Some Nearly Quantum Theories

Howard Barnum, Matthew Graydon and Alex Wilce

QPL XII, Oxford, July 2015
I. Euclidean Jordan Algebras
Euclidean Jordan Algebras

A euclidean Jordan algebra (EJA) is a finite-dimensional real inner product space $A$ with commutative bilinear product $a, b \mapsto a \cdot b$, having identity element $u$, and satisfying, $\forall a, b, c \in A$,

- $a \cdot (a^2 \cdot b) = a^2 \cdot (a \cdot b)$;
- $\langle a \cdot b, c \rangle = \langle a, b \cdot c \rangle$

Examples (and notation):

$\mathbb{R}^n := M_n(\mathbb{R})_{sa}, \quad \mathbb{C}^n := M_n(\mathbb{C})_{sa}, \quad \mathbb{Q}^n := M_n(\mathbb{H})_{sa}, \quad E_3 := M_3(\mathbb{O})_{sa}$

All with $a \cdot b = \frac{1}{2}(ab + ba)$ and $\langle a, b \rangle = \text{Tr}(ab)$. 
Thorem [Jordan, von Neumann, Wigner, 1934] All EJAs are direct sums of the following *simple* types:

- $\mathbb{R}_n$, $\mathbb{C}_n$, $\mathbb{Q}_n$, $E_3$ as above;

- **Spin Factors**: $V_n = \text{the euclidean space } \mathbb{R} \times \mathbb{R}^n$, with Jordan product

  $$(t, x) \bullet (s, y) = (ts + \langle x, y \rangle, ty + sx).$$

An EJA $A$ is **special** iff it’s a Jordan subalgebra of $M_n(\mathbb{C})_{sa}$ for some $n$. $A$ is special iff it has no $E_3$ summand.
EJAs as Probabilistic Models

An EJA $A$ is also an order-unit space with $A_+ = \{a^2|a \in A\}$ and order-unit $u$. Thus, $A$ can be viewed as a probabilistic model:

- **Effects** are elements $a \in A_+$ with $a \leq u$
- **States** are positive linear functionals $\alpha : A \to \mathbb{R}$ with $\alpha(u) = 1$; $\alpha(a) =$ probability of effect $a$ in state $\alpha$.
- **Observables** are sets $\{a_1, ..., a_n\}$ of effects summing to $u$.

Can also model continuous dynamics via one-parameter subgroups of $G(A) =$ identity component of group of order-automorphisms.

$R_n, C_n, Q_n = \mathbb{R}, \mathbb{C}, \mathbb{H}$ QM systems; $V_n =$ generalized bit with $n$-dimensional “Bloch sphere”. In particular:

$V_2 = R_2$ (rebit), $V_3 = C_2$ (qubit), $V_5 = Q_2$ (quabit).
EJAs as Probabilistic Models

The order-unit space structure determines the Jordan structure:

**Theorem [Koecher 1958; Vinberg 1961]** An order-unit space \((A, u)\) arises from an EJA as above iff the cone \(A_+\) is

- homogeneous (order-isomorphisms act transitively on interior of \(A_+\));
- self-dual (\(\exists\) inner product such that \(a \in A_+\) iff \(\langle a, b \rangle \geq 0\) for all \(b \in A_+\)).

Homogeneity and self-duality can be motivated in various ways (e.g., AW 2012, Barnum-Mueller-Ududec 14). So EJAs form a *reasonably natural* class of probabilistic models!
II. Composites of EJAs
Composites of EJAs

In general, there’s no reasonable way to make $A \otimes B$ into an EJA.

A *composite* of EJAs $A$ and $B$: an EJA $AB$, plus bilinear mapping $A \times B \to AB$, $(a, b) \mapsto a \circ b$, such that

(a) $u_A \circ u_B = u_{AB}$;
(b) $\langle a \circ b | x \circ y \rangle = \langle a | x \rangle \langle b | y \rangle$
(c) $\phi \in G(A), \psi \in G(B) \Rightarrow \exists \phi \circ \psi \in G(AB)$ with

$$(\psi \circ \phi)(a \circ b) = \psi(a) \circ \phi(b).$$

(d) $\{a \circ b | (a, b) \in A \times B\}$ generates $AB$ as a Jordan algebra.

**Theorem 1:** For $A, B$ nontrivial, $AB$ exists $\Rightarrow \ A, B, AB$ all special!
Embedded JC algebras

An embedded JC algebra (EJC) is a pair \((A, M_A)\):

- \(M_A\) a (unital) \(*\)-subalgebra of \(M_n(\mathbb{C})\) for some \(n\);
- \(A\) a (unital) Jordan sub-algebra of \((M_A)_{sa}\).

EJCs have a canonical product:

\[
(A, M_A) \circ (B, M_B) := (A \circ B, M_A \otimes M_B),
\]

\[
A \circ B = J(A \otimes B) \leq (M_A \otimes M_B)_{sa}
\]

where \(J(X) = \) Jordan subalgebra of \(M_{sa}\) generated by \(X \subseteq M_{sa}\).

**Notation:** From now on, overload \(A\) for \((A, M_A)\).

**Theorem 2:** \(A \circ B\) is a composite in our sense.

**Theorem 3:** \(A, B \mapsto A \circ B\) is naturally associative.
Standard and Universal Embeddings

Special EJAs have both standard and universal embeddings.

Standard embeddings for simple (special) EJAs:

- $R_n, C_n \leq M_n(\mathbb{C})_{sa}$;
- $H \leq M_2(\mathbb{C})$, so $Q_n \leq M_{2n}(\mathbb{C})_{sa}$;
- $V_n \leq M_{2k}(\mathbb{C})_{sa}$ for $n = 2k, 2k + 1$
  (Details: MG’s quantum lunch talk on the 24th).
Standard and Universal Embeddings

Universal embedding $A \mapsto C^*(A)_{sa}$: Jordan homomorphisms $A \rightarrow M_{sa}$, $M$ a complex $*$-algebra, factor uniquely through $*$-homomorphisms $C^*(A) \rightarrow M$.

Universal and standard embeddings agree except:

1. $C^*(C_n) = M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$
2. $C^*(Q_2) = M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$
3. $C^*(V_n) = M_{2k}(\mathbb{C}) \oplus M_{2k}(\mathbb{C})$ for $n = 2k + 1$

Notation: $A\tilde{\otimes}B = A \circ B$ for the universal embedding (Studied by Hanche-Olsen, 1983).
Standard and Universal Tensor Products

For $A, B \in \{ R_n, C_n, Q_n | n \geq 2 \}$, $A \tilde{\otimes} B$ can be computed explicitly (Hanche-Olsen 83):

$$A \tilde{\otimes} B = R_k \oplus C_k \oplus Q_k$$

<table>
<thead>
<tr>
<th>$A \tilde{\otimes} B$</th>
<th>$R_k$</th>
<th>$C_k$</th>
<th>$Q_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_n$</td>
<td>$R_{nk}$</td>
<td>$C_{nk}$</td>
<td>$Q_{nk}$</td>
</tr>
<tr>
<td>$C_n$</td>
<td>$C_{nk}$</td>
<td>$C_{nk} \oplus C_{nk}$</td>
<td>$C_{nk}$</td>
</tr>
<tr>
<td>$Q_n &gt; 2$</td>
<td>$Q_{nk}$</td>
<td>$C_{nk}$</td>
<td>$R_{nk}$</td>
</tr>
</tbody>
</table>

The case of two quaternionic bits or quabits is a bit special:

$$Q_2 \tilde{\otimes} Q_2 = R_{16} \oplus R_{16} \oplus R_{16} \oplus R_{16}.$$  

**Theorem 4:** $A, B$ simple $\Rightarrow$ $AB$ a direct summand of $A \tilde{\otimes} B$.

So the only possible composites are standard ones, except for $C_n C_k$ (two candidates) and $Q_2 Q_2$ (four candidates).
III. Categories of EJCs
CJP Mappings

Let \( \mathcal{C} \) be any class of EJCs closed under \( \circ \). For \( A, B \in \mathcal{C} \), a CP mapping \( \phi : M_A \rightarrow M_B \) is *completely Jordan preserving* (CJP) (rel. \( \mathcal{C} \)) iff, for all \( C \in \mathcal{C} \), \( \phi \otimes \text{id}_{M_C}(A \circ C) \subseteq B \circ C \).

**Theorem 5:** With relatively CJP mappings as morphisms, \((\mathcal{C}, \circ)\) is a SMC.

If \( \mathcal{C} \) contains any universally embedded \( V_n \) with \( n \neq 2, 3 \), then \( \mathcal{C}(A, I) = \{0\} \). Two nicer examples:

- **RSE:** Reversible, Standardly Embedded EJCs (no non-quantum \( V_n \));
- **URUE:** Universally Reversible, Universally Embedded EJCs (no \( V_n, n > 3 \) — so, no \( Q_2 \)!)
Compact structure

The category of finite-dimensional matrix $\ast$-algebras and CP maps has a natural compact structure: dual of $\mathcal{M}$ is $\overline{\mathcal{M}}$ (conjugate algebra); unit and counit given by

$$f_{\mathcal{M}} = \sum_i e_i \otimes \overline{e}_i \text{ and } \eta_{\mathcal{M}}(a \otimes \overline{b}) = \text{Tr}(ab^*) ,$$

where $\{e_i\}$ is any Tr-orthonormal basis for $\mathcal{M}$.

Theorem 6 Both RSE and URUE inherit this compact structure.
Conclusion and Speculation

- **RSE** unifies real, complex and quaternionic QM. **URUE** *almost* does so, except that it omits $Q_2$ and gives the wrong composite for two complex QM systems.


- A striking feature of both **RSE** and **URUE**:

  anything $\otimes$ complex $=$ complex.

Maybe a universe initially containing all three types of quantum systems would evolve into one in which complex systems predominate?
(also Foundations of Physics 42, 2012)

(also New J. Phys. 16, 2014)
