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Project: Understand thermodynamics abstractly by investigating properties necessary and/or sufficient for a Generalized Probabilistic Theory to have a well-behaved analogue of quantum thermodynamics, conceived of as a resource theory.

Aim for results analogous to “Second Laws of Quantum Thermo”, and Lostaglio/Jenner/Rudolph work on transitions between non-energy-diagonal states.

This talk: some groundwork. Assume spectra in order to have analogue to state majorization.

We give conditions sufficient for operationally-defined measurement entropies to be the spectral entropies.

Under these conditions we describe assumptions about which processes are thermodynamically reversible, sufficient to extend von Neumman’s argument that quantum entropy is thermo entropy to our setting.
**Theory**: Set of systems

**System**: Specified by bounded convex sets of allowed states, allowed measurements, allowed dynamics compatible with each measurement outcome. (Could view as a category (with “normalization process”).)

**Composite systems**: Rules for combining systems to get a composite system, e.g. tensor product in QM. (Could view as making it a symmetric monoidal category)

**Remark**: Framework (e.g. convexity, monoidality...) justified operationally. Very weakly constraining.
Normalized states of system $A$: Convex compact set $\Omega_A$ of dimension $d - 1$, embedded in $A \cong \mathbb{R}^d$ as the base of a regular cone $A_+$ of unnormalized states (nonnegative multiples of $\Omega_A$).

Measurement outcomes: linear functionals $A \to \mathbb{R}$ called effects whose values on states in $\Omega_A$ are in $[0, 1]$.

Unit effect $u_A$ has $u_A(\Omega_A) = 1$.

Measurements: Indexed sets of effects $e_i$ with $\sum_i e_i = u_A$ (or continuous analogues).

Effects generate the dual cone $A_+^*$, of functionals nonnegative on $A_+$. Sometimes we may wish to restrict measurement outcomes to a (regular) subcone, call it $A_+^\#$, of $A_+^*$. If no restriction, system saturated. ($A_+$ is regular: closed, generating, convex, pointed. It makes $A$ an ordered linear space (inequalities can be added and multiplied by positive scalars), with order $a \geq b := a - b \in A_+$.)

Dynamics are normalization-non-increasing positive maps.
In a *real* vector space $A$ an inner product $(\_ , \_ )$ is equivalent to a linear isomorphism $A \rightarrow A^*$. $y \in A$ corresponds to the functional $x \mapsto (y , x)$. GPT theories often represented this way (Hardy, Barrett...).

- **Internal dual** of $A_+$ relative to inner product:
  $A^{*\text{int}}_+ := \{ y \in A : \forall x \in A_+ (y , x) \geq 0 \}$ . (Affinely isomorphic to $A^*_+$).

- If there exists an inner product relative to which $A^{*\text{int}}_+ = A_+$, $A$ is called **self-dual**.

- Self-duality is stronger than $A_+$ affinely isomorphic to $A^*_+$!
  (examples)

- related to time reversal?
Examples

**Classical:** $A$ is the space of $n$-tuples of real numbers; $u(x) = \sum_{i=1}^{n} x_i$. So $\Omega_A$ is the probability simplex, $A_+$ the positive (i.e. nonnegative) orthant $x : x_i \geq 0, i \in 1, \ldots, n$.

**Quantum:** $A = \mathcal{B}_h(H) =$ self-adjoint operators on complex (f.d.) Hilbert space $H$; $u_A(X) = \text{Tr}(X)$. Then $\Omega_A =$ density operators. $A_+ =$ positive semidefinite operators.

**Squit (or P/Rbit):** $\Omega_A$ a square, $A_+$ a four-faced polyhedral cone in $\mathbb{R}^3$.

**Inner-product representations:**
- Quantum: $\langle X, Y \rangle = \text{tr } XY$
- Classical: $\langle x, y \rangle = \sum_i x_i y_i$

Quantum and classical cones are self-dual! Squit cone is not, but is isomorphic to dual.
Faces of convex sets

**Face** of convex $C$: subset $S$ such that if $x \in S$ & $x = \sum_i \lambda_i y_i$, where $y_i \in C$, $\lambda_i > 0$, $\sum_i \lambda_i = 1$, then $y_i \in S$.

**Exposed face**: intersection of $C$ with a supporting hyperplane. Classical, quantum, squit examples.

For effects $e$, $F^0_e := \{ x \in \Omega : e(x) = 0 \}$ and $F^1_e := \{ x \in \Omega : e(x) = 1 \}$ are exposed faces of $\Omega$. 
States $\omega_1, \ldots, \omega_n \in \Omega$ are perfectly distinguishable if there exist allowed effects $e_1, \ldots, e_n$, with $\sum_i e_i \leq u$, such that $e_i(\omega_j) = \delta_{ij}$.

Let $e_i, i \in \{1, \ldots, n\}$ be a submeasurement. $F^1_i (:= F^1_{e_i}) \subseteq F^0_j$ for $j \neq i$. So it distinguishes the faces $F^1_i$ from each other.

A list $\omega_1, \ldots, \omega_n$ of perfectly distinguishable pure states is called a frame or an n-frame.
Convex abstraction of QM’s Projection Postulate (Lüders version): $\rho \mapsto Q\rho Q$ where $Q$ is the orthogonal projector onto a subspace of Hilbert space $\mathcal{H}$. Helpful in abstracting interference.

**Filter**: Normalized positive linear map $P : A \to A$: $P^2 = P$, with $P$ and $P^*$ both complemented. 

*Complemented* means $\exists$ filter $P'$ such that $\text{im } P \cap A_+ = \ker P' \cap A_+$. 

*Normalized* means $\forall \omega \in \Omega \; u(P\omega) \leq 1$.

- Dual of Alfsen and Shultz’ notion of compression.
- Filters are neutral: $u(P\omega) = u(\omega) \implies P\omega = \omega$.
- $\Omega$ called projective if every face is the positive part of the image of a filter.
Perfection (and Projectivity)

A cone is **perfect** if every face is self-dual in its span according to the restriction of the same inner product.

- In a perfect cone the orthogonal (in self-dualizing inner product) projection onto the span of a face $F$ is positive. In fact it’s a filter.
The lattice of faces

- **Lattice**: partially ordered set such that every pair of elements has a least upper bound $x \lor y$ and a greatest lower bound $x \land y$.

- The faces of any convex set, ordered by set inclusion, form a lattice.

- **Complemented lattice**: bounded lattice in which every element $x$ has a **complement**: $x'$ such that $x \lor x' = 1$, $x \land x' = 0$. (Remark: $x'$ not necessarily unique.)

- **Orthocomplemented** if equipped with an order-reversing complementation: $x \leq y \implies x' \geq y'$. (Remark: still not necessarily unique.)

- Orthocomplemented lattices satisfy DeMorgan’s laws.
Orthomodularity: \( F \leq G \iff G = F \vee (G \wedge F') \).

For projective systems, define \( F' := \text{im} + P_F' \). Then \( ' \) is an orthocomplementation, and the face lattice is orthomodular. (Alfsen & Shultz)

OMLs are “Quantum logics”

OML’s are precisely those orthocomplemented lattices that are determined by their Boolean subalgebras.

Closely related to Principle of Consistent Exclusivity (A. Cabello, S. Severini, A. Winter, arxiv 1010.2163):

If a set of sharp outcomes \( e_i \) are pairwise jointly measurable, their probabilities sum to 1 or less in any state. Limit on noncontextuality.
Symmetry of transition probabilities

- Given projectivity, for each atomic projective unit $p = P^* u$ ($P$ an \textbf{atomic} (:= minimal nonzero) filter) the face $P\Omega$ contains a single pure state, call it $\hat{p}$.

$p \mapsto \hat{p}$ is 1:1 from atomic projective units onto extremal points of $\Omega$ (pure states).

- \textbf{Symmetry of transition probabilities}: for atomic projective units $a, b$, $a(\hat{b}) = b(\hat{a})$.

A self-dual projective cone has symmetry of transition probabilities.

\textbf{Theorem (Araki 1980; we rediscovered...)}

\textbf{Projectivity $\implies$ (STP $\equiv$ Perfection).}
Initial results relevant to thermo

(HB, Jonathan Barrett, Markus Mueller, Marius Krumm; in prep, some have appeared in M. Krumm’s masters thesis)

**Definition**

**Unique Spectrality**: every state has a decomposition into perfectly distinguishable pure states and all such decompositions use the same probabilities.

Stronger than Weak Spectrality (example).

**Definition**

For $x, y \in \mathbb{R}^n$, $x \prec y$, $x$ is majorized by $y$, means that $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$ for $k = 1, \ldots, n - 1$, and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$. 
A measurement \( \{ e_i \} \) is **fine-grained** if \( e_i \) are on extremal rays of \( A_+^* \).

**Theorem (H. Barnum, J. Barrett, M. Müller, M. Krumm)**

Let a system satisfy Unique Spectrality, Symmetry of Transition Probabilities, and Projectivity. (Equivalently, Unique Spectrality and Perfection.) Then for any state \( \omega \) and fine-grained measurement \( e_1, ..., e_n \), the vector \( p = [e_1(\omega), ..., e_n(\omega)] \) is majorized by the vector of probabilities of outcomes for a spectral measurement on \( \omega \).

**Corollary**

Let \( \omega' = \int_K d\mu(T) T_{\mu}(\rho) \), where \( d\mu(T) \) is a normalized measure on the compact group \( K \) of reversible transformations. Then \( \omega \preceq \omega' \).
Definition
A function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is called Schur-concave if for every \( \mathbf{v}, \mathbf{w} \in \mathbb{R}^n \), \( \mathbf{v} \) majorizes \( \mathbf{w} \) implies \( f(\mathbf{v}) \leq f(\mathbf{w}) \).

Entropy-like; mixing-monotone.

Proposition
Every concave symmetric function is Schur-concave.

Definition (Measurement, preparation, spectral “entropies”)
Let \( \chi \) be a Schur-concave function. Define
\[
\chi^{\text{meas}}(\omega) := \min_{\text{fine-grained measurements}} \chi([e_1(\omega), \ldots, e_{\#\text{outcomes}}(\omega)]).
\]
\[
\chi^{\text{prep}}(\omega) := \min_{\text{convex decompositions}} \chi(p)
\]
into pure states, of \( \chi(p) \).
\[
\chi^{\text{spec}}(\omega) := \chi(\text{spec}(\omega)).
\]
Rényi entropies

Definition (Rényi entropies)

\[ H_\alpha(p) := \frac{1}{1 - \alpha} \log \left( \sum_i p_i^\alpha \right) \]

for \( \alpha \in (0, 1) \cup (1, \infty) \).

\[ H_0(p) := \lim_{\alpha \to 0} H_\alpha(p) = -\log |\text{supp } p|. \]

\[ H_1(p) = \lim_{\alpha \to 1} H_\alpha(p) = H(p). \]

\[ H_\infty(p) = \lim_{\alpha \to \infty} H_\alpha(p) = -\log \max_j p_j. \]

Concave, Schur-concave.
Proposition (Corollary of “spectral probabilities majorize”.)

*In a perfect system (equivalently one with spectrality, projectivity, and STP), any concave and Schur-concave function of finegrained measurement outcome probabilities is minimized by the spectral measurement.*

So e.g. Rényi measurement entropy = spectral Rényi entropy.
Proposition

Assume Weak Spectrality, Strong Symmetry. Then $H_{2}^{\text{prep}} = H_{2}^{\text{meas}}$. ("Collision entropies").
Proposition

Assume Weak Spectrality, Strong Symmetry. If $H_{0}^{\text{prep}} = H_{0}^{\text{meas}}$ then No Higher-Order Interference holds (and vice versa). (So systems are Jordan-algebraic.)

Because $H_{0}^{\text{prep}} = H_{0}^{\text{meas}}$ is basically the covering law given the background assumptions.

Could enable some purification axiom that implies $H_{0}^{\text{prep}} = H_{0}^{\text{meas}}$ via steering (e.g. locally tomographic purification with identical marginals) to imply Jordan-algebraic systems.
Relative entropy

**Definition (Relative entropy)**

Assume **Strong Symmetry**, **Weak Spectrality**.

\[ S(\rho || \sigma) := -H^{spec}(\rho) - (\rho, \ln \sigma). \]

**Theorem**

\[ S(\rho || \sigma) \geq 0. \]

To Do: Define more information divergences/“distances”. Get monotonicity results. Use these in a resource theory.
Further observations:

Filters allow for **emergent classicality**: generalized *decoherence* onto classical subsets of the state space: \( \omega \mapsto P_1 \omega + P_2 \omega + \cdots + P_n \rho \), \( P_i \) filters.

**Open question**: the operator projecting out higher-order interference is a projector. Is it positive? If so, **higher-order decoherence** possible. Could make HOI more plausible as potential trans-quantum physics.

Filters might be useful in **information-processing** protocols like computation, data compression (“project onto typical subspace”), coding.
Characterization of quantum systems

HB, Markus Müller, Cozmin Ududec

1. **Weak Spectrality**: every state is in convex hull of a set of perfectly distinguishable pure (i.e. extremal) states

2. **Strong Symmetry**: Every set of perfectly distinguishable pure states transforms to any other such set of the same size reversibly.

3. **No irreducibly three-slit (or more) interference.**

4. **Energy observability**: Systems have nontrivial continuously parametrized reversible dynamics. Generators of one-parameter continuous subgroups ("Hamiltonians") are associated with nontrivial conserved observables.

- $1 - 4 \implies$ standard quantum system (over $\mathbb{C}$)
- $1 - 3 \implies$ irreducible Jordan algebraic systems, and classical.
- $1 - 2 \implies$ "projective" (filters onto faces), self-dual systems.
Jordan Algebraic Systems

- Pascual Jordan, (Z. Phys, 1932 or 1933):
  - **Jordan algebra**: abstracts properties of Hermitian operators.
  - Symmetric product $\bullet$ abstracts $A \bullet B = \frac{1}{2}(AB + BA)$.
  - Jordan identity: $a \bullet (b \bullet a^2) = (a \bullet b) \bullet a^2$.
  - **Formally real JA**: $a^2 + b^2 = 0 \implies a = b = 0$. Makes the cone of squares a candidate for unnormalized state space.

  - quantum systems (self-adjoint matrices) over $\mathbb{R}, \mathbb{C},$ and $\mathbb{H}$;
  - systems whose state space is a ball (aka “spin factors”);
  - $3 \times 3$ Hermitian octonion matrices (“exceptional” JA).

- f.d. homogeneous self-dual cones are precisely the cones of squares in f.d. formally real Jordan algebras. (Koecher 1958, Vinberg 1960)
Consequences of Postulates 1 and 2

Postulates 1 and 2 together have many important consequences including:

- **Saturation**: effect cone is *full* dual cone.
- **Self-duality**: (Mueller and Ududec, PRL: saturation plus special case of postulate 2, reversible transitivity on *pairs* of pure states $\implies$ self-duality.)
- **Perfection**: every face is self-dual in its span according to the restriction of the same inner product.
- Every face of $\Omega$ is generated by a frame. If $F \leq G$, a frame for $F$ extends to one for $G$. All frames for $F$ have same size.
- The orthogonal (in self-dualizing inner product) projection onto the span of a face $F$ is positive, in fact it’s a *filter* (defined soon).
Multi-slit interference I

To adapt Rafael Sorkin’s $k$-th order interference to our framework, need $k$-slit experiments.

**$k$-slit mask:** Set of filters $P_1, \ldots, P_k$ onto distinguishable faces. Define $P_J := \bigvee_{i \in J} P_i$. (Notation: $P_{ij \ldots n} = P_i \lor P_j \lor \cdots \lor P_n$.)

**In QM:** maps $\rho \mapsto Q_i \rho Q_i$, where $Q_i$ are projectors onto orthogonal subspaces $S_i$ of $\mathcal{H}$.

- **2nd-order interference** if for some 2-slit mask,
  \[ P_1 + P_2 \neq P_{12}. \] (1)

- **3rd-order interference** if for some 3-slit mask,
  \[ P_{12} + P_{13} + P_{23} - P_1 - P_2 - P_3 \neq P_{123}. \] (2)

(Zero in quantum theory; easy to check at Hilbert space/pure-state level.)
k-th order interference if for some mask \( M = \{P_1, \ldots, P_k\} \),

\[
\sum_{r=1}^{k-1} (-1)^{r-1} \sum_{|J|=k-r} P_J \neq P_M.
\] (3)

• Equivalently \( F_M = \text{lin} \cup_{|J|=k-1} F_J \) (no “k-th order coherence”).

(arxiv: 0909.4787) for \( k = 3 \), in prep. arbitrary \( k \) ( & CU thesis).)

Components of a state in \( F_M \setminus \text{lin} \cup_{|J|=k-1} F_J \) are k-th order “coherences”. In QM: off-block-diagonal density matrix elements.

• No k-th order \( \implies \) no \( k+1 \)-st order.
Characterizing Jordan algebraic systems

Theorem (Adaptation of Alfsen & Shultz, Thm 9.3.3)

Let a finite-dimensional system satisfy

(a) **Projectivity**: there is a filter onto each face
(b) **Symmetry of Transition Probabilities**, and
(c) **Filters Preserve Purity**: if $\omega$ is a pure state, then $P\omega$ is a nonnegative multiple of a pure state.

Then $\Omega$ is the state space of a formally real Jordan algebra.

Theorem (Barnum, Müller, Ududec)

(Weak Spectrality & Strong Symmetry) $\implies$ Projectivity & STP; WS & SS & No Higher Interference $\implies$ Filters Preserve Purity. Jordan algebraic system thus obtained must be either irreducible or classical. (All such satisfy WS, SS, No HOI.)