Some Graphical Aspects of Frobenius Structures

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The categorical flow of information in quantum physics and linguistics

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Theme: Why can we yank?

Let’s first see how it works...
Frobenius algebras (informal)

$k$ a commutative ring
$A$ fin. generated projective $k$-module
$m : A \times A \to A$ algebra s.t.
$A^* = \text{Hom}_k(A, k)$ dual module
with $A$-$A$ bimodule structure
$(afb)(x) = f(bxa)$
$\Rightarrow A A \cong A A^*, \quad A_A \cong A_A^*$

Example:
fin. group algebras $\mathbb{C}G$
Frobenius studied: $S_n$

**Img:** Ferdinand Georg Frobenius, 1849–1917
Finite Hopf algebras (informal)

$k$ a commutative ring
$H$ fin. generated projective module
$H$ is an algebra : product $m$
$H$ is a coalgebra: coproduct $\Delta$
$f, g, \text{Id} \in \text{Hom}_k(H, H)$ caries conv.
product : $f \star g := m; (f \otimes g); \Delta$
$\text{Id}_H$ is conv. invertible (antipode $S$)
compatibility axiom:

Example:
fine group algebras $\mathbb{C}G$

Img: Heinz Hopf, 1894–1971
Frobenius algebras (historical)

Let $A$ be finitely generated projective over $k \in \mathbf{cRing}$ i.e. $\exists \{x_i\}_{i=1}^n$ generators for $A$ (e.g. group algebra $\mathbb{C}S_n$)

regular representations

$x_i x_j = \sum_k f_{ij}^k x_k$ \quad $f_{ij}^k \in k$, mult. table $[f_{ij}^k]$

- $l_{x_i} \cong [f_i]^k_j = [f_{ij}^k]$ left reg. repr. $AA$ \quad $l_a \in \text{End}_k(AA)$
- $r_{x_j} \cong [f_j]^k_i = [f_{ij}^k]$ right reg. repr. $AA$ \quad $r_a \in \text{End}_k(AA)$
- $\sum_k [f_{ij}^k] a_k = [(P(a))_{ij}]$ parastrophic matrix \quad $(a_k \in k)$

**Thm. Frobenius:** If there exists $a_k \in k$ such that $[(P(a))_{ij}]$ is invertible then $AA \cong A_A$.

**Examples**

- $A = k[X, Y]/\langle X^2, Y^2 \rangle$ is Frobenius
- $A = k[X, Y]/\langle X^2, XY^2, Y^3 \rangle$ is not Frobenius
- $A = M_n(k)$, $k$ division ring, is Frobenius
Dualities: topological move

$X$ object in monoidal category $C$, rigid $\forall X$ if $\exists X^*, X^*$ such that:

- **right dual:**
  \[ev_X : X^* \times X \to 1_X\]
  \[cev_X : 1_X \to X^* \times X\]
  \[(1_X \times cev_X); (ev_X \times 1_X) = 1_X\]
  \[(ev_{X^*} \times 1_{X^*}); (1_{X^*} \times cev_{X^*}) = 1_{X^*}\]
  
  topological Reidemeister 0 move

- **left dual:**
  \[X ev : X \times X^* \to 1_X\]
  \[X cev : 1_X \to X \times X^*\]
  \[(X cev \times 1_X); (1_X \times X ev) = 1_X\]
  \[(1_{X^*} \times X ev); (X^* cev \times 1_{X^*}) = 1_{X^*}\]
  
  topological Reidemeister 0 move

- **symmetry (braiding):**
  \[\sigma_{X,Y} : X \times Y \to Y \times X\]
  \[(\sigma_{X,Y} \times 1); (1 \times \sigma_{X,Z}); (\sigma_{Y,Z} \times 1) =\]
  \[(1 \times \sigma_{Y,Z}); (\sigma_{Y,Z} \times 1); (1 \times \sigma_{X,Y})\]

  (this is not our yanking move...)
Graphical dualities: topological move, twist

$\xRightarrow{\text{ev}_X}$ $\xRightarrow{\text{cev}_X}$ $\xRightarrow{X\text{ev}}$ $\xRightarrow{X\text{cev}}$ if sym

$\sim$ $\sim$ $\sim$ $\sim$

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Bilinear forms

Regular associative bilinear forms $\text{Bil}^r_{\text{ass}}(A, k)$

- $\beta : A \times A \to k \in \text{Bil}^r_{\text{ass}}(A, k)$ if $\beta(ab, c) = \beta(a, bc)$ (≡ass.) and $\beta$ non-degenerate
- $\beta' \cong \beta$ (homothetic) if $\exists k \in k^\times, \exists V \in \text{Aut}_k(A)$ such that $\beta(a, b) = k \beta'(Va, Vb)$
- $\beta$ is symmetric if $\beta(a, b) = \beta(b, a), \forall a, b \in A$ (i.e. $A \neq A^{op}$)
- $\alpha \in \text{Aut}_{k-\text{alg}}(A)$ s.t. $\beta(a, b) = \beta(b, \alpha(a))$ Nakayama aut. unique up to inner aut., iff $\alpha = \text{Id} \iff \beta$ is symmetric
- $\beta(a, Vb) = \beta(V^t a, b)$ transposition: $(V^t)^t = \alpha V \alpha^{-1}$, i.e. not identity; $\alpha$ has finite order $n$ then $(\cdot)^{t^{2n}} = (\cdot)$
- $\lambda := \beta(1, -) = \beta(-, 1)$ is called Frobenius homomorphism
  If $\lambda(ab) = \lambda(ba)$ (≡ $\alpha = \text{Id}$) ‘trace form’
Bilinear forms cont.
Duality from bilinear forms in $\text{Bil}^r_{\text{ass}}(A, k)$

$[A \text{ unital algebra, fin. generated projective; generators } \{x_i\}; \beta \in \text{Bil}^r_{\text{ass}}(A, k) ]$

$\rhd \ r : \text{Bil}^r_{\text{ass}}(A, k) \overset{\sim}{\longrightarrow} \text{Hom}_k(A, A^*) :: \beta \mapsto r_\beta, \ r_\beta(a) = \beta(a, -)$

$\rhd \ l : \text{Bil}^r_{\text{ass}}(A, k) \overset{\sim}{\longrightarrow} \text{Hom}_k(A, A^*) :: \beta \mapsto l_\beta, \ l_\beta(a) = \beta(-, a)$

$\text{End}(A) \overset{\sim}{\rightarrow} A^e = A \otimes A^{op} \overset{\sim}{\rightarrow} A \otimes A^* \overset{\sim}{\rightarrow} A \otimes A$

$V \cong \sum_i x_i \otimes b_i^{op} \cong \sum_i x_i \otimes f_i \cong \sum_i x_i \otimes y_i$

$(\cdot)^{op}$ maps left to right modules (actions)

$f_i \in \text{Hom}_k(A, k)$ dual elements (indep. of choice)

$y_i \in A$ acts via $\beta$ (and $r_\beta, l_\beta$) (indep. of choice)

**Frobenius system:**

A Frobenius system for $A$ is a triple $(\beta, x_i, y_i)$ such that $\forall a \in A$:

$$\sum_i x_i \beta(y_i, a) = a = \sum_i \beta(a, x_i)y_i$$

this is the ‘yanking move’!... but wait a moment...
Separability and Frobenius

\[ k \in \mathbf{cRing}; \ A \text{ a } k\text{-algebra, } _A M_A \text{ an }(A, A)\text{-bimodule, i.e. an } A^e \text{ left module } \]

- \( D : A \to M \) s.t. \( D(ab) = D(a)b + aD(b) \) derivation
  \( \text{Der}_k(A, M) \) \( k \)-module of derivations
  \( D_m : A \to M : D_m(a) = am - ma \) for all \( m \in M \) inner der.

- \( D_m = 0 \) iff \( m \in M^A := \{ m \in M \mid am = ma, \ \forall a \in A \} \)

\[
0 \to M^A \to M \to \text{Der}_k(A, M) \quad \text{exact, also}
\]

\[
M^A \cong \text{Hom}_{A^e}(A, M), \ M \cong \text{Hom}_{A^e}(A^e, M), \ m_A : A^e \to A \text{ epi}
\]

\[
0 \to I(A) = \text{Ker}(m_A) \to A \otimes A^{op} \to A \to 0 \quad \text{exact}
\]

- \( \delta : A \to I(A) : a \mapsto \delta(a) = a \otimes 1 - 1 \otimes a \)
  \( A\delta(A) = I(a) = \delta(A)A \) is an ideal

Lemma

\[ \text{Hom}_{A^e}(I(A), M) \cong \text{Der}_k(A, M) \]
Separability and Frobenius, cont.

Apply $\text{Hom}_{A^e}(-, A)$ to exact seq., recall
$H^1(A, M) = \text{Ext}^1_{A^e}(A, M) \cong \text{Der}_k(A, M)/\text{InnDer}_k(A, M)$

1st. Hochschild cohomology grp.

**Thm:**
For $k$-algebras $A$ is equivalent:

- $A$ is projective as left $A^e$-module
- $0 \to I(A) \to A \otimes A^{op} \to A \to 0$ for $A^e$-modules is split
- $\exists e = \sum e_1 \otimes e_2 \in A \otimes A$ s.t. $\forall a \in A : ae = ea$ and $\sum e_1 e_2 = 1$ splitting idempotent

**Thm:**
Any projective separable $A$ over $k \in \texttt{cRing}$ is finitely generated.

**Thm:**
A separable algebra $A$ over a field is semisimple.
Frobenius algebra: characterisation

[recall: A fin. dim $k$-algebra is Frobenius if $A_A \cong A_A^*$ as right $A$-modules ]

**Thm:**
For an $n$-dim. algebra $A$, the following are equivalent:

- $A$ is Frobenius
- the representations $r, l : A \rightarrow M_n(k)$ are equivalent
- $\exists a \in k^n$ s.t. the parastrophic matrix $P_a$ is invertible
- $\exists \beta \in \text{Bil}^r_{\text{ass}}$ and hence a Frobenius homomorphims $\lambda$
- $\exists$ hyperplane of $A$ that does not contain any nonzero right ideals of $A$
- $\exists$ a Frobenius system $(\lambda, x_i, y_i)$, $\lambda \in A^*$, $(\lambda = \beta; m_A)$
  
  $e = \sum e_1 \otimes e_2 = \sum_i x_i \otimes y_i \in A \otimes A$ s.t.
  
  $ae = ea$, $(e \subset (A^e)^A)$  $\sum \lambda(e_1)e_2 = 1 = \sum e_1 \lambda(e_2)$
- and many more...
Frobenius extensions (needed for Jones idempotents and polynom)

- ring extension \( A/S \), homomorphism \( S \mapsto A \), \( Z(A) \) center
- algebra if: \( S \in \text{cRing} \) and \( i \) factors as \( S \to Z(A) \to A \)
- \( A/S \) central if \( i(S) = Z(A) \), proper if \( i \) is 1-1

Let \( \mathcal{M}_S \) and \( \mathcal{M}_A \) be the categories of right \( S \) resp \( A \) modules, \( R : \mathcal{M}_A \to \mathcal{M}_S \) restriction functor; Define functors:

- adjoint: \( T : \mathcal{M}_S \to \mathcal{M}_A : M_S \mapsto M_S \otimes_S A, f \mapsto f \otimes \text{Id}_A \)
- coadjoint: \( H : \mathcal{M}_S \to \mathcal{M}_A : M_S \mapsto \text{Hom}_S(A_S, M_S), ms \mapsto (as \mapsto mf(a)s) \)

\( (T, R) \) and \( (R, H) \) are adjoint pairs of functors

**Def:** A ring extension \( A/S \) is a Frobenius extension iff \( H, T \) are naturally adjoint functors from \( \mathcal{M}_S \to \mathcal{M}_A \). equival. to:

1) \( S A_A \cong (A A_S)^* \) and \( A_S \) fin. proj.
2) \( A A_S \cong ^*(S A_A) \) and \( S A \) fin. proj.
3) \( \exists \lambda \in \text{Hom}_{S \to S}(A, S), x_i, y_i \in A \) s.t. \( \forall a \in A \)
   \[ \sum x_i \lambda(y_i a) = a = \sum \lambda(ax_i)y_i \]
**λ-multiplication**

End\( (A_S) \cong A_S \otimes_S S^*A \) implies the multiplication:

\[
a_A; b_A = \sum a_i \otimes f_i; \sum b_j \otimes g_j = \sum a_i f_i(b_j) \otimes g_j
\]

**Thm:**

If \( A/S \) is a Frobenius extension with system \((\lambda, x_i, y_i)\), then \( A \otimes_S A \cong \text{End}(A_S) \) as rings, with \( \lambda \)-multiplication on \( A \otimes A \)

\[
(a \otimes b); (c \otimes d) := a \lambda(bc) \otimes d = a \otimes \lambda(bc)d
\]

**Cor:**

If \((\lambda, x_i, y_i)\) is a Frobenius system for \( A/S \), then

\[
e = \sum x_i \otimes y_i \in (A \otimes_S A)^A
\]

**Thm:**

Let \((\lambda, x_i, y_i)\) be a Frobenius system for \( A/S \), all other such systems are in 1-1 correspondence up to equivalence, for \( d \in \text{Cent}_A(S) \) invertible, by \((\lambda d, x_i, d^{-1}y_i)\).
Frobenius multiplication & ‘yanking’

We are now in the position to produce the ‘yanking move’:

- l.h.s: \( m : A \otimes A \to A \) in two versions, using dality via the (left) regular representation \( l(a) \in A \otimes A^* \), and the \( \lambda \)-multiplication from the Frobenius homomorphism \( \lambda(-) = \beta(1, -) = \beta(-, 1) \)

- r.h.s: duality expressed via Frobenius system
  [This is the archetypical move for ‘teleportation’]
Frobenius and Hopf

Let $k$ be a ring with trivial Picard group $\text{Pic}[k] = 0$ (e.g. field) $H$ fin. generated projective

- augmentation: $\epsilon : H \to k$ is a homomorphism
  $\epsilon(ab) = \epsilon(a)\epsilon(b)$

- right integral: $\int_H^r \ni 0 \neq \mu_r : H \to k \text{ s.t. } \forall a \in H : \mu_r a = \epsilon(a)\mu_r$
  $\int_H^r$ is an ideal in $H : H\int_H^r = \int_H^r H, \int_H^r \cong k$

- right norm: $n \in H \text{ s.t. for } \lambda \text{ Frob. hom. and } \lambda n = \epsilon, n \in \int_H^r$
  $\lambda(nax) = (\lambda n)(ax) = \epsilon(ax) = \epsilon(a)\epsilon(x) = \lambda(n\epsilon(a)x)$

[careful: e.g. Clifford algebras don’t have naively such structures... ]
**Frobenius and Hopf, cont**

**Thm: Larson-Sweedler-Pareigis**

$H$ is a fin. proj. Hopf algebra over $\mathbb{k}$, Pic[$\mathbb{k}$] = 0 then:

- there is a right Hopf module structure on $H^*$
- there exists a left integral $\mu_l \in \int^l_H$ such that $\Theta : H \to H^*$ defined by
  $$\Theta(x)(y) = \mu_l(y S(x))$$

is a right Hopf module isomorphism

- The antipode $S$ is bijective ($\exists S^{-1}$)
- $H$ is a Frobenius algebra with Frobenius homomorphism $\mu_l$

If $\exists e = \sum x_i \otimes y_i$ separable: $\sum a x_i \otimes y_i = \sum x_i \otimes y_i a$, we obtain:

\[ \sum x_i \otimes y_i \sim \sum x_i \otimes y_i \]

Frobenius coproduct
Frobenius and Hopf: commonalities and differences

Kupperberg ladder, invertible
Hopf loop sing., isospectral
Frobenius loop sing., isospectral

Bialgebra property
If special

\[ \sum x_i y_i = 1 \]  
reg. Trace
q-Bit: Clifford view

\[ V = \mathbb{R}^3, \; \beta = \delta, \text{ generators } \sigma_1, \sigma_2, \sigma_2, \text{ real algebra } C\ell_{3,0}(V, \delta) \]

[actually a comodule algebra, Grassman alg. deformation, ask Majid]

idempotents: \( f_0 = \frac{1}{2}(1 + \sigma_3), \; f_1 = \frac{1}{2}(1 - \sigma_3), \; f_0 + f_1 = 1, \text{ primitive} \)

▷ regular rep. \( I_a \) 8-dim \( \subset M_8(\mathbb{R}) \)

▷ center \( C_\mathbb{R} = \mathbb{R} \oplus \mathbb{R} \), as real algebra \( \langle 1, \sigma_1\sigma_2\sigma_3 \equiv \sigma_{123} = i \rangle \)

▷ spinor rep. by left ideals: \( C\ell_{3,0} f_i \cong C\ell_{3,0} S^i_C \cong \langle f_i, \sigma_1 f_i \rangle \cong \langle |0\rangle^i, |1\rangle^i \rangle \)

▷ \( \mathbb{R} \rightarrow \mathbb{C} \rightarrow C\ell_{3,0} \) factors through center, \( f_i \) splitting idemp.

▷ matrix rep: \( [\sigma_1]^i = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^i, [\sigma_2]^i = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}^i, [\sigma_3]^i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^i \)

▷ trace: \( \lambda(xy) = \beta(x, y) = \sum_i \langle i | xy | i \rangle \in \mathbb{C} \)

▷ \( e = \sum_i x_i \otimes y_i = |i\rangle \otimes |i\rangle \)

▷ Frobenius product: \( m(|i\rangle \otimes |j\rangle) = \delta_{i,j} |j\rangle \)

▷ Frobenius coproduct: \( \delta(|i\rangle) = |i\rangle \otimes |i\rangle \)

... as you knew before
Teleportation again

$|\psi\rangle$

[Diagram of a person and a quantum state $|\psi\rangle$.]
What you should know, but weren’t told.

- Hattori-Stallings ranks, Higman trace, module of trace forms
- Azumaya algebras, quadratic algebras, arithmetic Witt groups
- Cartan map, Casimir elements
- relation $K_0(A) \to G_0(A)$, Grothendieck groups
- relation to character theory, regular characters, irreducible characters
- relation to non-linear (solvable) differential equations
- Frobenius manifolds, Chazy equation, Egoroff metric
- topological quantum field theory, Gromov-Witten invariants
- character theory of filtered/graded algebras (of polynomial type)
- quantum cohomology (Vershik, Olshanski, Okounkov et al.)
- symmetric functions and Grothendieck $\lambda$-rings, Macdonald polynomials
- spherical categories and generalized hyper geometric spherical functions
- Frobenius functors, separable functors and entwined modules: (Doi-Koppinen, Yetter-Drinfeld, Hopf, and Long modules...)
Literature: where it was taken, where to go..., a partial & subjective view

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