Categories in control

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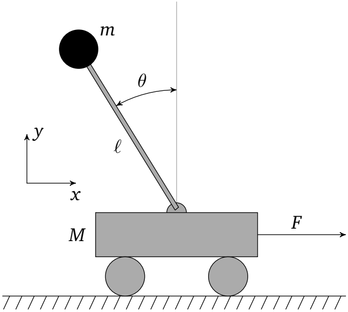

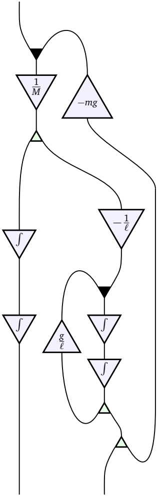

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Control theory is concerned with manipulating systems to induce them to enter a desired range of states. Modelling a system helps us understand what is happening and what manipulations can be made. Control theorists use the visual language of signal-flow diagrams as an effective way of communicating system models.

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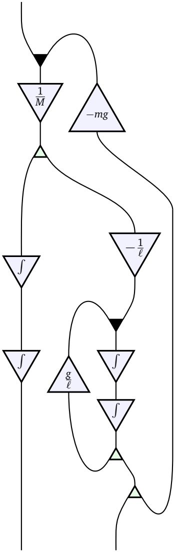
Despite working at the classical level, categories of signal-flow diagrams have striking similarities to categories of quantum systems.

Two prominent features:

- Integration
- Feedback

Signal-flow diagrams in control theory are systems of linear differential equations with a user-friendly package.

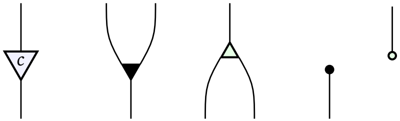


Lemma (Baez, E.)

The category FinVect$_k$, *with*

- *finite dimensional vector spaces over k as objects,*
- *linear maps as morphisms,*

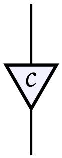
is a symmetric monoidal category with $\oplus$ *as its tensor product instead of* $\otimes$. FinVect$_k$ *is generated as a symmetric monoidal category by one object, k, together with the morphisms*



where $c \in k$.

Scalar multiplication

1. For each $c \in k$ we get a linear map for multiplying numbers by c:



$$\begin{array}{rccc} c\colon & k & \to & k \\ & x & \mapsto & cx \end{array}$$

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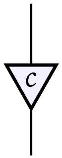
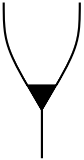


$$c\colon \quad \begin{array}{ccc} k & \to & k \\ x & \mapsto & cx \end{array}$$

By taking Laplace transforms, engineers reduce integration to multiplication by $\frac{1}{s}$. This makes integration a special case of scalar multiplication when we take $k = \mathbb{R}(s)$.

Addition

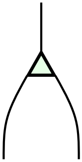
2. We can add two numbers together:



$$\begin{array}{rrcl} +: & k \oplus k & \to & k \\ & (x, y) & \mapsto & x + y \end{array}$$

Duplication

3. We can **duplicate** a number to get two copies of it:



$$\begin{aligned} \Delta\colon \quad k &\to k \oplus k \\ x &\mapsto (x, x) \end{aligned}$$

Zero

4. We have the number zero:

$$
\begin{aligned}
0\colon \quad \{0\} &\to k \\
0 &\mapsto 0
\end{aligned}
$$

Deletion

5. We can **delete** a number:

$$\begin{array}{rccc} !: & k & \to & \{0\} \\ & x & \mapsto & 0 \end{array}$$

These morphisms obey relations that we can state succinctly as

Theorem (Baez, E.)

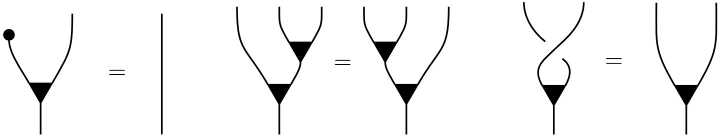
$\mathrm{FinVect}_k$ *is the free symmetric monoidal category on a bicommutative bimonoid over k.*

Simon Wadsley and Nick Woods later demonstrated this also holds for finitely generated free modules over any commutative rig k.

Expanded, this theorem lists the relations obeyed by the generating morphisms:

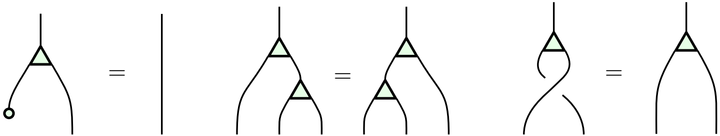
1–3 Commutative monoid

$(k, +, 0)$ is a commutative monoid:



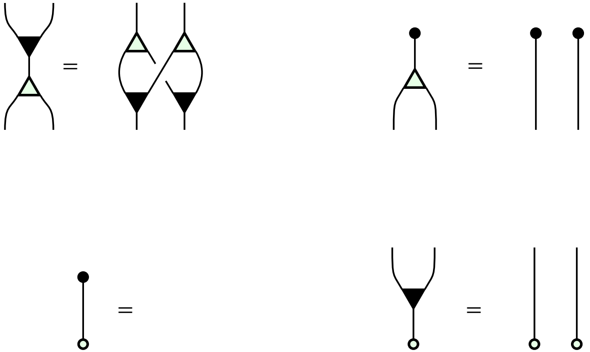
4–6 Cocommutative comonoid

$(k, \Delta, !)$ is a cocommutative comonoid:



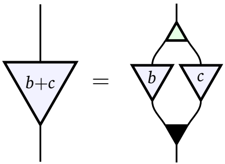
7–10 Bimonoid

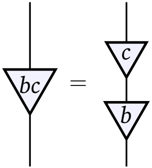
$(k, +, 0, \Delta, !)$ is a bimonoid:

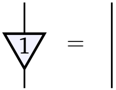


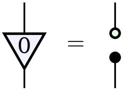
11–14 Rig structure

The rig structure of k can be recovered from the generators:



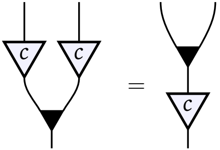


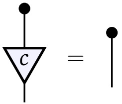


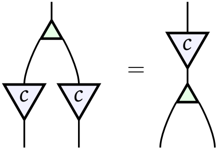


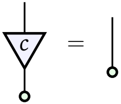
15–18 Scalar multiplication

Scalar multiplication by $c \in k$ commutes with the generators:









Linear maps only flow one-way. Since we want to describe feedback, we need something better.

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Linear relations!

A **linear relation** $F\colon U \nrightarrow V$ from a vector space U to a vector space V is a linear subspace $F \subseteq U \oplus V$.

When we compose linear relations $F\colon U \nrightarrow V$ and $G\colon V \nrightarrow W$, we get a linear relation $G \circ F\colon U \nrightarrow W$:

$$G \circ F = \{(u,w)\colon \exists v \in V \quad (u,v) \in F \text{ and } (v,w) \in G\}.$$

A linear map $\phi\colon U \to V$ gives a linear relation $F\colon U \nrightarrow V$, namely the graph of that map:

$$F = \{(u, \phi(u)) \colon u \in U\}.$$

In this way, composing linear maps is a special case of composing linear relations.

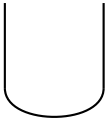
There is a category FinRel_k with finite-dimensional vector spaces over the field k as objects and linear relations as morphisms.

FinRel_k becomes symmetric monoidal using $\oplus$. It has $\mathrm{FinVect}_k$ as a symmetric monoidal subcategory.

Fully general signal-flow diagrams are pictures of morphisms in FinRel_k.

Baez and I showed that starting with the generators of $\mathrm{FinVect}_k$, we only need two more morphisms to generate FinRel_k, namely:

6. The 'cup':



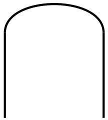
This is the linear relation

$$\cup \colon k \oplus k \nrightarrow \{0\}$$

given by:

$$\cup = \{(x, x, 0) : x \in k\} \subseteq k \oplus k \oplus \{0\}.$$

7. The ‘cap’:



This is the linear relation

$$\cap \colon \{0\} \nrightarrow k \oplus k$$

given by:

$$\cap = \{(0, x, x) : x \in k\} \subseteq \{0\} \oplus k \oplus k.$$

Lemma (Baez, E.)

The category FinRel_k*, with*

- *finite dimensional vector spaces over k as objects,*
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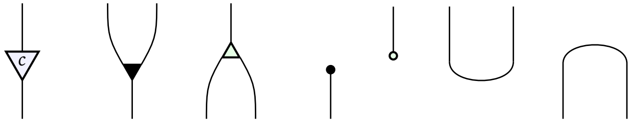
is a symmetric monoidal category with $\oplus$ *as its tensor product instead of* $\otimes$*.* FinRel_k *is generated as a symmetric monoidal category by one object, k, together with the morphisms*



where $c \in k$*.*

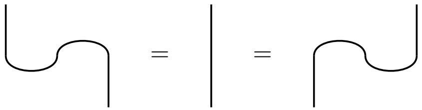
The relations governing these morphisms can be briefly stated as

Theorem (Baez–E., Bonchi–Sobociński–Zanasi)

FinRel_k *is the free symmetric monoidal category on a pair of interacting bimonoids over* k*.*

Expanded to a list, this theorem says we have the following relations in addition to the ones already seen:

19–20 Zigzag relations



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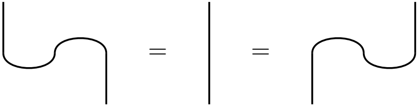
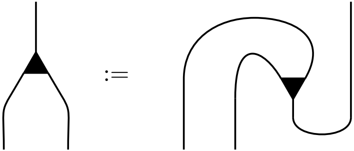


These relations allow us to 'turn morphisms around'. *E*.g. **coaddition** is addition turned around:

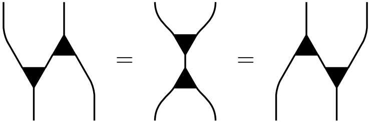


$$+^{\dagger}\colon k \nrightarrow k^2$$

$$+^{\dagger} = \{(x,y,z) \,:\; x = y + z\} \subseteq k \oplus k^2$$

21–24 Frobenius relations

‘Dark’ morphisms:
$(k, +, 0, +^\dagger, 0^\dagger)$ is a Frobenius monoid:



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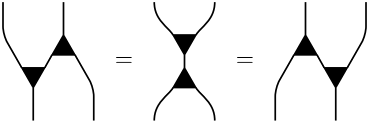
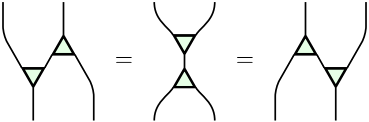
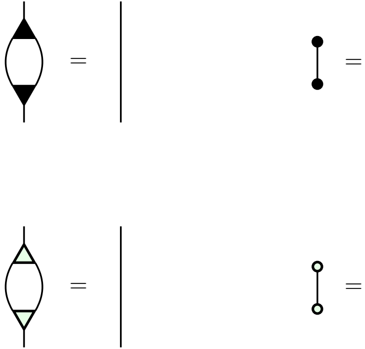


'Light' morphisms:
$(k, \Delta^{\dagger}, !^{\dagger}, \Delta, !)$ is a Frobenius monoid:



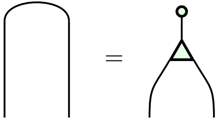
25–28 Extra-special structure

Both Frobenius monoids are extra-special:

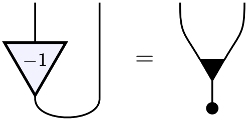


29–30 Cap and Cup

$\cap$ can be expressed in terms of Δ and $!^\dagger$:

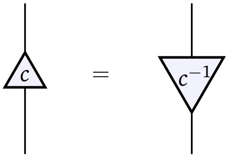


$\cup$ with a factor of -1 inserted can be expressed in terms of $+$ and $0^\dagger$:



31 Reciprocal scalar multiplication

For any $c \in k$, $c \neq 0$, scalar multiplication by c^{-1} is the adjoint of scalar multiplication by c:



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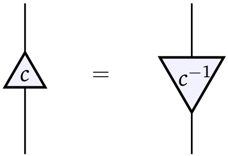

This list of relations was independently discovered by Bonchi, Sobociński and Zanasi. They noted the Frobenius relations can be seen as coming from the interaction of the bimonoids over k.

The categories $\mathrm{FinVect}_k$ and FinRel_k are beautiful exhibits of the category theory lurking within control theory. What other remarkable structures can we uncover?