Unordered Tuples in Quantum Computation

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July 15, 2015
What we did

Computed algebras for several unordered quantum types. (e.g., unordered pair, cycles)

(After discussing paper of Pagani, Selinger, Valiron with Sam Staton.)
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The heavy lifting
The heavy lifting

Schur

Weyl
Quantum types as algebras

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<tbody>
<tr>
<td>qubit</td>
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<td>ordered pair of bits</td>
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He replied: "Fermions or Bosons?"

1. Bosons:

$$|00\rangle, |11\rangle, |01\rangle + |10\rangle$$

2. Fermions:

$$C|01\rangle - |10\rangle$$

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**Unordered pair of qubits**

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So what about CoEq(id, swap)?

\[ t \otimes t f \rightarrow s (f \circ \text{swap} = f) \]

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So what about $\text{CoEq}(\text{id}, \text{swap})$?

\[
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\text{CoEq}(\text{id}, \text{swap}) & \xrightarrow{f'} s
\end{align*}
\]
CoEq(id, swap)
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\[ M_3 \oplus \mathbb{C} \]

comes from \(|00\rangle, |11\rangle, |10\rangle + |01\rangle\). 

\(C\) corresponds to \(|01\rangle - |10\rangle\), which is symmetric up to global phase.
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(In $\text{fd-CStar}_{\text{cPsU}}^{\text{op}}$)
CoEq(id, swap)

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(In Selinger’s Q)
CoEq(id, swap)

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\((\text{In } \text{fd-} \text{CStar}_{\text{cpSU}}^{\text{op}})\)

\(M_3\) comes from \(|00\rangle\), \(|11\rangle\) and \(|10\rangle + |01\rangle\).
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The coequalizer is easy to describe:

\[ E = \{ a; \ a \in M_2 \otimes M_2; \ \text{swap}(a) = a \} \]
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**Crux:** \( E \cong M_3 \oplus \mathbb{C} \).
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Has simple $\frac{1}{2}$-page proof, which led to . . .
Remainder of this talk

1. Unordered tuples
   - Sketch of proof
2. Cycles
3. Unordered words
Remainder of this talk

1. Unordered tuples
   ▶ Sketch of proof
2. Cycles
3. Unordered words
Result 1: unordered tuples

\[ \text{unordered tuples of } d\text{-level systems} \]
\[ \bigoplus_{\lambda \in Y^n,d} M^\lambda \]
where \( Y^n,d \) denotes the \( n \)-block Young diagrams of height at most \( d \) and \( M^\lambda \) the dimension of the corresponding representation of \( \text{GL}(d) \).

Or explicitly:
\[ Y^n,d = \{ \lambda; \lambda \in \mathbb{N}^d; \lambda_1 \geq \ldots \geq \lambda_d \geq 0; \lambda_1 + \ldots + \lambda_d = n \} \]
and
\[ M^\lambda = \prod_{1 \leq i < j \leq d} \lambda_i - \lambda_j + j - i. \]
Result 1: unordered tuples

Unordered $n$-tuples of $d$-level systems

$$\bigoplus_{\lambda \in Y_{n,d}} M_{m\lambda}$$
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Unordered \( n \)-tuples of \( d \)-level systems

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where $Y_{n,d}$ denotes the $n$-block Young diagrams of height at most $d$ and $m_\lambda$ the dimension of the corresponding representation of $\text{GL}(d)$. Or explicitly: $Y_{n,d} = \left\{ \lambda; \lambda \in \mathbb{N}^d; \left[ \begin{array}{c} \lambda_1 \geq \cdots \geq \lambda_d \geq 0 \\ \lambda_1 + \cdots + \lambda_d = n \end{array} \right] \right\}$
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Examples
Examples

Unordered triple of qutrits
Examples

Unordered triple of qutrits  \[ M_{10} \oplus M_8 \oplus \mathbb{C} \]
Examples

Unordered triple of qutrits \( M_{10} \oplus M_8 \oplus \mathbb{C} \)

Unordered pair of ququads
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Unordered pair of ququads \( M_{10} \oplus M_6 \)
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Unordered triple of qutrits \( M_{10} \oplus M_8 \oplus \mathbb{C} \)
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Unordered quad of qubits
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Unordered triple of qutrits $M_{10} \oplus M_{8} \oplus \mathbb{C}$
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Unordered quad of qubits  $M_{5} \oplus M_{3} \oplus \mathbb{C}$

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Proof, setting up

acts on $H = (C^d)^{\otimes n}$ in the obvious way. Also on $B(H)$ by $\pi(a) = \pi^{-1}a\pi$.

We wish to compute their equalizer $E = \{a; a \in B(H); \pi^{-1}a\pi = a \forall \pi \in S_n\}$.
Proof, setting up

\[ S_n \text{ acts on } H = (\mathbb{C}^d)^\otimes n \text{ in the obvious way.} \]
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$$E = \{a; a \in B(H); \pi^{-1}a\pi = a \ \forall \pi \in S_n\}$$
Proof, crucial observation

\[ E = \{ a; a \in B \left( H \right); \pi^{-1}a\pi = a \forall \pi \in S_n \} = \text{Rep} S_n \left( H, H \right) \]
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\[ E = \{a; a \in B(H); \quad \pi^{-1} a \pi = a \quad \forall \pi \in S_n\} \]
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The equalizer coincides with the representation endomorphisms of \( H \)!
Proof, basic representation theory

\[ \mathcal{H} \cong (C^d \otimes n) \sim = \bigoplus \lambda U_{\lambda} \]

where \( U_{\lambda} \) distinct irreducible representations.

Schur's lemma:
\[ \text{Rep}(U_{\lambda}, U_{\mu}) = \begin{cases} C_{\mu} = \lambda & \text{if } \mu = \lambda \\ 0 & \text{otherwise} \end{cases} \]
Proof, basic representation theory

\[ H = (\mathbb{C}^d)^\otimes n \cong \bigoplus_\lambda U^m_\lambda \]

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Proof, basic representation theory

\[ H = (\mathbb{C}^d)^n \cong \bigoplus_{\lambda} U^{m_{\lambda}}_{\lambda} \]

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Schur’s lemma:

\[
\text{Rep}(U_\lambda, U_\mu) = \begin{cases} 
\mathbb{C} & \mu = \lambda \\
0 & \mu \neq \lambda 
\end{cases}
\]
Proof, putting it together

\[ E = \text{Rep} S_n(H, H) \cong \bigoplus_{\lambda, \mu} \text{Rep} S_n(U^\lambda, U^\mu) \cong \bigoplus_{\lambda} M^\lambda m^\lambda \]

What are the irreducible representations \( U^\lambda \) and their multiplicities \( m^\lambda \)?

Answer is given by Schur-Weyl duality.
Proof, putting it together

\[ E = \text{Rep}_{S_n}(H, H) \]
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1. Unordered tuples
   ▶ Sketch of proof
2. Cycles
3. Unordered words
3-cycle of bits is a 4dit:

\{000, 001 = 010 = 100, 011 = 101 = 110, 111\}

What about a 3-cycle of qubits?

(= coequalizer of obvious action of $C_3$ on $B(C_2 \oplus C_2 \oplus C_2)$.)
3-cycle

A 3-cycle of bits is a 4dit:
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\{000,
3-cycle

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\(\text{coequalizer of obvious action of } C_3 \text{ on } B (C_2 \oplus C_2 \oplus C_2).\)
A 3-cycle of bits is a 4dit:

\[ \{000, 001 = 010 = 100, 011 = 101 = 110, \} \]
A 3-cycle of bits is a 4dit:

\{000, 001 = 010 = 100, 011 = 101 = 110, 111\}
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What about a 3-cycle of qubits?
A 3-cycle of bits is a 4dit:
{000, 001 = 010 = 100, 011 = 101 = 110, 111}

What about a 3-cycle of qubits?

(= coequalizer of obvious action of $C_3$ on $B(\mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2)$.)
Quantum 3-cycle
Quantum 3-cycle

\[ M_4 \oplus M_2 \oplus M_2 \]
Quantum 3-cycle

\[ M_4 \oplus M_2 \oplus M_2 \]

\[ |001\rangle + |010\rangle + |100\rangle \]
Quantum 3-cycle

\[ M_4 \oplus M_2 \oplus M_2 \]

given by

\[ |001\rangle + |010\rangle + |100\rangle \]

and

\[ |001\rangle + e^{\frac{2\pi i}{3}} |010\rangle e^{\frac{4\pi i}{3}} |100\rangle \]
Quantum 3-cycle

$$M_4 \oplus M_2 \oplus M_2$$

- $|001\rangle + |010\rangle + |100\rangle$
- $|001\rangle + e^{\frac{2\pi i}{3}} |010\rangle e^{\frac{4\pi i}{3}} |100\rangle$
- $|001\rangle + e^{\frac{4\pi i}{3}} |010\rangle e^{\frac{2\pi i}{3}} |100\rangle$
Arbitrary cycles
Schur-Weyl does not apply.
Arbitrary cycles

Schur-Weyl does not apply.
How to compute multiplicities?
Arbitrary cycles

Schur-Weyl does not apply.
How to compute multiplicities?
By computing the character table.
Result 2: arbitrary cycles
Result 2: arbitrary cycles

\[ m_k = \sum_{0 \leq j < n} e^{\frac{2\pi ijk}{n}} d^{\gcd(j,n)} \]
Result 2: arbitrary cycles

\[ m_k = \sum_{0 \leq j < n} e^{\frac{2\pi ijk}{n}} d^{\gcd(j,n)} \]

With some number theory:
Result 2: arbitrary cycles

\[ m_k = \sum_{0 \leq j < n} e^{\frac{2\pi i j k}{n}} d^{\gcd(j, n)} \]

With some number theory:

\[ m_k = \frac{1}{n} \sum_{\ell \mid n} d_{\ell}^n \mu\left(\frac{\ell}{\gcd(\ell, k)}\right) \frac{\phi(\ell)}{\phi\left(\frac{\ell}{\gcd(\ell, k)}\right)}. \]
1. Unordered tuples
   - Sketch of proof

2. Cycles

3. Unordered words
Result 3: quantum unordered words
Result 3: quantum unordered words

\[ \prod_n S_n \text{ acts on } B(\bigoplus_n (\mathbb{C}^d) \otimes n). \]
Result 3: quantum unordered words

\[ \prod_n S_n \text{ acts on } B(\bigoplus_n (\mathbb{C}^d)^{\otimes n}). \]

With care we can compute the coequalizer:
Result 3: quantum unordered words

\[ \prod_n S_n \text{ acts on } B(\bigoplus_n (\mathbb{C}^d)^\otimes n). \]

With care we can compute the coequalizer:

\[ B(\ell^2) \bigoplus \prod_{\lambda \in Y^*} M_{m\lambda}. \]

\( Y^* \): Young diagrams of height at least 2.
Recap

1. Algebras for unordered types are given by coequalizers.
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1. Algebras for unordered types are given by coequalizers.
2. They are more interesting than expected.
Recap

1. Algebras for unordered types are given by coequalizers.
2. They are more interesting than expected.
3. Representation theory of finite groups is a perfect fit to study them.
Thanks!
Thanks!

Questions?