# A Bestiary of Sets and Relations

arXiv:1506.05025

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17 July 2015

# Introduction

Today, in this talk: a veritable bestiary of sets and relations.

Credit: Aberdeen Bestiary

# Section 1

# Pure State Quantum Mechanics

# Pure State Quantum Mechanics in fRel

Looks like fdHilb, but something is not quite right...

Credit: *Chimera*, Giovannag, DeviantArt

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  (distributive enrichment over finite commutative monoids)
- Scalars form a semiring $(\{\emptyset, id_1\}, \vee, \times) \cong \mathbb{B}$

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- The classical points are the states $|G_\lambda\rangle : 1 \to X$

# Classical Computation

1. Morphisms of classical structures are used to embed partial functions (and thus classical computation) in fdHilb:

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2. Morphisms of classical structures $\bigoplus_{\lambda\in\Lambda} G_\lambda \to \bigoplus_{\gamma\in\Gamma} H_\gamma$ can be used to embed all partial functions $f : \Lambda \rightharpoonup \Gamma$ in fRel:

$$R_f := \bigvee_{\lambda \in \text{dom } f} |H_{f(\lambda)}\rangle\langle G_\lambda|$$

# Classical Computation

3. However, the correspondence in fRel is not 1-to-1.
For example, consider a family $(\Phi_\lambda : G_\lambda \to H_{f(\lambda)})_{\lambda \in \Lambda}$ of isomorphisms of abelian groups and embed $f : \Lambda \rightharpoonup \Gamma$ as:

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4. Non-uniqueness is a consequence of the fact that most classical structures don't have enough classical points.
5. These additional degrees of freedom could be related to microscopic degrees of freedom in computation using the groupoid framework of [Bar&Vicary (2014)].

# Discrete structures

On each finite set $X$, the **discrete structure** is given by:

 $= (x, y) \mapsto \delta_{xy} x$ (partial function)

 $= 1 \times X$

 $= x \mapsto (x, x)$ (total function)

 $= x \mapsto \star$ (total function)

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- It has the singletons $\{x\}$ as its classical points.
- It is the only classical structure with enough classical points.
- It gives the usual 1-to-1 embedding of partial functions:

$$R_f := \{(x, f(x)) | x \in \mathrm{dom}\, f\}$$

# Isometries and Unitaries

- A morphism $F : X \to Y$ in fRel is an isometry iff it is in the form, for some classical structure $\bigoplus_{\gamma \in \Gamma} H_\gamma$ on $Y$

$$F = \bigvee_{x \in X} |H_{f(x)}\rangle\langle\{x\}|$$

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- Isometries are in the form $F = R_f$ for some total injection $f$
- Forces discrete structure on $X \Rightarrow$ more restrictive than fdHilb
- Indeed this forces unitaries = bijections

Section 2

# CPM and Decoherence

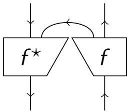
# Decoherence in fRel

One look at it and things turns to stone. Very classical stone.

Credit: *Medusa*, Miragenathalen, DeviantArt

# The category CPM[fRel]

Morphisms in CPM[fRel] take the usual doubled-up form:




where the compact-closed structure on fRel is given by:

$$\cap_X := (x,y) \mapsto \delta_{xy} : X \times X \to 1$$
$$\cup_X := \Delta_X : 1 \to X \times X$$

We call $\cap_X$ the **discarding map** $X \xrightarrow{CPM} 1$, and we will say that a CPM morphism $R : X \xrightarrow{CPM} Y$ is **causal** iff $\cap_Y \cdot R = \cap_X$ .

# Graphs for CPM

[Marsden 2015] A clever graph-theoretic formalism for CPM[fRel]:

- States $\rho : 1 \xrightarrow{CPM} X$ in CPM[fRel] correspond to subgraphs $\mathcal{G}_\rho$ of the complete graph $\mathrm{K}_X$ on $X$.

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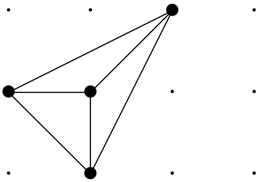
- Morphisms $R : X \xrightarrow{CPM} Y$ correspond to subgraphs $\mathcal{G}_R$ of the complete graph $\mathrm{K}_{X \times Y}$.
- Composition is done by lifting and projecting edges.

# Graphs of CPM states (example)



a pure state in a 12 element set

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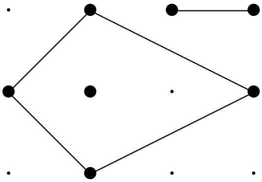

a non-pure state in a 12 element set

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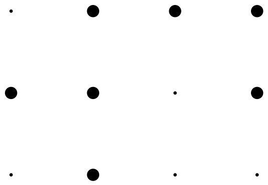
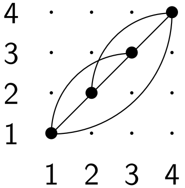
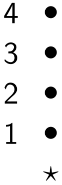


a discrete state in a 12 element set

# Graphs of CPM maps (example)




identity on 4 element set $X$
as a graph on $X \times X$




$\cap_X$ on 4 element set $X$
as a graph on $X \times 1$

# Graph composition (example)

Computing the image $R\rho : 1 \xrightarrow{CPM} Y$ of a CPM state $\rho : 1 \xrightarrow{CPM} X$ under a CPM map $R : X \xrightarrow{CPM} Y$ using the associated graphs.

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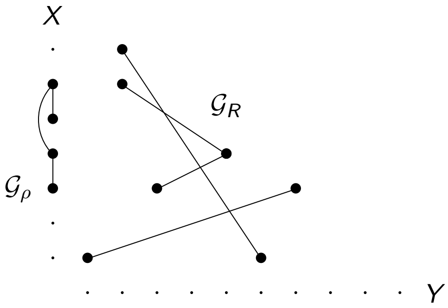

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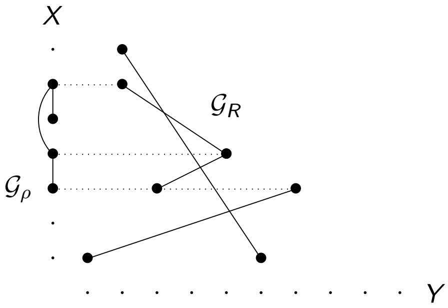

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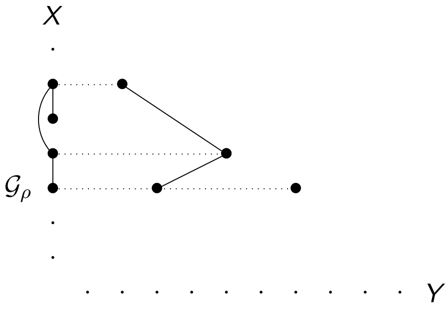

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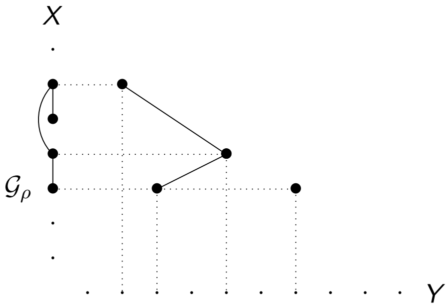

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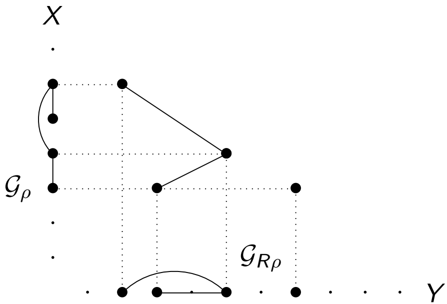

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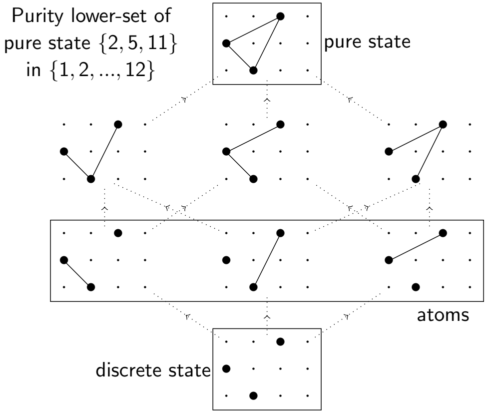
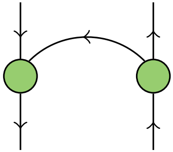
- Therefore every pure state $\rho$ (clique $\mathcal{G}_\rho$) can be expressed as a convex combination of non-pure states (the atoms of $\rho\downarrow$).

# Purity of states (example)



# Decoherence Maps in †-SMCs

Let ● be a classical structure on some object $X$ of a compact closed †-SMC. The ● **-decoherence** map dec(●) is the following causal CPM morphism $X \xrightarrow{CPM} X$:



# Decoherence Maps in fdHilb

In fdHilb, the decoherence map sends any (causal) CPM state to a (probabilistic) convex combination of ∘-classical points:

$$\mathrm{dec}(\circ)\rho = \sum_z \langle z|\rho|z\rangle \; |z\rangle\langle z|$$

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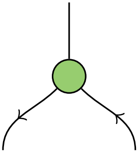
This also justifies the following quantum-classical notation when all operations after the decoherence are ∘-classical:



# Decoherence Maps in fRel

This convex combination assumption fails in fRel:

- The result of decohering a CPM state $\rho$ to dec($\circ$)$\rho$ is <u>not</u> in general a convex combination of $\circ$-classical points.

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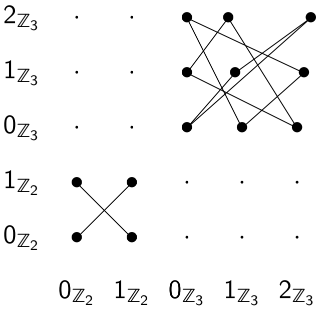
- In the case of fRel, the CPM category cannot be interpreted as a category of mixed states in the usual sense.

# Decoherence Maps in fRel (example)

Let $X$ be a 5 element set, and $\circ$ the classical structure of groupoid $\mathbb{Z}_2 \oplus \mathbb{Z}_3$. Then $\mathcal{G}_{\mathrm{dec}(\circ)}$ is the following subgraph of $\mathsf{K}_{X\times X}$:




Tq

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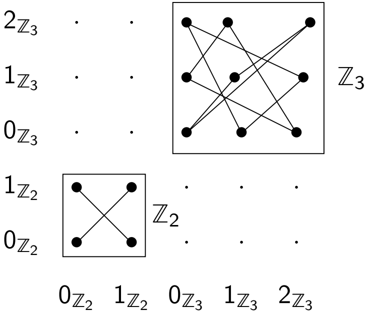
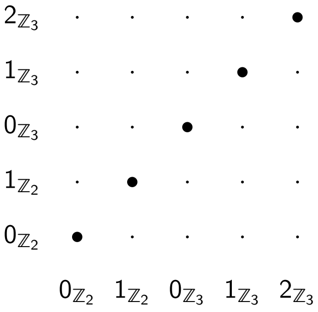



The cliques on the boxed sets of nodes are the $\circ$-classical states.

# Decoherence Maps in fRel (example)

However, the decoherence of the discrete structure always yields a convex combination of singletons (i.e. it eliminates all edges):




This is because the discrete structure has enough classical points.

Section 3

# Measurements and Locality

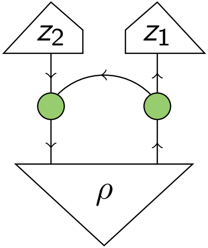
# Measurements and Locality in fRel

The riddle with no apparent answer. We should ask Oedipus.

Credit: *Sphynx*, Snaketoast, DeviantArt

# Testing against classical states

- Testing against ∘-classical points yields a more familiar scenario for decoherence:

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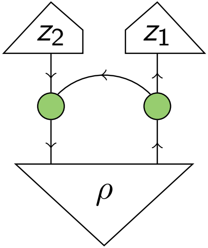
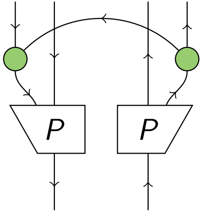



$$= \quad 0 \qquad \text{if } z_1 \neq z_2$$

- However quotienting by equivalence in testing against states trivializes the CPM construction entirely.

# Non-Demolition Measurements

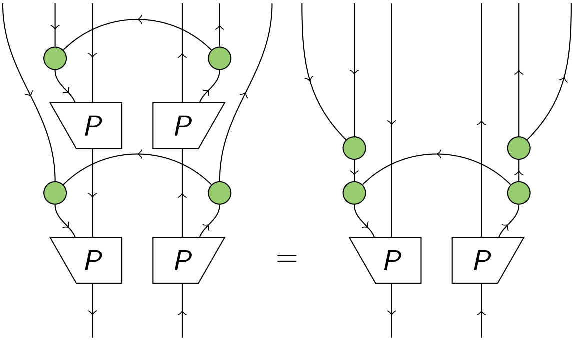
Let ● be a classical structure in a compact-closed †-SMC on some object $Z$. A ●-valued **non-demolition measurement** on some object $X$ is a causal CPM morphism $M : X \xrightarrow{CPM} X \otimes Z$ taking the following form, and which is ●-idempotent and ●-self-adjoint:




Causality is equivalent to $P : X \to X \otimes Z$ being an isometry.

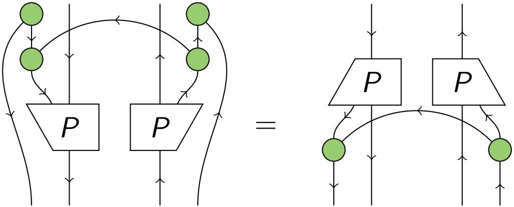
# Non-Demolition Measurements: idempotence

The required ●-idempotence is defined by the following equation:



# Non-Demolition Measurements: self-adjointness

The required ●-self-adjointness is defined by the following equation:



# Demolition Measurements

- If $M : X \xrightarrow{CPM} X \otimes Z$ is a non-demolition measurement, the **demolition measurement** $\bar{M}$ is defined by discarding $X$:

$$\bar{M} := (\cap_X \otimes id_Z) \cdot M : X \xrightarrow{CPM} Z$$

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- Because of the convex-combination issues with decoherence in fRel, we are forced to test the demolition measurement $\bar{M}$ against classical points of $\circ$ to get the classical outcomes:

$$\left( \bar{M}_\lambda := \rho^\dagger_{G_\lambda} \cdot \bar{M} \right)_{\lambda \in \Lambda}$$

# Demolition Measurements

Testing against classical points makes things a bit boring:

- The same classical outcomes of a $\textcolor{green}{\bullet}$-valued demolition measurement $\bar{M} : X \xrightarrow{CPM} Z$ can be obtained by using a decoherence dec($\textcolor{red}{\bullet}$) on $X$, followed by a classical map:

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# Demolition Measurements

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- On the plus side, we only need to consider empirical models coming from decoherences in our proof of locality. This leads to the simplified definition of empirical model that follows.

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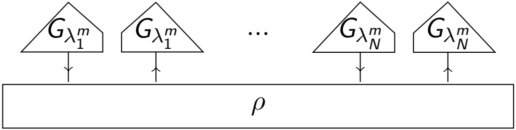
- Let $(\bullet_j^m)_{m=1,...,M}$ be a family of classical structures on $X_j$
- Let $(\Lambda_j^m)_{jm}$ be the sets indexing the classical points
- The **empirical model** is the family of boolean functions $\Phi^m(\lambda_1^m, ..., \lambda_N^m) : \Lambda_1^m \times ... \times \Lambda_N^m \to \{\perp, \top\}$ defined as follows

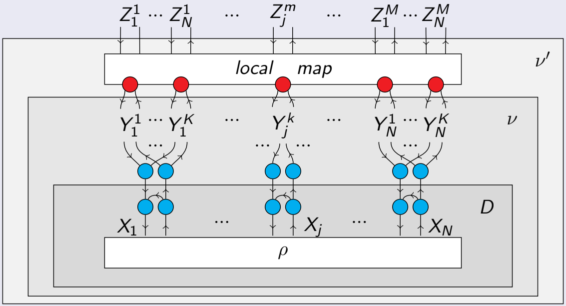
$\Phi^m(\lambda_1^m, ..., \lambda_N^m) \quad :=$



# Locality

## Theorem

*Every empirical model $(\Phi^m)_m$ admits a local hidden variable $\nu$:*
*(i) the mixed state $\rho$ is decohered in the discrete structures* ●
*(ii) the discrete classical data is appropriately copied*



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Key points of the proof (in a nice graph flavour):

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- Decoherence in the discrete structure $\circ$ turns any CPM state $\rho$ into the discrete state $\tau \preceq \rho$ (i.e. removes all edges).
- A discrete state is a convex combination of classical points of the discrete structure $\circ$, and can be appropriately copied.

# Conclusions

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# Conclusions

- fRel, with all the fundamental ingredients and many exotic features, still provides an excellent sandbox for CQM.
- The issues with decoherence invite a deeper reflection on the quantum-classical boundary in CQM, and on the operational interpretation of CPM as a category of mixed states.
- In a framework where decoherence doesn't return convex combinations, testing against classical points may not be physically sound. Measurements/locality need revisiting.

# Thank You!

## Thanks for Your Attention!
## Any Questions?

[Pavlovic (2009)] Quantum and classical structures in nondeterministic computation

[Bar&Vicary (2014)] Groupoid Semantics for Thermal Computing

[Marsden (2015)] A Graph Theoretic Perspective on CPM(Rel)