Complementarity in categorical quantum mechanics

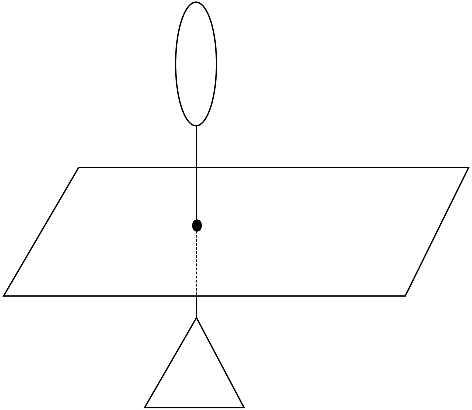
Chris Heunen

May 29, 2010

Complementarity

- Bohr: knowledge of a quantum system can only be attained through examining classical subsystems
- Bohr: *two* incompatible classical subsystems can be 'complementary'
- we will consider *all* classical subsystems ('complete complementarity')
- slogan: complete knowledge of a quantum system can only be attained through examining all of its classical subsystems

Three levels



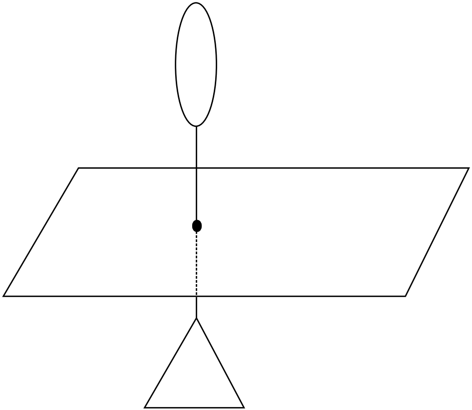
von Neumann algebra
of operators on H

Hilbert space H

orthomodular lattice
of closed subspaces of H

Three levels

complete complementarity means considering (interaction of) all



commutative von Neumann algebras of operators on H

Hilbert spaces H
with a chosen basis

Boolean lattices
of closed subspaces of H

Three levels, categorically

Recently studied:

- commutative von Neumann subalgebras form interesting topos
- basis of a Hilbert space = H*-algebra in **Hilb**
- closed subspaces = dagger kernels in **Hilb**

Also:

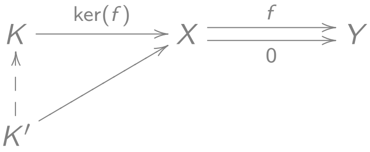
- any von Neumann algebra is a colimit of its commutative subalgebras
- any orthomodular lattice is a colimit of its Boolean sublattices

Dagger kernel categories

A *dagger kernel category* is a category $\mathbf{D}$ with:

- a dagger $\dagger \colon \mathbf{D}^{\mathrm{op}} \to \mathbf{D}$;
  ($X^\dagger = X$ and $f^{\dagger\dagger} = f$)
- a zero object $0 \in \mathbf{D}$;
  ($\mathbf{D}(0, X) = \{0\}$)
- kernels $\ker(f)$ which are dagger monic

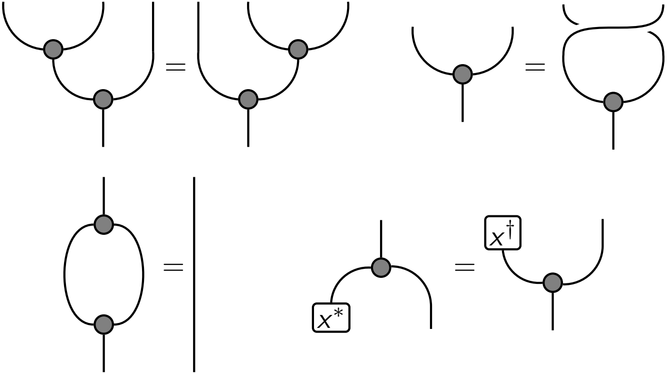
  (



  with $\ker(f)^\dagger \circ \ker(f) = \mathrm{id}$)

Classical structures

A *classical structure* in a dagger symmetric monoidal category **D** is a commutative semigroup $\delta\colon X \to X \otimes X$ that satisfies $\delta^\dagger \circ \delta = \mathrm{id}$ and the *H*-axiom*: there is an involution $*\colon \mathbf{D}(I,X)^{\mathrm{op}} \to \mathbf{D}(I,X)$ such that $\delta^\dagger \circ (x^* \otimes \mathrm{id}) = (x^\dagger \otimes \mathrm{id}) \circ \delta$.



Kernels and tensor products

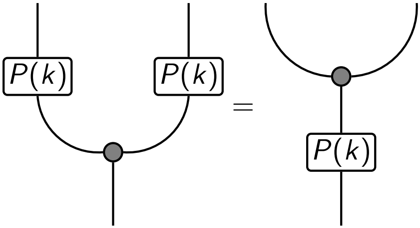
Consider categories **D** that is simultaneously a dagger kernel category and a dagger symmetric monoidal category. Additionally:

$$\ker(f) \otimes \ker(g) = \ker(f \otimes \mathrm{id}) \wedge \ker(\mathrm{id} \otimes g)$$

- e.g. **Hilb** and **Rel** satisfy this property
  ($\ker(f) \otimes \ker(g) = \{x \otimes y \mid f(x) = 0 \text{ and } g(y) = 0\}$)
- $\ker(f \otimes g) = \ker(f) \otimes \ker(g)$ is too strong
  (take $g = 0$: any morphism is zero)
- it does follow that $\ker(f \otimes f) = \ker(f) \otimes \ker(f)$

Copyability

A morphism $k\colon K \to X$ is *copyable* (along a classical structure δ) when $\delta \circ P(k) = P(k \otimes k) \circ \delta$, where $P(k) = k \circ k^\dagger$.



- point-free: works for any $k\colon K \to X$, not just points $I \to X$

Copyability, examples

- in any **D**: $0, \mathrm{id}$ are copyable
- in **Hilb**:
  - classical structure is orthonormal basis
  - kernel is closed subspace
  - kernel is copyable iff it is closed linear span of subset basis
- in **Rel**:
  - classical structure is (disjoint union of) Abelian group(s)
  - kernel is subset
  - kernel is copyable iff it is 0 or id

Copyability, examples

- in any **D**: $0, \mathrm{id}$ are copyable
- in **Hilb**:
  - classical structure is orthonormal basis
  - kernel is closed subspace
  - kernel is copyable iff it is closed linear span of subset basis
- in **Rel**:
  - classical structure is (disjoint union of) Abelian group(s)
  - kernel is subset
  - kernel is copyable iff it is 0 or id

  but definition of copyability works for any projection
  - projection is partial equivalence relation $\sim$,
  - and is copyable iff it is a 'groupoid congruence':

$$xy \sim z \Longleftrightarrow \exists_{x',y'}[x \sim x', y \sim y', x'y' = z]$$

Copyability and morphisms of classical structures

- **Lemma** A dagger monic k is copyable if and only if there is a (unique) morphism δ_k making the following diagram commute:

$$\begin{array}{ccccc}
X & \xrightarrow{k^\dagger} & K & \xrightarrow{k} & X \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta_k} & & \downarrow{\scriptstyle \delta} \\
X \otimes X & \xrightarrow[k^\dagger \otimes k^\dagger]{} & K \otimes K & \xrightarrow[k \otimes k]{} & X \otimes X.
\end{array}$$

Copyability and morphisms of classical structures

- **Lemma** A dagger monic k is copyable if and only if there is a (unique) morphism δ_k making the following diagram commute:

$$\begin{array}{ccccc} X & \xrightarrow{k^\dagger} & K & \xrightarrow{k} & X \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta_k} & & \downarrow{\scriptstyle\delta} \\ X \otimes X & \xrightarrow[k^\dagger \otimes k^\dagger]{} & K \otimes K & \xrightarrow[k \otimes k]{} & X \otimes X. \end{array}$$

- **Lemma** If k is a copyable dagger monic, δ_k is a classical structure.
- **Corollary** A dagger monic k is copyable if and only if its domain carries a classical structure δ_k and k is simultaneously a (non-unital) monoid homomorphism and a (non-unital) comonoid homomorphism.

Complementarity and mutual unbiasedness

- a morphism $x\colon U \to X$ is *unbiased* (for δ) when $P(x^\dagger \circ k) = P(x^\dagger \circ l)$ for all copyable kernels k and l
- two classical structure are *mutually unbiased* if a nontrivial kernel is unbiased for one whenever it is copyable along the other
- two classical structures are *partially complementary* if no nontrivial kernel is simultaneously copyable along both
- mutual unbiasedness $\substack{\Longrightarrow \\ \not\Longleftarrow}$ partial complementarity

Boolean subalgebras

Recall that kernels $K \to X$ form an orthomodular lattice.

Theorem Copyable kernels $K \to X$ form a *Boolean* lattice.

- $k \wedge l$ is copyable when k and l are
- $k^{\perp}$ is copyable when k is
- copyable kernels are distributive

Boolean subalgebras

Recall that kernels $K \to X$ form an orthomodular lattice.

Theorem Copyable kernels $K \to X$ form a *Boolean* lattice.

- $k \wedge l$ is copyable when k and l are
- $k^\perp$ is copyable when k is
- copyable kernels are distributive

Only possible if copyability ignores (co)units:

$$\varepsilon = \varepsilon \circ P(k^\perp) = \varepsilon \circ P(k) \circ P(k^\perp) = \varepsilon \circ P(k \wedge k^\perp) = 0$$

Boolean subalgebras, categorically

- taking kernels is a functor to a dagger category of orthomodular lattices
- taking classical structures gives an idempotent comonad **HStar** on the category of dagger monoidal kernel categories
- could formulate result as

$$\begin{array}{ccc}
\mathbf{HStar}[\mathbf{D}] & \xrightarrow{\mathrm{KSub}} & \mathbf{BoolLatGal} \\
\downarrow & & \downarrow \\
\mathbf{D} & \xrightarrow[\mathrm{KSub}]{} & \mathbf{OMLatGal}
\end{array}$$

Von Neumann algebras

Lemma Commutative subalgebras C of $A = \mathbf{Hilb}(H, H)$ correspond to Boolean sublattices of $\mathrm{KSub}(H)$

- $\mathrm{Proj}(A) = \{p \in A \mid p^\dagger = p = p^2\}$ is a complete, atomic, atomistic, orthomodular lattice
- there is an order isomorphism $\mathrm{Proj}(A) \cong \mathrm{KSub}(H)$
- von Neumann algebras are generated by projections, so $C = \mathrm{Proj}(C)''$.
- since C subalgebra of A, also $\mathrm{Proj}(C)$ sublattice of $\mathrm{Proj}(A)$
- because C commutative, $\mathrm{Proj}(C)$ is a Boolean lattice

Von Neumann algebras

Theorem Denote by $\mathcal{C}(A)$ the collection of commutative subalgebras of $A = \mathbf{Hilb}(H,H)$. Then:

$$\mathcal{C}(A) \cong \{L \subseteq \mathrm{KSub}(H) \mid L \text{ orthocomplemented sublattice}, \\ \exists_{\delta\colon H \to H \otimes H} \forall_{l \in L} [l \text{ copyable along } \delta]\}.$$

Von Neumann algebras

Theorem Denote by $\mathcal{C}(A)$ the collection of commutative subalgebras of $A = \mathbf{Hilb}(H, H)$. Then:

$$\mathcal{C}(A) \cong \{L \subseteq \mathrm{KSub}(H) \mid L \text{ orthocomplemented sublattice}, \\ \exists_{\delta\colon H \to H \otimes H} \forall_{l \in L} [l \text{ copyable along } \delta]\}.$$

If H is finite-dimensional, this can be completely characterized in terms of classical structures:

$$\mathcal{C}(A) \cong \{(\delta_i)_{i \in I} \mid \delta_i, \delta_j \text{ partially complementary classical structures}, \\ \exists_\delta \forall_i \exists_{k_i\colon \delta_i \to \delta} [k_i \text{ morphism of classical structures}]\}.$$

Hence $\mathcal{C}(A)$ is isomorphic to the collection of cocones in the category of classical structures on H that are pairwise partially complementary.

Concluding remarks

Tentative definition: A collection of classical structures is *completely complementary* when its members are pairwise partially complementary and jointly epic.

- Morphisms in $\mathcal{C}(A)$: direction, beyond poset?
- Logic on dagger monoidal (kernel) categories $\mathbf{D}$ such as $\mathbf{Hilb}$:
  - transfer from orthomodular lattices
  - transfer from topos of functors $\mathcal{C}(A)$ or its characterization in $\mathbf{D}$
- Tensor products and $\mathcal{C}(A)$!
- Interaction with compactness?
- Fibration of classical structures over $\mathbf{D}$?