Towards a Paraconsistent Quantum Set Theory

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Introduction

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Topos Quantum Theory (TQT)

(Contravariant) TQT initiated by Isham and Butterfield in late 1990's. Aims to provide realist reformulation of quantum theory, replacing quantum logic with intuitionistic logic. Lattice of physical propositions in TQT forms a Heyting algebra. Uses the internal logic/language of presheaf toposes.

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Quantum Set Theory (QST)

Dates back to Takeuti (1978), who discovered models of set theory in which the set of all Dedekind reals is isomorphic to a given set of self adjoint operators. Attempts to represent physical information about quantum systems using the internal language/logic of these models.

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Our basic aim will be to make some first steps towards unifying QST and TQT. In order to do this, we'll need to study a new form of paraconsistent logic that arises quite naturally in TQT and allows for the replication of important results from QST in the context of TQT.

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- 1: TQT and a New Paraconsistent Logic

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- and takes an inclusion arrow $i : B \subseteq B'$ to the restriction mapping $|_{B',B} : B' \to B$ between $\Omega(B)'$ and $\Omega(B)$, i.e. $|_{B',B}(\lambda) = \lambda|_B$, for any $\lambda \in \Omega(B')$

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Daseinisation and Clopen Subobjects

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- By Stone duality, $\underline{\delta(a)}_B$ is a clopen subset of $\underline{\Omega(L)}_B$, for any $B \in B(L)$. So we say that $\underline{\delta(a)}_B$ is a 'clopen subobject' of $\underline{\Omega(L)}$.

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- It is easily shown that the lattice $Sub_{cl}(\underline{\Omega})$ of clopen subobjects of the spectral presheaf is a complete Heyting algebra under context-wise union and intersection (taking interiors and closures). So we can think of $\underline{\delta}$ as an injection of L into $Sub_{cl}(\underline{\Omega})$.

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- We can define an equivalence relation on $Sub_{cl}(\underline{\Omega})$ by $\underline{S} \sim \underline{T} \leftrightarrow \varepsilon(\underline{S}) = \varepsilon(\underline{T})$. Let E be the quotient class of $Sub_{cl}(\underline{\Omega})$ under $\sim$. E can be turned into a complete lattice by defining $\bigwedge_{i\in I}[\underline{S}_i] = [\bigwedge_{i\in I} \underline{S}_i]$, $[\underline{S}] \leq [\underline{T}] \leftrightarrow [\underline{S}] \wedge [\underline{T}] = [\underline{S}]$ and $\bigvee_{i\in I}[\underline{S}_i] = \bigwedge\{[\underline{T}] \| [\underline{S}_i] \leq [\underline{T}]\ \forall i \in I\}$

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- Cannon and Döring showed that E and L are isomorphic as complete lattices.

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- However, by using ε, we can define another logical structure on $Sub_{cl}(\underline{\Omega})$. Specifically, given $\underline{S} \in Sub_{cl}(\underline{\Omega})$, define $\underline{S}^* = \underline{\delta(\varepsilon(S)^{\perp})}$, i.e. $\underline{S}^*$ is the daseinisation of the orthocomplement of $\varepsilon(\underline{S})$

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- Using this negation, we can extend the isomorphism between E and L, so that they are now isomorphic *as complete ortholattices*.
- We have that $\underline{S} \wedge \underline{S}^* = \underline{S} \wedge \underline{\delta(\varepsilon(S)^{\perp})} \geq \underline{\delta(\varepsilon(S))} \wedge \underline{\delta(\varepsilon(S)^{\perp})} \geq \underline{\delta(\varepsilon(S) \wedge \varepsilon(S)^{\perp})} = \underline{\delta(0)} = \perp$, i.e. $*$ is a paraconsistent negation.

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Properties of *

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(i) $\underline{S} \vee \underline{S^*} = \top$
(ii) $\underline{S}^{**} \leq \underline{S}$
(iii) $\underline{S}^{***} = \underline{S}^*$
(iv) $\underline{S} \wedge \underline{S}^* \geq \bot$
(v) $(\underline{S} \wedge \underline{T})^* = \underline{S}^* \vee \underline{T}^*$
(vi) $(\underline{S} \vee \underline{T})^* \leq \underline{S}^* \wedge \underline{T}^*$
(vii) $\varepsilon(\underline{S}) \vee \varepsilon(\underline{S}^*) = 1$
(viii) $\varepsilon(\underline{S}) \wedge \varepsilon(\underline{S}^*) = 0$
(ix) $\underline{S} \leq \underline{T}$ implies $\underline{S}^* \geq \underline{T}^*$

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- These properties ensure that, equipped with the $*$ negation and an implication defined by $\underline{S} \Rightarrow \underline{T} = \underline{S}^* \Rightarrow \underline{T}$, $Sub_{cl}(\underline{\Omega})$ is a model of 'dialectical logic with quantifiers' (DKQ), a well known form of paraconsistent relevance logic.

QST - a Very Quick Overview

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The Model

- Fix a Hilbert space H and let $P(H)$ be the orthomodular lattice of projection operators on H. Then, for any Boolean subalgebra B of $P(H)$, we can make the following recursive definition,
  $V_\alpha^{(B)} = \{x : func(x) \wedge ran(x) \subseteq B \wedge \exists \xi < \alpha (dom(x) \subseteq V_\xi^{(B)})\}$,
  $V^{(B)} = \{x : \exists \alpha (x \in V_\alpha^{(B)})\}$

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- $V^{(B)}$ is what is known as a Boolean valued model of ZFC.

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- Subsequently, Ozawa and others have studied the structure $V^{(P(H))}$ (defined analogously to $V^{(B)}$, but with the whole of $P(H)$ playing the role of the truth value algebra, rather than just some Boolean subalgebra). This is not a model of full ZFC (due to the non-distributivity of $P(H)$), but it does model various fragments on the theory in quite a sophisticated way. This allows us to extend Takeuti's isomorphism so that the set $\mathbb{R}^{(P(H))}$ of all Dedekind reals in $V^{(P(H)}$ is in bijection with the full set $SA(H)$ of self adjoint operators on H.

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- Ozawa has shown how, by using this full correspondence between real numbers in $V^{(P(H))}$ and $SA(H)$, we can represent a lot of physical information about the quantum system whose state space is given by H inside of $V^{(P(H))}$.

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- Since $Sub_{cl}(\underline{\Omega})$ is a complete Heyting algebra, we can think of $V^{(Sub_{cl}(\underline{\Omega}))}$ as a Heyting valued model of intuitionistic set theory, and attempt to reconstruct Takeuti/Ozawa's bijection between Dedekind reals in this model and $SA(H)$.

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- One advantage of this approach is that the distributivity of $Sub_{cl}(\underline{\Omega})$ appears to solve a number of technical problems in QST. However, I have not been able to reconstruct anything like Takeuti/Ozawa's results using the Heyting or co-Heyting algebraic structure of $Sub_{cl}(\underline{\Omega})$.

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- But if we consider $Sub_{cl}(\underline{\Omega})$ as being equipped with $*$ and the corresponding implication connective, it is possible to reconstruct an approximation of these results (indeed, I only thought of defining $*$ for the purpose of solving this problem).

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- In recent years, the project of building a meaningful set theory over paraconsistent logics has attracted a lot of interest. We will be interested in one particular project, first developed by Weber (2012). Specifically, Weber showed that it is possible to develop a very rich and interesting set theory (PST) over the logic DKW. There are strong non-triviality proofs for the resulting theory and a great number of classical set theoretic ideas have been developed in PST (for example, ordinal and cardinal arithmtic, large fragments of real analysis etc).

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- We know that, when equipped with $*$, $Sub_{cl}(\underline{\Omega})$ is a model of DKQ. So it is natural to think of $V^{(Sub_{cl}(\underline{\Omega}))}$ as a model of PST. However, the model theory of this kind of theory is still being worked out (Lowe and Tarafder (2015) contains significant first steps in this repsect).

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- Theorem: For any $a, b \in \mathbb{R}$, $[a, b]^{(Sub_{cl}(\Omega))} \sim BSA(H)_{[a,b]}$
- This is a kind of 'bounded version' of Takeuti/Ozawa's results, and it allows us to build most of the usual machinery of QST inside of $V^{(Sub_{cl}(\Omega))}$.

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- Provides a way of representing operator inequalities in TQT. Given $X, Y \in SA(H)$, we can find canonical representations $\tilde{X}, \tilde{Y} \in V^{(Sub_{cl}(\Omega))}$ and we can show that $\|\tilde{X} \leq \tilde{Y}\| = \top$ holds if and only if X is spectrally smaller than Y.

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  Where $P_{\psi}^{X,Y}(x, y)$ represents the joint probability of obtaining the outcomes $Y = y$, $X = x$ from the successive projective measurements of X and Y (X measured after Y) when the system is prepared in the state $|\psi\rangle$

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- We can also define canonical maps that translate between $V^{(Sub_{cl}(\underline{\Omega}))}$ and $V^{(P(H))}$.

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