Total and Partial Computation in Categorical Quantum Foundations

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Background

“Effectus Theory”

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My work: partial computation in effectuses

Outline

1. Effectus Theory [Jacobs, New Directions]
2. Partial Computation in Effectuses
3. Conclusions and Future Work

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2. Partial Computation in Effectuses

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Basic idea and quantum example

An effectus has 1 and $(+, 0)$, by definition

Idea. In an effectus:

- $\omega\colon 1 \to X$ state
- $p\colon X \to 1+1$ predicate
- Validity $(\omega \vDash p) := (1 \xrightarrow{\omega} X \xrightarrow{p} 1+1)$
- $1 \to 1+1$ scalar

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- Quantum example: the category $\mathbf{Cstar}_{\mathrm{PU}}$ of (unital) C*-algebras and positive unital (= PU) maps
- The opposite $\mathbf{Cstar}_{\mathrm{PU}}^{\mathrm{op}}$ is an effectus
  - $1 = \mathbb{C}$, the algebra of complex numbers

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  - $1 = \mathbb{C}$, the algebra of complex numbers
- $\mathbf{Cstar}_{\mathrm{CPU}}^{\mathrm{op}}$, $\mathbf{Wstar}_{\mathrm{PU}}^{\mathrm{op}}$, $\mathbf{Wstar}_{\mathrm{CPU}}^{\mathrm{op}}$ are also effectuses
  - CPU = completely positive unital

States and predicates in C*-algebras

States

$$\frac{\omega\colon 1 \longrightarrow A \text{ in } \mathbf{Cstar}_{\mathsf{PU}}^{\mathsf{op}}}{\text{PU-functional } \omega\colon A \longrightarrow \mathbb{C}}$$

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$(*)$ $e = p(1,0)$ and $p(\lambda_1,\lambda_2) = \lambda_1 e + \lambda_2(1-e)$

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- Validity $(\omega \vDash e) = \omega(e) \in [0,1]$ is the Born rule
- State of $\mathcal{M}_n := \mathbb{C}^{n\times n}$ is of the form $\mathrm{tr}(\rho \cdot -)$ for a density matrix ρ, hence the validity $\mathrm{tr}(\rho \cdot e)$

Effect algebras

Fact.

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Def. A partial commutative monoid (PCM) is

- set M, with 'zero' $0 \in M$, partial 'sum' $\ovee\colon M \times M \rightharpoonup M$

s.t. $\ovee$ is associative, commutative, and $x \ovee 0 = x$.
Orthogonality $x \perp y \overset{\text{def}}{\iff} x \ovee y$ is defined

Def. An effect algebra is a PCM $(E, 0, \ovee)$ with a 'top' 1 and unique 'orthocomplements' $x^\perp$ s.t. $x \ovee x^\perp = 1$.

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For $x, y \in [0,1]_A$ ($x, y \in A$ with $0 \le x, y \le 1$)

- $x \perp y \iff x + y \le 1$
- $x \ovee y = x + y$
- $x^\perp = 1 - x$

Effect algebras, facts and examples

- Every effect algebra is a poset via

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Examples, besides effects $[0,1]_A$ in a C*-algebra

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- Any orthomodular lattice is an effect algebra
  - $x \perp y \iff x \leq y^{\perp}$
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  - $x \perp y \iff x \leq y^{\perp} \iff x \wedge y = 0$
- $[0,1]$, and fuzzy predicates $[0,1]^X$

State-and-effect triangle

$\mathbf{Cstar}_{\mathrm{PU}}^{\mathrm{op}}$

State-and-effect triangle

effect algebras

$\mathbf{EA}^{\mathrm{op}}$

$\mathbf{Cstar}_{\mathrm{PU}}^{\mathrm{op}}(-, 1+1) = \mathit{Pred}$

$[0,1]_{(-)} \cong$

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State-and-effect triangle

effect modules

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- Effect module = effect algebra E with a scalar multiplication $[0,1] \times E \to E$

State-and-effect triangle

effect modules	convex sets
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$[0,1]_{(-)} \cong$

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$\mathit{Stat} = \mathbf{Cstar}_{\mathrm{PU}}^{\mathrm{op}}(1, -)$

$= \mathbf{Cstar}_{\mathrm{PU}}(-, \mathbb{C})$

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- The distribution monad $\mathcal{D} \colon \mathbf{Set} \to \mathbf{Set}$

$$\mathcal{D}X = \{\text{probability distributions on } X\}$$
$$\cong \{\text{formal convex sums } \textstyle\sum_i r_i |x_i\rangle\}$$

State-and-effect triangle

Note: $[0,1]$ is an object sitting in the two categories

effect modules | convex sets

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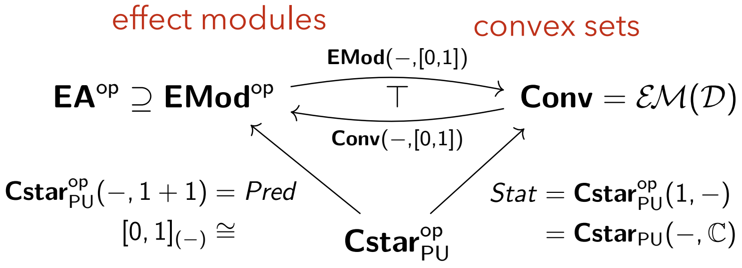

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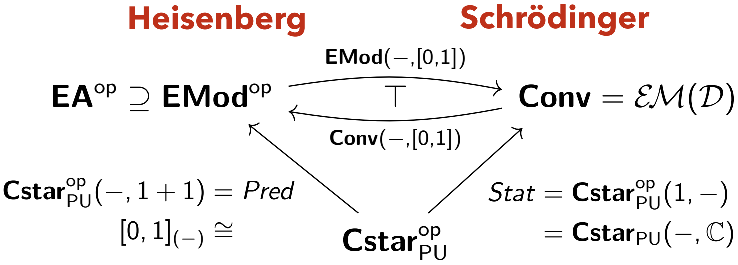

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- squares of the following form are pullbacks;

$$\begin{array}{ccc} A+X & \xrightarrow{\mathrm{id}+f} & A+Y \\ {\scriptstyle g+\mathrm{id}}\downarrow & & \downarrow{\scriptstyle g+\mathrm{id}} \\ B+X & \xrightarrow{\mathrm{id}+f} & B+Y \end{array} \qquad \begin{array}{ccc} A & = & A \\ {\scriptstyle \kappa_1}\downarrow & & \downarrow{\scriptstyle \kappa_1} \\ A+X & \xrightarrow{\mathrm{id}+f} & A+Y \end{array}$$

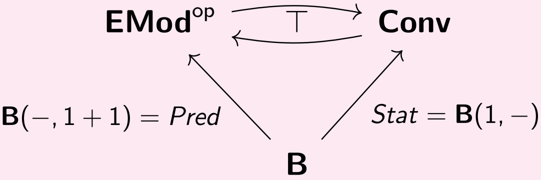
- the arrows $1+1+1 \underset{[\kappa_2,\kappa_1,\kappa_2]}{\overset{[\kappa_1,\kappa_2,\kappa_2]}{\rightrightarrows}} 1+1$ are jointly monic.

disjoint & universal coproducts (extensive cat.) $\Longrightarrow$ effectus $\Longrightarrow$ disjoint coprod. & strict initial

State-and-effect triangles over effectuses

Theorem. *Let* **B** *be an effectus.*

- *States* $1 \to X$ *form a convex set*
- *Predicates* $X \to 1+1$ *form an effect module*
- *We have a state-and-effect triangle:*



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Theorem. *Let* $\mathbf{B}$ *be an effectus.*

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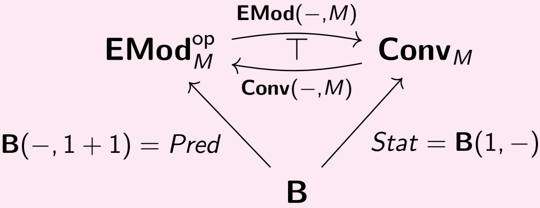

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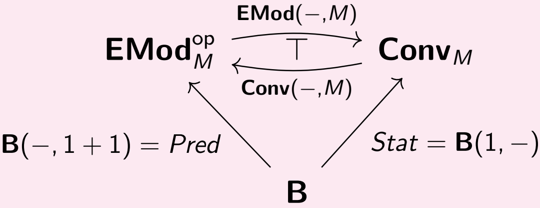

Later we use: $\mathit{Pred}(X) = \mathbf{B}(X, 1+1)$ is an effect algebra

More examples

Effectus	State $1 \stackrel{\omega}{\to} X$	Predicate $X \stackrel{p}{\to} 1+1$	Validity $\omega \models p$	Scalars $1 \to 1+1$
classical **Set**				
probabilistic $\mathcal{K}\ell(\mathcal{D})$				
quantum $\mathbf{Cstar}_{\mathrm{PU}}^{\mathrm{op}}$	state $X \stackrel{\omega}{\to} \mathbb{C}$	effect $p \in [0,1]_X$	$\omega(p)$	$[0,1]$

More examples

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probabilistic $\mathcal{K}\ell(\mathcal{D})$	prob. distribution $\sum_i s_i \lvert x_i\rangle$	fuzzy predicate $X \overset{p}{\to} [0,1]$	$\sum_i s_i p_i(x_i)$	$[0,1]$
quantum $\mathbf{Cstar}^{\mathrm{op}}_{\mathrm{PU}}$	state $X \overset{\omega}{\to} \mathbb{C}$	effect $p \in [0,1]_X$	$\omega(p)$	$[0,1]$

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quantum $\mathbf{Cstar}^{\mathrm{op}}_{\mathrm{PU}}$	state $X \xrightarrow{\omega} \mathbb{C}$	effect $p \in [0,1]_X$	$\omega(p)$	$[0,1]$

- Any extensive category with a final object
- $\mathcal{K}\ell(\mathcal{G})$, for the Giry monad $\mathcal{G}\colon \mathbf{Meas} \to \mathbf{Meas}$
- $\mathbf{Cstar}^{\mathrm{op}}_{\mathrm{CPU}}$, $\mathbf{Wstar}^{\mathrm{op}}_{\mathrm{PU}}$, $\mathbf{Wstar}^{\mathrm{op}}_{\mathrm{CPU}}$
- $\mathbf{DistLat}^{\mathrm{op}}$, $\mathbf{BoolAlg}^{\mathrm{op}}$, $\mathbf{Ring}^{\mathrm{op}}$, ...

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Total vs partial computation

- ‘Terminating’ vs ‘possibly non-terminating’

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	Total computation		Partial computation
classical **Set**	(total) function	**Pfn**	partial function

Total vs partial computation

- 'Terminating' vs 'possibly non-terminating'

	Total computation		Partial computation
classical **Set**	(total) function	**Pfn**	partial function
probabilistic $\mathcal{K}\ell(\mathcal{D})$	'stochastic relation' $X \to \mathcal{D}Y$	$\mathcal{K}\ell(\mathcal{D}_{\leq 1})$	'substochastic relation' $X \to \mathcal{D}_{\leq 1}Y$

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	Total computation		Partial computation
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probabilistic $\mathcal{K}\ell(\mathcal{D})$	'stochastic relation' $X \to \mathcal{D}Y$	$\mathcal{K}\ell(\mathcal{D}_{\leq 1})$	'substochastic relation' $X \to \mathcal{D}_{\leq 1}Y$
quantum $\mathbf{Cstar}^{\mathrm{op}}_{\mathrm{PU}}$	PU-map	$\mathbf{Cstar}^{\mathrm{op}}_{\mathrm{PSU}}$	PSU-map

- SU = subunital $f(1) \leq 1$

Total vs partial computation

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	Total computation		Partial computation
classical **Set**	(total) function	**Pfn**	partial function
probabilistic $\mathcal{K}\ell(\mathcal{D})$	‘stochastic relation’ $X \to \mathcal{D}Y$	$\mathcal{K}\ell(\mathcal{D}_{\leq 1})$	‘substochastic relation’ $X \to \mathcal{D}_{\leq 1}Y$
quantum $\mathbf{Cstar}^{\mathrm{op}}_{\mathrm{PU}}$	PU-map	$\mathbf{Cstar}^{\mathrm{op}}_{\mathrm{PSU}}$	PSU-map
$\mathbf{Wstar}^{\mathrm{op}}_{\mathrm{CPU}}$	‘quantum channel’ normal CPU-map	$\mathbf{Wstar}^{\mathrm{op}}_{\mathrm{CPSU}}$	‘quantum operation’ normal CPSU-map

- SU = subunital $f(1) \leq 1$, CP = completely positive

Total vs partial computation

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Effectus	Total computation		Partial computation
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Effectus $\mathbf{B}$	Total computation $X \to Y$		Partial computation $X \to Y + 1$ in $\mathbf{B}$
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probabilistic $\mathcal{K}\ell(\mathcal{D})$	'stochastic relation' $X \to \mathcal{D}Y$	$\mathcal{K}\ell(\mathcal{D}_{\leq 1})$	'substochastic relation' $X \to \mathcal{D}_{\leq 1}Y$
quantum $\mathbf{Cstar}^{\mathrm{op}}_{\mathrm{PU}}$	PU-map	$\mathbf{Cstar}^{\mathrm{op}}_{\mathrm{PSU}}$	PSU-map
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Effectus $\mathbf{B}$	Total computation $X \to Y$	$\mathbf{B}_{+1}$	Partial computation $X \to Y + 1$ in $\mathbf{B}$
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probabilistic $\mathcal{K}\ell(\mathcal{D})$	'stochastic relation' $X \to \mathcal{D}Y$	$\mathcal{K}\ell(\mathcal{D}_{\leq 1})$	'substochastic relation' $X \to \mathcal{D}_{\leq 1}Y$
quantum $\mathbf{Cstar}^{\mathrm{op}}_{\mathrm{PU}}$	PU-map	$\mathbf{Cstar}^{\mathrm{op}}_{\mathrm{PSU}}$	PSU-map
$\mathbf{Wstar}^{\mathrm{op}}_{\mathrm{CPU}}$	'quantum channel' normal CPU-map	$\mathbf{Wstar}^{\mathrm{op}}_{\mathrm{CPSU}}$	'quantum operation' normal CPSU-map

Notation.
$\mathbf{B}_{+1}$: the Kleisli category of the lift monad $(-) + 1$ on $\mathbf{B}$.

How the lift monads work

$$\frac{\text{function } X \longrightarrow Y+1}{\text{partial function } X \rightharpoonup Y}$$

$\mathbf{Set}_{+1} \cong \mathbf{Pfn}$

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How the lift monads work

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$$\frac{A \longrightarrow B + 1 \text{ in } \mathbf{Cstar}^{\mathrm{op}}_{\mathrm{PU}}}{\dfrac{\text{PU-map } f \colon B \times \mathbb{C} \longrightarrow A}{\dfrac{\text{PSU-map } g \colon B \longrightarrow A}{A \longrightarrow B \text{ in } \mathbf{Cstar}^{\mathrm{op}}_{\mathrm{PSU}}}} \; (*)}$$

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$(*)$ $g(x) = f(x, 0)$ and $f(x, \lambda) = g(x) + \lambda(1 - g(1))$

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Problem and result

Effectus $\mathbf{B}$	Total computation $X \to Y$	$\mathbf{B}_{+1}$	Partial computation $X \to Y + 1$ in $\mathbf{B}$
classical **Set**	(total) function	**Pfn**	partial function
probabilistic $\mathcal{K}\ell(\mathcal{D})$	'stochastic relation' $X \to \mathcal{D}Y$	$\mathcal{K}\ell(\mathcal{D}_{\le 1})$	'substochastic relation' $X \to \mathcal{D}_{\le 1}Y$
quantum $\mathbf{Cstar}^{op}_{PU}$	PU-map	$\mathbf{Cstar}^{op}_{PSU}$	PSU-map
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$$\Big(\text{effectuses}\Big) \underset{(-)_t}{\overset{(-)_{+1}}{\rightleftarrows}} \simeq \Big(\text{FinPACs with effects}\Big)$$

Partially additive category (PAC)

Def. (Arbib & Manes, 1980) A partially additive category is a category with countable coproducts that is enriched over partially additive monoids, satisfying:

- *(Compatible sum axiom)* A countable family $(f_i\colon X \to Y)_i$ is summable whenever there exists $f\colon X \to \coprod_i Y$ such that $\forall i.\, \triangleright_i \circ f = f_i$.
- *(Untying axiom)* If a countable family $(f_i\colon X \to Y)_i$ is summable, then $(\kappa_i \circ f_i\colon X \to \coprod_i Y)_i$ is summable too.

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Partially additive category (PAC)

Def. (Finite variant of PAC) A **finitely** partially additive category is a category with finite coproducts that is enriched over partial commutative monoids, satisfying:

- *(Compatible sum axiom)* A finite family $(f_i : X \to Y)_i$ is summable whenever there exists $f : X \to \coprod_i Y$ such that $\forall i.\, \rhd_i \circ f = f_i$.
- *(Untying axiom)* If a finite family $(f_i : X \to Y)_i$ is summable, then $(\kappa_i \circ f_i : X \to \coprod_i Y)_i$ is summable too.

- *Partial additivity* involves *countable partial sum*
- Abbrev: FinPAC = finitely partially additive category

Partially additive structure in effectuses

Proposition. *Let* $\mathbf{B}$ *be an effectus. The Kleisli category* $\mathbf{B}_{+1}$ *of the lift monad is a FinPAC.*

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Partially additive structure in effectuses

Proposition. *Let* **B** *be an effectus. The Kleisli category* $\mathbf{B}_{+1}$ *of the lift monad is a FinPAC.*

- $\mathbf{B}_{+1}$ inherits coproducts from **B**
- Each homset $\mathbf{B}_{+1}(X, Y) = \mathbf{B}(X, Y + 1)$ is a PCM via:

$$0_{XY} \colon X \to Y \;:=\; \left(X \xrightarrow{!} 1 \xrightarrow{\kappa_2} Y + 1 \text{ in } \mathbf{B}\right)$$

$$f \perp g \iff \begin{array}{ccccc} & & X & & \\ & \swarrow^{f} & \downarrow{\scriptstyle \exists b} & \searrow^{g} & \\ Y & \xleftarrow[\triangleright_1 := [\mathrm{id},0]]{} & Y + Y & \xrightarrow[\triangleright_2 := [0,\mathrm{id}]]{} & Y \end{array} \quad \text{in } \mathbf{B}_{+1}$$

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Examples, as a consequence

Classical $\mathbf{Set}_{+1} \cong \mathbf{Pfn}$ partial functions $f, g \colon X \rightharpoonup Y$

- $f \perp g \iff \mathrm{dom}(f) \cap \mathrm{dom}(g) = \varnothing$
- $(f \ovee g)(x) = \begin{cases} f(x) & x \in \mathrm{dom}(f) \\ g(x) & x \in \mathrm{dom}(g) \end{cases}$

Probabilistic $\mathcal{K}\ell(\mathcal{D})_{+1} \cong \mathcal{K}\ell(\mathcal{D}_{\le 1})$ $f, g \colon X \to \mathcal{D}_{\le 1} Y$

- $f \perp g \iff \sum_y f(x)(y) + \sum_y g(x)(y) \le 1$ for all $x \in X$
- $(f \ovee g)(x)(y) = f(x)(y) + g(x)(y)$

Quantum $(\mathbf{Cstar}^{\mathrm{op}}_{\mathrm{PU}})_{+1} \cong \mathbf{Cstar}^{\mathrm{op}}_{\mathrm{PSU}}$ PSU-maps $f, g \colon A \to B$

- $f \perp g \iff f(1) + g(1) \le 1$
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Def. A FinPAC with effects is a FinPAC $\mathbf{C}$ with a special object $I \in \mathbf{C}$ such that

- $\mathbf{C}(X, I)$ is an effect algebra for each $X \in \mathbf{C}$

satisfying the two conditions in the lemma.

FinPAC *with effects* ≈ FinPAC + EA

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$(\mathbf{C}, I)$: FinPAC with effects

Def. $f : X \to Y$ in $\mathbf{C}$ is total if $1_Y \circ f = 1_X$.
The subcategory $\mathbf{C}_{\mathrm{t}} \subseteq \mathbf{C}$ of total arrows, with all objects.

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Proposition. $\mathbf{C}_{\mathrm{t}}$ *is an effectus.*
Moreover, $(\mathbf{C}_{\mathrm{t}})_{+1} \cong \mathbf{C}$ *and* $(\mathbf{B}_{+1})_{\mathrm{t}} \cong \mathbf{B}$.

Main result

$\mathbf{B}$: effectus

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Theorem. *We have a 2-equivalence of 2-categories.*

$$\Big(\text{effectuses}\Big) \underset{(-)_{\mathrm{t}}}{\overset{(-)_{+1}}{\rightleftarrows}} \Big(\text{FinPACs with effects}\Big) \quad (\simeq)$$

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Problem and result (repeated)

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$$\Big(\text{effectuses}\Big) \underset{(-)_t}{\overset{(-)_{+1}}{\rightleftarrows}} \simeq \Big(\text{FinPACs with effects}\Big)$$

Outline

1. Effectus Theory [Jacobs, New Directions]
2. Partial Computation in Effectuses
3. Conclusions and Future Work

Conclusions and future work

Main result: the 2-equivalence of 2-categories

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total computation — partial computation

- Effectus ≈ finite partial additivity + effect algebra
- Total and partial computation are 'interchangeable'

Conclusions and future work

Main result: the 2-equivalence of 2-categories

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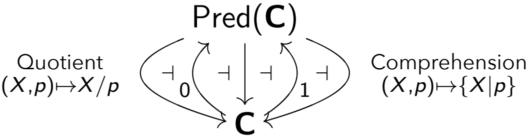
Future work: quotients, comprehension and measurements in effectuses / FinPACs with effects



instrument map $X \xrightarrow{\mathrm{instr}_p} X + X$