

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2. **Find better normal forms**.
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   Everything disconnectable should be disconnected!
3. **Understand how expressive each layer is**.
   Layer two already contains both 3-qubit SLOCC classes.

Extensions

1. **From integers to real numbers**.
   Signed metric on wires?

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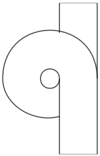
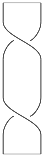
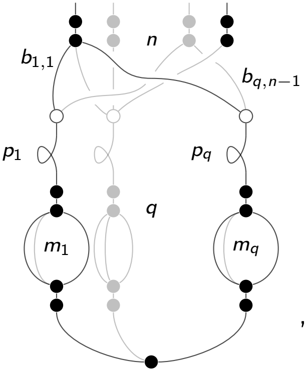
2. **Complex phases.**
   Topology might again give some suggestions!

π phase $=$  $=$  $\rightsquigarrow$  as $\dfrac{\pi}{2}$?

Thank you for your attention!



,

$$\sum_{i=1}^{q} (-1)^{p_i}\, m_i\, \left| b_{i,1} \ldots b_{i,n} \right\rangle$$