Entropy, majorization and thermodynamics in general probabilistic systems

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Introduction and Summary

- Project: Understand thermodynamics abstractly by investigating properties necessary and/or sufficient for a Generalized Probabilistic Theory to have a well-behaved analogue of quantum thermodynamics, conceived of as a *resource theory*.
- Aim for results analogous to "Second Laws of Quantum Thermo", and Lostaglio/Jenner/Rudolph work on transitions between non-energy-diagonal states.
- This talk: some groundwork. Assume spectra in order to have analogue to state majorization.
- We give conditions sufficient for operationally-defined measurement entropies to be the spectral entropies.
- Under these conditions we describe assumptions about which processes are thermodynamically reversibile, sufficient to extend von Neumman's argument that quantum entropy is thermo entropy to our setting.

Probabilistic Theories

Theory: Set of systems
System: Specified by bounded convex sets of allowed states, allowed measurements, allowed dynamics compatible with each measurement outcome. (Could view as a category (with "normalization process").)
Composite systems: Rules for combining systems to get a composite system, e.g. tensor product in QM. (Could view as making it a symmetric monoidal category)
Remark: Framework (e.g. convexity, monoidality...) justified operationally. Very weakly constraining.

State spaces and measurements

Normalized states of system A: Convex compact set Ω_A of dimension $d-1$, embedded in $A \simeq \mathbb{R}^d$ as the base of a regular **cone** A_+ of unnormalized states (nonnegative multiples of Ω_A).
Measurement outcomes: linear functionals $A \to \mathbb{R}$ called **effects** whose values on states in Ω_A are in $[0, 1]$.
Unit effect u_A has $u_A(\Omega_A) = 1$.
Measurements: Indexed sets of effects e_i with $\sum_i e_i = u_A$ (or continuous analogues).
Effects generate the **dual cone** A_+^*, of functionals nonnegative on A_+.
Sometimes we may wish to restrict measurement outcomes to a (regular) subcone, call it $A_+^\#$, of A_+^*. If no restriction, system **saturated**.
(A_+ is **regular**: closed, generating, convex, pointed. It makes A an **ordered linear space** (inequalities can be added and multiplied by positive scalars), with order $a \geq b := a - b \in A_+$.)
Dynamics are normalization-non-increasing positive maps.

Inner products, internal representation of the dual and self-duality

In a *real* vector space A an inner product $(\_,\_)$ is equivalent to a linear isomorphism $A \rightarrow A^*$. $y \in A$ corresponds to the functional $x \mapsto (y, x)$. GPT theories often represented this way (Hardy, Barrett...).

- **Internal dual** of A_+ relative to inner product: $A_+^{*int} := \{y \in A : \forall x \in A_+ (y, x) \geq 0\}$. (Affinely isomorphic to A_+^*).
- If there *exists* an inner product relative to which $A_+^{*int} = A_+$, A is called **self-dual.**
- Self-duality is stronger than A_+ affinely isomorphic to A_+^*! (examples)
- related to time reversal?

Examples

Classical: A is the space of n-tuples of real numbers; $u(x) = \sum_{i=1}^{n} x_i$. So Ω_A is the probability simplex, A_+ the positive (i.e.nonnegative) orthant $x : x_i \geq 0, i \in 1, ..., n$

Quantum: $A = \mathscr{B}_h(\mathbf{H})$ = self-adjoint operators on complex (f.d.) Hilbert space $\mathbf{H}$; $u_A(X) = \mathrm{Tr}(X)$. Then Ω_A = density operators. A_+ = positive semidefinite operators.

Squit (or P/Rbit): Ω_A a square, A_+ a four-faced polyhedral cone in $\mathbb{R}^3$.

Inner-product representations: $\langle X, Y \rangle = \mathrm{tr}\, XY$ (Quantum)
$\langle x, y \rangle = \sum_i x_i y_i$ (Classical)

Quantum and classical cones are self-dual! Squit cone is not, but is isomorphic to dual.

Faces of convex sets

Face of convex C: subset S such that if $x \in S$ & $x = \sum_i \lambda_i y_i$, where $y_i \in C$, $\lambda_i > 0$, $\sum_i \lambda_i = 1$, then $y_i \in S$.

Exposed face: intersection of C with a supporting hyperplane. Classical, quantum, squit examples.

For effects e, $F_e^0 := \{x \in \Omega) : e(x) = 0\}$ and $F_e^1 := \{x \in \Omega : e(x) = 1\}$ are exposed faces of Ω.

Distinguishability

States $\omega_1, ..., \omega_n \in \Omega$ are **perfectly distinguishable** if there exist allowed effects $e_1, ..., e_n$, with $\sum_i e_i \leq u$, such that $e_i(\omega_j) = \delta_{ij}$.

Let $e_i, i \in \{1, ..., n\}$ be a submeasurement. $F_i^1 (:= F_{e_i}^1) \subseteq F_j^0$ for $j \neq i$. So it distinguishes the faces F_i^1 from each other.

A list $\omega_1, ..., \omega_n$ of perfectly distinguishable *pure* states is called a **frame** or an **n-frame**.

Filters

Convex abstraction of QM's Projection Postulate (Lüders version): $\rho \mapsto Q\rho Q$ where Q is the orthogonal projector onto a subspace of Hilbert space $\mathscr{H}$.
Helpful in abstracting interference.

Filter := Normalized positive linear map $P : A \to A$: $P^2 = P$, with P and P^* both complemented.
Complemented means $\exists$ filter P' such that $\mathrm{im}\, P \cap A_+ = \ker P' \cap A_+$.
Normalized means $\forall \omega \in \Omega \;\; u(P\omega) \leq 1$.

- Dual of Alfsen and Shultz' notion of **compression**.
- Filters are **neutral**: $u(P\omega) = u(\omega) \implies P\omega = \omega$.
- Ω called **projective** if every face is the positive part of the image of a filter.

Perfection (and Projectivity)

A cone is **perfect** if every face is self-dual in its span according to the restriction of the same inner product.

- In a perfect cone the orthogonal (in self-dualizing inner product) projection onto the span of a face F is positive. In fact it's a filter.

The lattice of faces

- **Lattice**: partially ordered set such that every pair of elements has a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$.
- The faces of any convex set, ordered by set inclusion, form a lattice.
- **Complemented lattice**: bounded lattice in which every element x has a **complement**: x' such that $x \vee x' = 1$, $x \wedge x' = 0$. (Remark: x' not necessarily unique.)
- **orthocomplemented** if equipped with an order-reversing complementation: $x \leq y \implies x' \geq y'$. (Remark: still not necessarily unique.)
- Orthocomplemented lattices satisfy DeMorgan's laws.

Orthomodularity

- **Orthomodularity**: $F \leq G \implies G = F \vee (G \wedge F')$. (draw)
- For projective systems, define $F' := \mathrm{im}_+ P'_F$. Then $'$ is an orthocomplementation, and the face lattice is orthomodular. (Alfsen & Shultz)
- OMLs are "Quantum logics"
- OML's are precisely those orthocomplemented lattices that are determined by their Boolean subalgebras.
- Closely related to **Principle of Consistent Exclusivity** (A. Cabello, S. Severini, A. Winter, `arxiv 1010.2163`):
  If a set of sharp outcomes e_i are pairwise jointly measurable, their probabilities sum to 1 or less in *any* state.
  Limit on noncontextuality.

Symmetry of transition probabilities

• Given projectivity, for each atomic projective unit $p = P^* u$ (P an **atomic** (:= minimal nonzero) filter) the face $P\Omega$ contains a single pure state, call it $\hat{p}$.

$p \mapsto \hat{p}$ is 1:1 from atomic projective units onto extremal points of Ω (pure states).

• **Symmetry of transition probabilities**: for atomic projective units a, b, $a(\hat{b}) = b(\hat{a})$.

A self-dual projective cone has symmetry of transition probabilities.

Theorem (Araki 1980; we rediscovered...)

Projectivity $\implies$ (**STP** $\equiv$ **Perfection**).

Initial results relevant to thermo

(HB, Jonathan Barrett, Markus Mueller, Marius Krumm; in prep, some have appeared in M. Krumm's masters thesis)

Definition

Unique Spectrality: every state has a decomposition into perfectly distinguishable pure states and all such decompositions use the same probabilities.

Stronger than Weak Spectrality (example).

Definition

For $x, y \in \mathbb{R}^n$, $x \prec y$, x **is majorized by** y, means that $\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow}$ for $k = 1, ..., n-1$, and $\sum_{i=1}^{n} x_i^{\downarrow} = \sum_{i=1}^{n} y_i^{\downarrow}$.

Spectral measurement probabilities majorize

A measurement $\{e_i\}$ is **fine-grained** if e_i are on extremal rays of A_+^*.

Theorem (H. Barnum, J. Barrett, M. Müller, M. Krumm)

Let a system satisfy Unique Spectrality, Symmetry of Transition Probabilities, and Projectivity. (Equivalently, Unique Spectrality and Perfection.) Then for any state ω and fine-grained measurement $e_1, ..., e_n$, the vector $\mathbf{p} = [e_1(\omega), ..., e_n(\omega)]$ is majorized by the vector of probabilities of outcomes for a spectral measurement on ω.

Corollary

Let $\omega' = \int_K d\mu(T) T_\mu(\rho)$, where $d\mu(T)$ is a normalized measure on the compact group K of reversible transformations. Then $\omega \preceq \omega'$.

Definition

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *Schur-concave* if for every $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, $\mathbf{v}$ majorizes $\mathbf{w}$ implies $f(\mathbf{v}) \leq f(\mathbf{w})$.

Entropy-like; mixing-monotone.

Proposition

Every concave symmetric function is Schur-concave.

Definition (Measurement, preparation, spectral "entropies")

Let χ be a Schur-concave function. Define

$\chi^{meas}(\omega) := \min_{fine-grained\ measurements} \chi([e_1(\omega), ..., e_{\#outcomes}(\omega)])$.

$\chi^{prep}(\omega) :=$ minimum over convex decompositions of $\omega = \sum_i p_i \omega_i$) of ω into pure states, of $\chi(\mathbf{p})$.

$\chi^{spec}(\omega) := \chi(spec(\omega))$.

Rényi entropies

Definition (Rényi entropies)

$$H_\alpha(\mathbf{p}) := \frac{1}{1-\alpha} \log\left(\sum_i p_i^\alpha\right)$$

for $\alpha \in (0,1) \cup (1,\infty)$.

$$H_0(\mathbf{p}) := \lim_{\alpha\to 0} H_\alpha(\mathbf{p}) = -\log|\mathrm{supp}\ \mathbf{p}|.$$

$$H_1(\mathbf{p}) = \lim_{\alpha\to 1} H_\alpha(\mathbf{p}) = H(\mathbf{p}).$$

$$H_\infty(\mathbf{p}) = \lim_{\alpha\to\infty} H_\alpha(\mathbf{p}) = -\log\max_j p_j.$$

Concave, Schur-concave.

Proposition (Corollary of "spectral probabilities majorize".)

In a **perfect** *system (equivalently one with* **spectrality**, **projectivity**, *and* **STP***), any concave and Schur-concave function of finegrained measurement outcome probabilities is minimized by the spectral measurement.*

So e.g. Rényi measurement entropy = spectral Rényi entropy.

Proposition

Assume **Weak Spectrality**, **Strong Symmetry**. *Then* $H_2^{prep} = H_2^{meas}$. *("Collision entropies".)*

Proposition

Assume **Weak Spectrality**, **Strong Symmetry**. *If* $H_0^{prep} = H_0^{meas}$ *then* **No Higher-Order Interference** *holds (and vice versa). (So systems are Jordan-algebraic.)*

Because $H_0^{prep} = H_0^{meas}$ is basically the covering law given the background assumptions.

Could enable some purification axiom that implies $H_0^{prep} = H_0^{meas}$ via steering (e.g. locally tomographic purification with identical marginals) to imply Jordan-algebraic systems.

Relative entropy

Definition (Relative entropy)

Assume **Strong Symmetry**, **Weak Spectrality**.
$S(\rho||\sigma) := -H^{spec}(\rho) - (\rho, \ln \sigma)$.

Theorem

$S(\rho||\sigma) \geq 0$.

To Do: Define more information divergences/“distances”. Get monotonicity results. Use these in a resource theory.

Further observations:

Filters allow for **emergent classicality**: generalized *decoherence* onto classical subsets of the state space: $\omega \mapsto P_1\omega + P_2\omega + \cdots + P_n\rho$, P_i filters.

Open question: the operator projecting out higher-order interference is a projector. Is it positive? If so, **higher-order decoherence** possible. Could make HOI more plausible as potential trans-quantum physics.

Filters might be useful in **information-processing** protocols like computation, data compression ("project onto typical subspace"), coding.

Characterization of quantum systems

HB, Markus Müller, Cozmin Ududec

1. **Weak Spectrality**: every state is in convex hull of a set of perfectly distinguishable pure (i.e. extremal) states
2. **Strong Symmetry**: Every set of perfectly distinguishable pure states transforms to any other such set of the same size **reversibly**.
3. **No irreducibly three-slit** (or more) **interference.**
4. **Energy observability**: Systems have nontrivial continuously parametrized reversible dynamics. Generators of one-parameter continuous subgroups (“Hamiltonians”) are associated with nontrivial conserved observables.

•$1 - 4 \implies$ standard quantum system (over $\mathbb{C}$)

•$1 - 3 \implies$ irreducible Jordan algebraic systems, and classical.

•$1 - 2 \implies$ “projective” (filters onto faces), self-dual systems

Reference

H. Barnum, M. Müller, C. Ududec, "Higher order interference and single system postulates characterizing quantum theory," *New J. Phys* **16** 123029 (2014). Open access. Also `arxiv:1403.4147`.

Jordan Algebraic Systems

- Pascual Jordan, (Z. Phys, 1932 or 1933):
  - **Jordan algebra:** abstracts properties of Hermitian operators.
  - Symmetric product $\bullet$ abstracts $A \bullet B = \frac{1}{2}(AB + BA)$.
  - Jordan identity: $a \bullet (b \bullet a^2) = (a \bullet b) \bullet a^2$.
  - **Formally real JA:** $a^2 + b^2 = 0 \implies a = b = 0$. Makes the cone of squares a candidate for unnormalized state space.
- Jordan, von Neumann, Wigner (Ann. Math., **35**, 29-34 (1934)): irreducible f.d. formally real Jordan algebras are:
  - quantum systems (self-adjoint matrices) over $\mathbb{R}, \mathbb{C}$, and $\mathbb{H}$;
  - systems whose state space is a ball (aka "spin factors");
  - 3×3 Hermitian octonionic matrices ("exceptional" JA).
- f.d. homogeneous self-dual cones are precisely the cones of squares in f.d. formally real Jordan algebras. (Koecher 1958, Vinberg 1960)

Consequences of Postulates 1 and 2

Postulates 1 and 2 together have many important consequences including:

- *Saturation*: effect cone is *full* dual cone.
- *Self-duality*. (Mueller and Ududec, PRL: saturation plus special case of postulate 2, reversible transitivity on *pairs* of pure states $\implies$ self-duality.)
- **Perfection**: every face is self-dual in its span according to the restriction of the same inner product
- Every face of Ω is generated by a frame. If $F \leq G$, a frame for F extends to one for G. All frames for F have same size.
- The orthogonal (in self-dualizing inner product) projection onto the span of a face F is positive, in fact it's a *filter* (defined soon).

Multi-slit interference I

To adapt Rafael Sorkin's k-th order interference to our framework, need k-slit experiments.

k-slit mask: Set of filters $P_1, ..., P_k$ onto distinguishable faces. Define $P_J := \bigvee_{i \in J} P_i$. (Notation: $P_{ij...n} = P_i \vee P_j \vee \cdots \vee P_n$.)

In QM: maps $\rho \mapsto Q_i \rho Q_i$, where Q_i are projectors onto orthogonal subspaces S_i of $\mathscr{H}$.

- **2nd-order interference** if for some 2-slit mask,

$$P_1 + P_2 \neq P_{12}. \tag{1}$$

- **3rd-order interference** if for some 3-slit mask,

$$P_{12} + P_{13} + P_{23} - P_1 - P_2 - P_3 \neq P_{123}. \tag{2}$$

(Zero in quantum theory; easy to check at Hilbert space/pure-state level.)

Multi-slit interference II

k-th order interference if for some mask $M = \{P_1, .., P_k\}$,

$$\sum_{r=1}^{k-1} (-1)^{r-1} \sum_{|J|=k-r} P_J \neq P_M \,. \qquad (3)$$

• Equivalently $F_M = \mathrm{lin} \cup_{|J|=k-1} F_J$ (no "*k*-**th order coherence**"). (Ududec, Barnum, Emerson, *Found. Phys.* **46**: 396-405 (2011). (`arxiv: 0909.4787`) for $k = 3$, in prep. arbitrary *k* (& CU thesis).)

Components of a state in $F_M \setminus \mathrm{lin} \cup_{|J|=k-1} F_J$ are *k*-th order "coherences". In QM: off-block-diagonal density matrix elements.

• No *k*-th order $\implies$ no $k+1$-st order.

Characterizing Jordan algebraic systems

Theorem (Adaptation of Alfsen & Shultz, Thm 9.3.3)

Let a finite-dimensional system satisfy

(a) **Projectivity***: there is a filter onto each face*

(b) **Symmetry of Transition Probabilities***, and*

(c) **Filters Preserve Purity***: if ω is a pure state, then $P\omega$ is a nonnegative multiple of a pure state.*

Then Ω is the state space of a formally real Jordan algebra.

Theorem (Barnum, Müller, Ududec)

*(***Weak Spectrality** *&* **Strong Symmetry***)* $\implies$ **Projectivity** *&* **STP***;*
WS *&* **SS** *&* **No Higher Interference** $\implies$ **Filters Preserve Purity***.*
Jordan algebraic system thus obtained must be either irreducible or classical. (All such satisfy **WS, SS, No HOI***.)*