Doing dependent types wrong without going wrong

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What are dependent types?

Types that depend on elements of other types.

- **Examples:**
  - vec n – type of lists of length in `
  - Generalized tries
  - PADS
  - Type of ASTs that represent well-typed code

- **Statically enforce expressive program properties**
  - BST ops preserve BST invariants
  - CompCert compiler
## Two sorts

<table>
<thead>
<tr>
<th><strong>Full Spectrum</strong></th>
<th><strong>Phase-sensitive</strong></th>
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<tbody>
<tr>
<td>Types indexed by actual computations</td>
<td>Types indexed by a pure language, separate from computations</td>
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<tr>
<td>Easier to connect type system to actual computation, harder to extend computation language</td>
<td>Index language may have minimal similarity to computation language</td>
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<td>Includes &quot;strong eliminators&quot; if x=3 then Bool else Int</td>
<td>May or may not not include strong eliminators</td>
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<tr>
<td>Examples: Cayenne, Coq, Epigram, Agda2, Guru</td>
<td>Examples: DML, ATS, (\mathcal{O}) mega, Haskell</td>
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Let’s do it wrong...

- Cayenne is *only* language that deliberately allows nonterminating terms in types
  - Nothing proved about it!
- Primary Goal: prove *type soundness* for a language with impure computations in types.
  - Note: type checking may be undecidable
- Secondary Goals:
  - CBV language
  - "Modular" metatheory
Full spectrum: Pure type system

- No distinction between types and terms

\[ s, t, A, B, k ::= x \mid \lambda x. t \mid s \ t \mid (x:A) \to B \mid T \]
\[ \mid * \mid [] \mid c \mid \text{case } s \{ \ c \ x \Rightarrow t \ \} \]

- One set of formation rules

\[ \Gamma \vdash t : A \]

- Conversion rule uses type equivalence

\[ \Gamma \vdash t : A \quad \Gamma \vdash B : s \quad A \sim B \]
\[ \Gamma \vdash t : B \]

- Term equivalence is fixed by type system (and defined to be the same as type equivalence).
New vision

- Syntactic distinction between terms and types, but still full spectrum
  \[ k ::= \ast \mid (x:A) \to k \]
  \[ A ::= (x:A1) \to A2 \mid T \mid A \wedge \mid \text{let } x = t \text{ in } A \]
  \[ \mid \text{case } t \text{ of } \{ c \ x \Rightarrow A \} \]
  \[ t ::= x \mid \lambda x. t \mid t \wedge \mid \text{let } x = t \text{ in } t \]
  \[ \mid c \wedge \mid \text{case } t \text{ of } \{ c \ x \Rightarrow t \} \]
  \[ \mid \text{fix } f(x). t \]
  \[ w ::= x \mid \lambda x. t \mid \text{fix } f(x). t \mid c \wedge \]

- Key changes:
  - Term language explicitly includes non-termination
  - CBV – only pure terms (w) substituted for variables
  - Type system parameterized by term equality
Parameterized term equality

- Given a list of equality assumptions about terms:
  - $\Delta ::= \cdot \mid \Delta, t_1 = t_2$

- Assume the existence of two functions:
  - $\text{con}(\Delta)$ in $\{\text{maybe, false}\}$
  - $\text{isEq}(\Delta, t_1, t_2)$ in $\{\text{true, maybe}\}$

- Equality is untyped
  - No guarantee that $t_1$ and $t_2$ have the same type
  - No assumptions about the types of the free variables
  - Types don’t require terms appearing in them to be well-typed
Type equivalence (excerpt)

\[
\text{con } (\Delta) = \text{false}
\]
\[
\Delta \vdash A_1 = A_2
\]

\[
\Delta \vdash A_1 = A_2 \quad \text{isEq } (\Delta, w_1 w_2) = \text{true}
\]
\[
\Delta \vdash A_1 w_1 = A_2 w_2
\]

\[
\text{isEq } (\Delta, t, c_i w_i) = \text{true}
\]
\[
\Delta \vdash \text{case } t \text{ of } \{ c_i x_i \Rightarrow A_i \} = A_i \{ w_i / x_i \}
\]

\[
\Delta, x = t \vdash A = B \quad x \not\in \Delta, B
\]
\[
\Delta \vdash \text{let } x = t \text{ in } A = B
\]
Typing rules (excerpt)

\[ \Gamma \Delta \vdash t : (x:A) \rightarrow B \quad \Gamma \Delta \vdash w : A \]
\[ \Gamma \Delta \vdash tw : B \{ w / x \} \]

\[ \Gamma \Delta \vdash t1 : A \quad \Gamma, x:A \quad \Delta, x=t1 \vdash t2 : B \]
\[ \Gamma \Delta \vdash \text{let } x = t1 \text{ in } t2 : B \]

\[ \Gamma \Delta \vdash t : T \quad t' \quad \Delta \vdash B : * \]
\[ \text{ci} : (xi : Ai) \rightarrow T \quad ti' \]
\[ \Gamma, x_i:A_i \quad \Delta, t = \text{ci } x_i, ti' = t' \vdash ti : B \]
\[ \Gamma \Delta \vdash \text{case } t \text{ of } \{ \text{ci } x_i \Rightarrow ti \} : B \]

\[ \Gamma \Delta \vdash t : A \quad \Delta \vdash A = B \quad \Delta \vdash B : * \]
\[ \Gamma \Delta \vdash t : B \]
Questions to answer

- What properties of isEq & Con must we assume to show preservation & progress?

- What instantiations of isEq & Con satisfy these properties?
Necessary assumptions (con)

- Don’t start inconsistent
  \( \text{con}(.\) = maybe

- Once inconsistent, stay inconsistent through weakening, substitution, cut and conversion
  \[
  \begin{align*}
  &\text{con}\ (\Delta) = \text{false} \implies \text{con}\ (\Delta \ \Delta’) = \text{false} \\
  &\text{con}\ (\Delta) = \text{false} \implies \text{con}\ (\Delta \ {w/x}\ ) = \text{false} \\
  &\text{con}\ (\Delta \ (e1 = e2) \ \Delta’) = \text{false} \land \text{isEq}\ (\Delta, e1, e2) \implies \text{con}\ (\Delta \ \Delta’) = \text{false} \\
  &\text{con}(\Delta) = \text{false} \land (\Delta = \Delta’) \implies \text{con}(\Delta’) = \text{false}
  \end{align*}
  \]
Necessary assumptions (isEq)

- isEq is an equivalence class
- Holds for evaluation: If \( e \rightarrow e' \) then isEq (\( \Delta, e, e' \) )
- Constructors are injective, for (possibly) consistent contexts
  \[
  \text{con}(\Delta) = \text{maybe} \land \text{isEq}(\Delta, ci \ e_1, cj \ e_2) \Rightarrow \\
  \text{isEq}(\Delta, e_1, e_2) \land i=j
  \]
- Preserved by substitution
  \[
  \text{isEq}(\Delta'\Delta', e_1, e_2) \Rightarrow \text{isEq}(\Delta, w, w') \Rightarrow \\
  \text{isEq}(\Delta'\{w/x\}, e_1\{w/x\}, e_2\{w'/x\})
  \]
- Preserved under contextual operations (weakening, cut, conversion)
  \[
  \text{isEq}(\Delta \ (e = e') \Delta', e_1, e_2) \land \text{isEq}(\Delta, e, e') \Rightarrow \\
  \text{isEq}(\Delta'\Delta', e_1, e_2)
  \]
What satisfies these properties?

- Compare normal forms, ignoring equalities in the context
  - Above plus equalities in the context
- Contextual equivalence
  - Contextual equivalence modulo $\Delta$
- Some strange equalities that identify nonterminating terms with terminating terms
  - Sound to conclude $\text{isEq(let } x = \text{loop in 3, 3)}$ as long as we don’t conclude $\text{isEq(let } x = \text{loop in 3, loop)}$
  - Sound to say $\text{isEq(loop,3)}$ as long as we don’t say $\text{isEq(loop, 4)}$
What about termination?

- Termination analysis not required for type soundness
  - Decidable approximation of `isEq` is type sound, but doesn’t satisfy preservation
    - Any types system that checks strictly fewer terms than a sound type system is sound.
- However, like most type systems, only get partial correctness results:
  - “If this expression terminates, then it produces a value of type `t`”
- Termination analysis permits proof erasure
More questions

- Is untyped equivalence strong enough?
  - Have we accomplished anything?

- Can we give more information about typing to Con and isEq?
  - For now, we want to make axiomatization of isEq independent of the type system, but does that buy us anything?

- Can we add a predicate to control what expressions are compared for equality?
  - Limit domain of isEq for stronger properties

- What about more computational effects: state/control effects?
  - Can we use effect typing to strengthen equivalence?