Categories of Relations as Models of Quantum Theory

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- Quantum-like behaviour without superposition.

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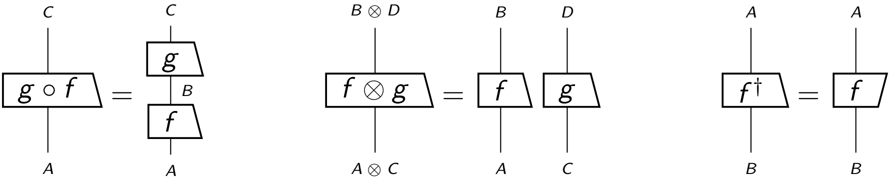

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$\mathbf{Rel}(\mathbf{Set}) = \mathbf{Rel}$.
$\mathbf{Rel}(\mathbf{Grp})$: subgroups $R \leq G \times H$.
$\mathbf{Rel}(\mathbf{Vect}_k)$: subspaces $R \leq V \oplus W$ - see 'Categories in Control'.

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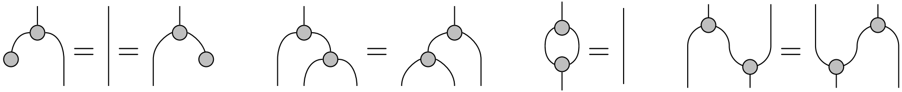
A special dagger Frobenius structure $(A,$  $,$  $)$ satisfies:



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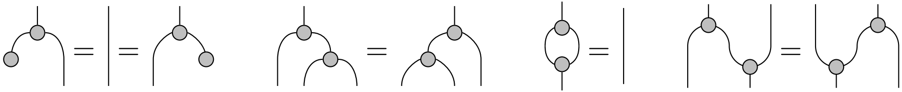

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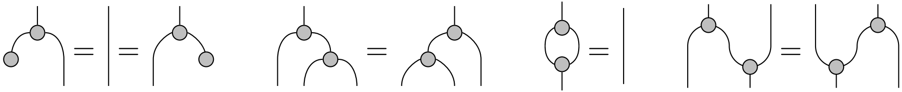

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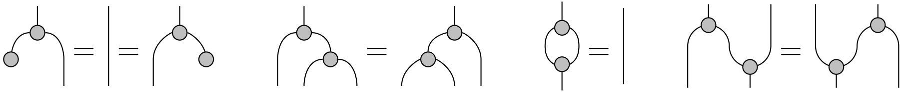

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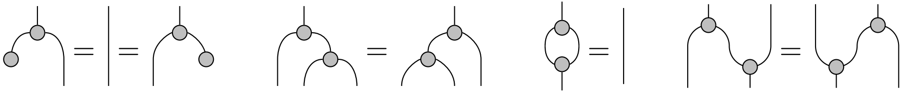

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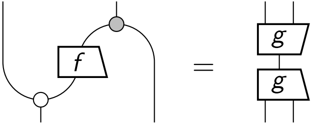
Examples

Set: small groupoids.
Grp: strict 2-groups (Baez-Lauda) $\Longleftrightarrow$ crossed modules.
$\mathbf{Vect}_k$: 2-vector spaces (Baez-Crans).

Completely Positive Relations

CP(D): Frobenius  in **D** & completely positive $(A, \circ) \xrightarrow{f} (B, \bullet)$



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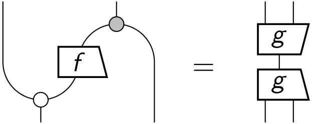

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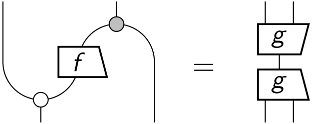

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CP(**FHilb**): finite-dimensional C*-algebras & completely positive maps.
CP(**Rel**): groupoids & relations such that

$$R(a, b) \Rightarrow R(a^{-1}, b^{-1}) \wedge R(\mathrm{id}_{\mathsf{dom}(a)}, \mathrm{id}_{\mathsf{dom}(b)})$$

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$\mathbf{CP}(\mathbf{Rel}(\mathbf{Grp})) \simeq \mathbf{Rel}(\mathbf{CrossedModules})$
$\mathbf{CP}(\mathbf{Rel}(\mathbf{Vect}_k)) \simeq \mathbf{Rel}(\mathbf{2Vect}_k)$

Quantum Properties of **Rel**(**C**)

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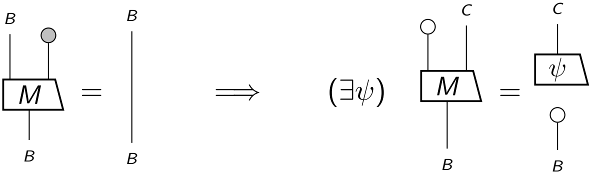
Heisenberg Uncertainty Principle:



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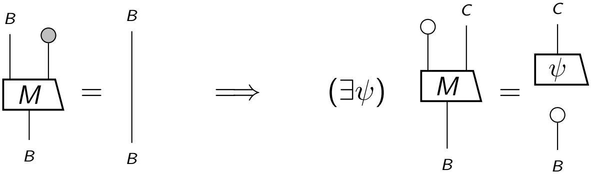

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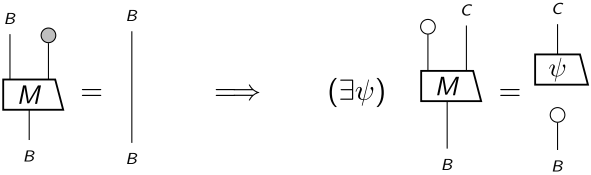

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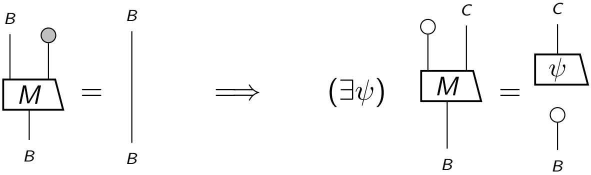

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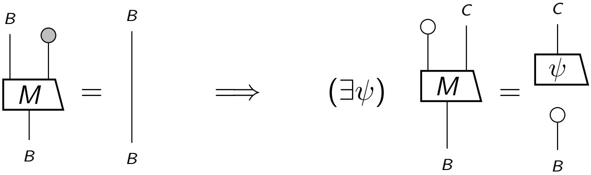

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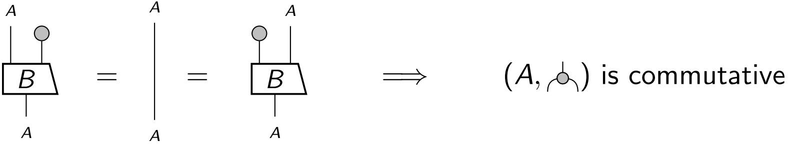

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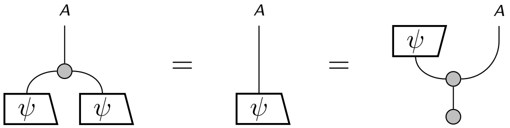

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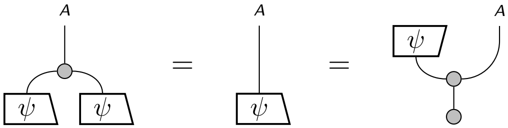
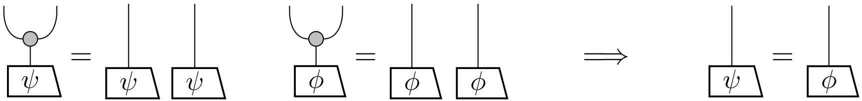


C has zero object (e.g. **Grp**, $\mathbf{Vect}_k$) $\rightsquigarrow$ no distinct classical data:



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