Axiomatizing complete positivity

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Model quantum mechanics by $\text{FdHilb}$ (Dagger Compact Symmetric Monoidal Category).

Can model mixed states in two ways

- $\text{CPM}[\text{FdHilb}]$: Same objects, morphisms are completely positive maps.
- $\text{CP}^*[\text{FdHilb}]$: Category of C*-algebras and completely positive maps.

Generalises to functors $\text{CPM, CP}^*: \text{DCSMC} \rightarrow \text{DCSMC}$.

Canonical inclusions $\mathbb{C} \hookrightarrow \text{CPM}[\mathbb{C}] \hookrightarrow \text{CP}^*[\mathbb{C}]$

Question: When is $\mathbb{D}$ equal to some $\text{CPM}[\mathbb{C}]$ or $\text{CP}^*[\mathbb{C}]$?
Categories

Diagram

Object:
Finite-dim Hilbert space $A$

Morphism:
Linear map $f : A \to B$

Composition:
Composition of linear maps

Example: $\text{FdHilb}$
Symmetric Monoidal Categories

Monoidal product: $A \otimes A'$

Tensor product $A \otimes A'$

On morphisms: $f \otimes g$

Swap: Canonical map $A \otimes A' \rightarrow A' \otimes A$
Compact Monoidal Categories

Dual object: $A$

Dual vector space $A^*$

Cup: $\cup : \mathbb{C} \to \text{End}(A) \cong A \otimes A^*$

$z \mapsto zI$

Cap: $\cap : A^* \otimes A \cong \text{End}(A) \to \mathbb{C}$

Can define transpose: $f$ denoted as $\bar{f}$
Dagger Categories

Dagger: 

\[ \begin{array}{c}
B \\
A
\end{array} \rightarrow \begin{array}{c}
A \\
B
\end{array} \]

\[ f^\dagger \text{ s.t. } \langle fv, w \rangle = \langle v, f^\dagger w \rangle \]

Gives a contravariant functor:

\[ \begin{array}{c}
C \\
A
\end{array} \rightarrow \begin{array}{c}
A \\
C
\end{array} \]

Hence have all four orientations:

Original  Transpose  Adjoint  Conjugate

\[ f \quad f \quad f \quad f \]
In QM, mixed states are given by positive matrices:

These correspond to states of $A^\ast \otimes A$:

Generalising this, the Choi–Jamiolkowski isomorphism says that the completely positive maps are precisely the ones of the form:
The CPM construction

(P. Selinger Dagger Compact Closed Categories and Completely Positive Maps)
CPM[D] has objects $ob(D)$
Morphisms $A \rightarrow B$ are

\[ D \text{ embeds into } CPM[D] \text{ by} \]

Question: Given $C$, when is $C \sim CPM[D]$ for some $D$?
Answer: B. Coecke  **Axiomatic Description of Mixed States**  
From Selinger’s CPM-construction

**Theorem**

Suppose $C$ has a subcategory $C^{\text{pure}}$ (with the same objects) and for each object a map $\hat{\cdot}$ satisfying

\[
\begin{align*}
A \otimes C &= A \otimes C \quad \text{and} \quad I = I \quad \text{and} \quad A = A \\
\begin{array}{c}
\text{Diagram 1}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{Diagram 2}
\end{array}
\end{align*}
\]

For all $f \in C$ we have $g \in C^{\text{pure}}$ such that

\[
\begin{array}{c}
\text{Diagram 3}
\end{array}
\]

Then $C \sim CPM[C^{\text{pure}}]$
Proof Sketch

Consider functors between $\mathbf{C}$ and $\mathbf{CPM}[\mathbf{C}^{\text{pure}}]$ given by

\[ g \leftrightarrow \overline{g} \]

(”Folding in half”)

The axioms are precisely what is needed to prove that these are well defined DCSM-functors. They are clearly inverse to each other.
A special dagger Frobenius structure is an object $A$ equipped with morphisms $\langle \alpha \rangle : A \otimes A \to A$ and $\bullet : I \to A$ satisfying:

The special dagger Frobenius structures in $\text{FdHilb}$ correspond precisely to the finite dimensional $\text{C}^*$-algebras. (B. Coecke et al. Categories of Quantum and Classical Channels)
(Shorthand: Write $\phi\downarrow\phi$ for $\phi\downarrow\phi\phi$, and $\phi\downarrow\phi$ for $\phi\downarrow\phi\phi$.)

A state is positive when it is of the form:

![Diagram of a state positive]

Generalising this, a morphism is completely positive when it is of the form:

![Diagram of a morphism completely positive]
The CP* construction

CP*[D] has as objects the special dagger Frobenius algebras 
\((A, \otimes, \dagger)\) in D

Morphisms \((A, \otimes, \dagger) \rightarrow (B, \otimes, \dagger)\) are

\[ A \rightarrow (A \otimes A^*, \otimes, \dagger) \]

D embeds into CP*[D] by

\[ A \rightarrow (A \otimes A^*, \otimes, \dagger) \]
Problem: We can’t talk about $\circlearrowleft$ in $\text{CP}^*[\mathbf{D}]$ because it isn’t itself a completely positive map (from $(A \otimes A, \ldots)$ to $(A, \ldots)$).

Idea: $\circlearrowleft$ is completely positive when considered as going from $(A \otimes A^*, \ldots)$ to $(A, \ldots)$. 

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Answer:

**Theorem**

Suppose $\mathbf{C}$ has a subcategory $\mathbf{C}^{\text{pure}}$ and

- for each object $A \in \mathbf{C}^{\text{pure}}$ we have specified a morphism $\triangleleft : A \to I$,

- for each special dagger Frobenius algebra $A = (A, \bigtriangleup, \bigtriangledown)$ in $\mathbf{C}^{\text{pure}}$ we have specified an object $F_A \in \mathbf{C}$ and a map $\uparrow : A \to F_A$.

If the following axioms are satisfied then $\mathbf{C} \sim \text{CP}^*[\mathbf{C}^{\text{pure}}]$
\[ A \otimes C = A \quad \text{and} \quad I = I \quad \text{and} \quad A = A \quad \text{(1)} \]

\[ f = g \iff f = g \quad \text{for } f, g \in \mathbf{C}^{\text{pure}} \quad \text{(2)} \]

\[ F_{A \otimes B} = F_A F_B \quad \text{and} \quad F_I = I \quad \text{(3)} \]

\[ = \quad \text{and} \quad = \quad \text{(4)} \]

For all \( f \in \mathbf{C} \) we have \( g \in \mathbf{C}^{\text{pure}} \) such that \( f = g \quad \text{(5)} \)
The isomorphism is

\[ f \leftrightarrow f \]