Additive monotones for resource theories of parallel-combinable processes with discarding

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Resource theory framework by B. Coecke, T. Fritz, R. W. Spekkens

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The **preordered monoid** is the structure at the core of the Resource Theory formalism.

Ordered monoid from Symmetric Monoidal Category $\mathbf{C}$

Given $\mathbf{C}$ we can always get a preordered monoid.

Ordered monoid from Symmetric Monoidal Category **C**

Given **C** we can always get a preordered monoid.

Theorem

Let **C** be a symmetric monoidal category, and $f \sim g$ in **C** if there $\exists f \to g$ and $g \to f$. This defines an equivalence relation.

Write $[f]$ for the equivalence class of f; we also write $|\mathbf{C}|$ for the set of equivalence classes of objects in **C**.

Then there exists an ordered monoid $(|\mathbf{C}|, \succeq, \otimes)$ on the set of these equivalence classes, with $[f] \succeq [g]$ if $\exists f \to g$ in **C**, and using the monoidal product in **C** to define $[f] \otimes [g] = [f \otimes g]$. Moreover, this monoid is commutative.

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This free / non-free separation is modelled by a **partitioned resource theory** $(\mathbf{C}_{\mathrm{free}}, \mathbf{C})$ which consists of

1. A symmetric monoidal category $\mathbf{C}$, and
2. An all-object-including symmetric monoidal subcategory $\mathbf{C}_{\mathrm{free}}$

$$\mathbf{C}_{\mathrm{free}} \subseteq \mathbf{C}$$

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Examples

$(\mathbf{Bij}, \mathbf{Set}), (\mathbf{Inj}, \mathbf{Set}), (\mathbf{Bij}_{\sqcup}, \mathbf{Set}_{\sqcup}), (\mathbf{Inj}_{\sqcup}, \mathbf{Set}_{\sqcup})$

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Result 1

Given the relation $\succeq$ defined in the next frame which stands for *can be transformed into*, we find the complete family of "consistent pricing functions" of morphisms of two Resource Theories:

$$(\mathbf{Bij}_\sqcup, \mathbf{Set}_\sqcup) \text{ and } (\mathbf{Inj}_\sqcup, \mathbf{Set}_\sqcup)$$

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Result 2

We also give a general theorem that can be used as a tool for finding these pricing functions in general resource theories.

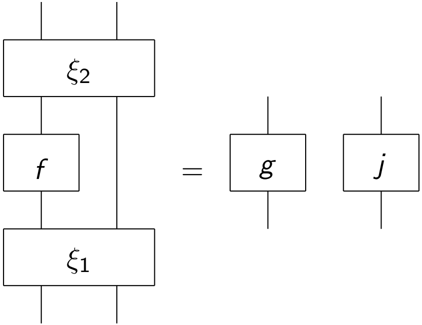
This family of consistent pricing functions onto the reals is called a *complete family of monotones.*
To our knowledge, complete family of monotones haven't worked out yet making use of this $\succeq$ relation.

For $f, g \in \mathrm{Mor}(\mathbf{C})$ we set

$$f \succeq g$$

whenever $\exists Z \in |\mathbf{C}|$, $\xi_1, \xi_2 \in \mathrm{Mor}(\mathbf{C}_{\mathrm{free}})$, $j \in \mathrm{Mor}(\mathbf{C})$ such that

$$\xi_2 \circ (f \otimes 1_Z) \circ \xi_1 = g \otimes j. \tag{1}$$



Definition

Let $(X, \succeq)$ be a partially ordered set. A monotone is an order-preserving function $M : (X, \succeq) \to (\mathbb{R}, \geq)$. It is called complete if for all $x, y \in X$ we have

$$x \succeq y \quad \text{if and only if} \quad M(x) \geq M(y).$$

Definition

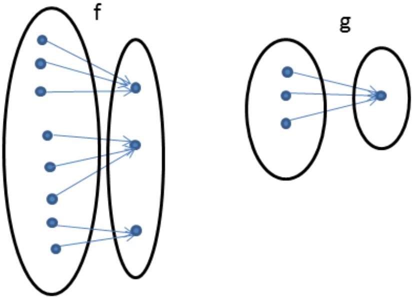
Given a partially ordered set $(X, \succeq)$, we call a collection $\{M_i\}_{i \in I}$ of monotones on $(X, \succeq)$ a complete family of monotones if for all $x, y \in X$ we have

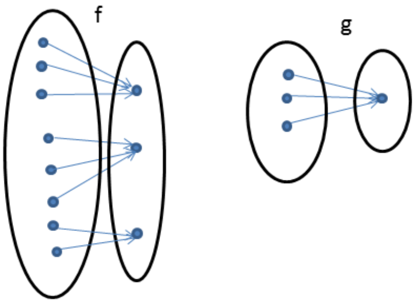
$$x \succeq y \quad \text{if and only if} \quad M_i(x) \geq M_i(y) \text{ for all } i \in I.$$

Complete family of additive monotones of $(\mathbf{Bij}_\sqcup, \mathbf{Set}_\sqcup)$

For $i \in \mathbb{N}$, define functions:

$$\begin{aligned} \varphi_i : \mathrm{Mor}(\mathbf{Set}_\sqcup) &\longrightarrow \mathbb{N}; \\ (f : X \to Y) &\longmapsto \#\{y \in Y \mid \#f^{-1}(y) = i\}. \end{aligned}$$


f
g



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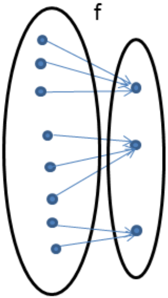
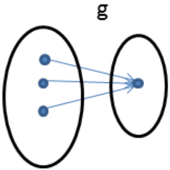
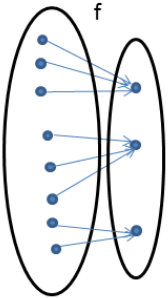
f	...	0	2	1
g	...	0	1	0

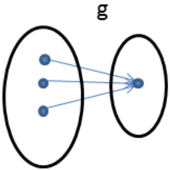




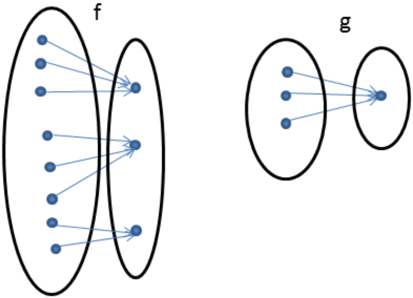
	...	4	3	2
f	...	0	2	1
g	...	0	1	0
		$0 \geq 0$		

✓

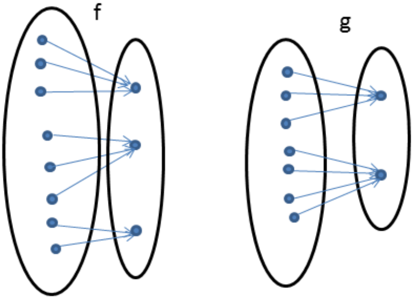




	...	4	3	2
f	...	0	2	1
g	...	0	1	0
		$0 \geq 0$	$2 \geq 1$	



	...	**4**	**3**	**2**
f	...	0	2	1
g	...	0	1	0
		$0 \geq 0$	$2 \geq 1$	$1 \geq 0$

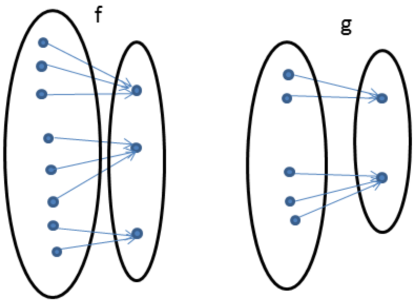


	...	**4**	**3**	**2**
f	...	0	2	1
g	...	1	1	0
		$0 \ngeq 1$	$2 \geq 1$	$1 \geq 0$

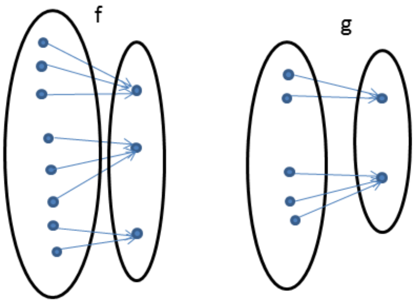
Complete family of additive monotones of $(\mathbf{Inj}_\sqcup, \mathbf{Set}_\sqcup)$

For $i \in \mathbb{N}$, define functions

$$\gamma_i : \mathrm{Mor}(\mathbf{Set}_\sqcup) \longrightarrow \mathbb{N};$$
$$(f : X \to Y) \longmapsto \#\{y \in Y \mid \#f^{-1}(y) \geq i\}$$

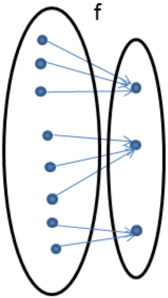


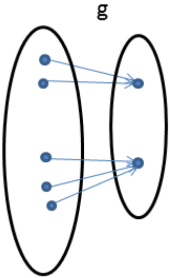
	...	4	3	2
f	...	0	2	1
g	...	0	1	1



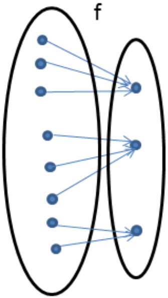
	...	4	3	2
f	...	0	2	1
g	...	0	1	1
		$0 \geq 0$		

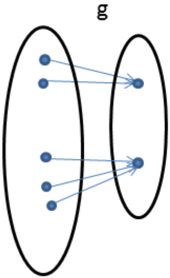
✓





	...	4	3	2
f	...	0	2	1
g	...	0	1	1
		$0 \geq 0$	$0+2 \geq 0+1$	





	...	4	3	2
f	...	0	2	1
g	...	0	1	1
		$0 \geq 0$ ✓	$0+2 \geq 0+1$ ✓	$0+2+1 \geq 0+1+1$ ✓

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Our theorem eases the task of finding this complete family by reducing it to only finding 3 properties:

(i) $\mu(f \otimes g) = \mu(f) \cdot \mu(g)$;

(ii) $\mu(1_Z) = 1$; and

(iii) $\mu(f) \geq \mu(\xi \circ f)$ and $\mu(f) \geq \mu(f \circ \xi)$ whenever it makes sense.

for all $Z \in |\mathbf{C}|$, $f, g \in \mathrm{Mor}(\mathbf{C})$, and $\xi \in \mathrm{Mor}(\mathbf{C}_{\mathrm{free}})$

Theorem

Let $(\mathbf{C}, \mathbf{C}_{free})$ be a PRT and let $(X, \geq, \cdot)$ be a non-negative ordered monoid. A function $\mu : \mathrm{Mor}(\mathbf{C}) \to X$ induces an order-preserving monoid homomorphism

$$M : (|\mathrm{PCD}(\mathbf{C}, \mathbf{C}_{free})|, \succeq, \otimes) \longrightarrow (X, \geq, \cdot)$$
$$[f] \longmapsto \mu(f)$$

iff for all $Z \in |\mathbf{C}|$, $f, g \in \mathrm{Mor}(\mathbf{C})$, and $\xi \in \mathrm{Mor}(\mathbf{C}_{free})$ we have

(i) $\mu(f \otimes g) = \mu(f) \cdot \mu(g)$;

(ii) $\mu(1_Z) = 1$; *and*

(iii) $\mu(f) \geq \mu(\xi \circ f)$ *and* $\mu(f) \geq \mu(f \circ \xi)$ *whenever such composites are well-defined.*

Moreover, this gives a one-to-one correspondence: every order-preserving monoid homomorphism on $(|\mathrm{PCD}(\mathbf{C}, \mathbf{C}_{free})|, \succeq, \otimes)$ arises from a unique such function μ.

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- Find complete families of monotones for more interesting pairs of monoidal categories. **Rel**, **Vect**, **Hilb**, etc.
- Seek for properties of physical (and chemical, biological) interest that this theory could predict.
- Extend the theory so that it can measure properties currently incommensurable, like the irreversibility of a Markov process (by taking FinStoch as the main Category) or the irreducibility. Neither irreversible nor irreducible matrices form a category.