CONSTRUCTING RED-BLACK TREES

RALF HINZE

Institut für Informatik III, Universität Bonn, Römerstraße 164
53117 Bonn, Germany
Email: ralf@informatik.uni-bonn.de
Hompage: http://www.informatik.uni-bonn.de/~ralf

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Problem

Given an ascending sequence of elements, construct, in linear time, a red-black tree that contains the elements in symmetric order.

Approach #1: analyse and improve the standard method, which works by repeatedly inserting elements into an empty initial tree.

Approach #2: build upon well-known algorithms for constructing trees of minimum height. *Subproblem:* given an arbitrary binary search-tree, is there a way of coloring the nodes such that a red-black tree emerges?

Recap: Chris Okasaki's functional red-black trees

A red-black tree is a binary tree whose nodes are colored either red or black.

$$\begin{array}{lcl} \textbf{data}\ \mathit{Color} & = & R \mid B \\ \textbf{data}\ \mathit{RBTree}\ a & = & E \mid N\ \mathit{Color}\ (\mathit{RBTree}\ a)\ a\ (\mathit{RBTree}\ a) \end{array}$$

Red condition: Each red node has a black parent.
Black condition: Each path from the root to an empty node contains exactly the same number of black nodes (this number is called the tree's *black height*).

Recap: insertion

$$\begin{array}{lcl} \mathit{insert} & :: & (\mathit{Ord}\ a) \Rightarrow a \to \mathit{RBTree}\ a \to \mathit{RBTree}\ a \\ \mathit{insert}\ a\ t & = & \mathit{blacken}\ (\mathit{ins}\ t) \\ \quad \textbf{where} & & \\ \quad \mathit{ins}\ E & = & N\ R\ E\ a\ E \\ \quad \mathit{ins}\ (N\ c\ l\ b\ r) & & \\ \qquad \mid a < b & = & \mathit{bal}\ c\ (\mathit{ins}\ l)\ b\ r \\ \qquad \mid a == b & = & N\ c\ l\ a\ r \\ \qquad \mid a > b & = & \mathit{bal}\ c\ l\ b\ (\mathit{ins}\ r) \\ \mathit{blacken}\ (N\ _\ l\ a\ r) & = & N\ B\ l\ a\ r \end{array}$$

Recap: balancing

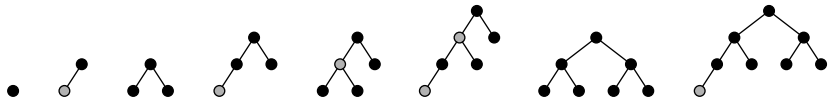
$$
\begin{array}{lcl}
bal\ B\ (N\ R\ (N\ R\ t_1\ a_1\ t_2)\ a_2\ t_3)\ a_3\ t_4 & & \\
\qquad = & N\ R\ (N\ B\ t_1\ a_1\ t_2)\ a_2\ (N\ B\ t_3\ a_3\ t_4) & \\
bal\ B\ (N\ R\ t_1\ a_1\ (N\ R\ t_2\ a_2\ t_3))\ a_3\ t_4 & & \\
\qquad = & N\ R\ (N\ B\ t_1\ a_1\ t_2)\ a_2\ (N\ B\ t_3\ a_3\ t_4) & \\
bal\ B\ t_1\ a_1\ (N\ R\ (N\ R\ t_2\ a_2\ t_3)\ a_3\ t_4) & & \\
\qquad = & N\ R\ (N\ B\ t_1\ a_1\ t_2)\ a_2\ (N\ B\ t_3\ a_3\ t_4) & \\
bal\ B\ t_1\ a_1\ (N\ R\ t_2\ a_2\ (N\ R\ t_3\ a_3\ t_4)) & & \\
\qquad = & N\ R\ (N\ B\ t_1\ a_1\ t_2)\ a_2\ (N\ B\ t_3\ a_3\ t_4) & \\
bal\ c\ l\ a\ r & = & N\ c\ l\ a\ r
\end{array}
$$

Recap: constructing a red-black tree

$$
\begin{array}{lcl}
top\text{-}down & :: & (Ord\ a) \Rightarrow [a] \to RBTree\ a \\
top\text{-}down & = & foldr\ insert\ E
\end{array}
$$

Approach #1: a closer look at $top\text{-}down$

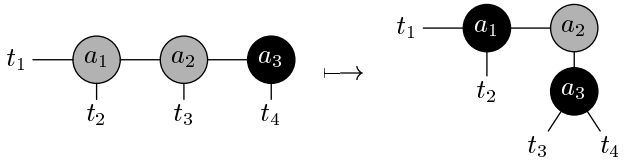
The following trees are generated by $top\text{-}down\,[1\,..\,i]$ for $1 \leqslant i \leqslant 8$ ('○' is a red node and '●' is a black node).



NB. ins always traverses the *left spine* of the tree to the leftmost leaf.

Approach #1: a closer look at $top\text{-}down$

If we draw the left spine horizontally, the balancing operation (first equation of bal) takes the following form.

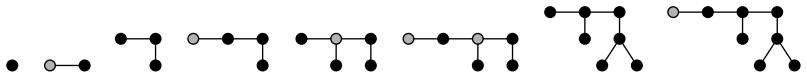


☞ The trees below the left spine (t_2, t_3 and t_4) must be perfectly balanced binary trees (perfect trees for short). Thus, the generated red-black trees correspond to sequences of *topped perfect trees* or *pennants*.

☞ A pennant of rank r contains exactly 2^r nodes.

Approach #1: a closer look at *top-down*

It is helpful to redraw the examples according to the *left-spine view*.



Let r be the rank of the rightmost pennant; the black condition implies that a pennant of rank i appears either once or twice for all $0 \leqslant i \leqslant r$.

☞ The red-black trees generated by *top-down* correspond to 'binary numbers' composed of the digits 1 and 2.

Approach #1: the 1-2 number system

Recall that the value of the radix-2 number $(b_{n-1} \ldots b_0)_2$ is $\sum_{i=0}^{n-1} b_i 2^i$.

Each natural number has a unique representation in the 1-2 number system.

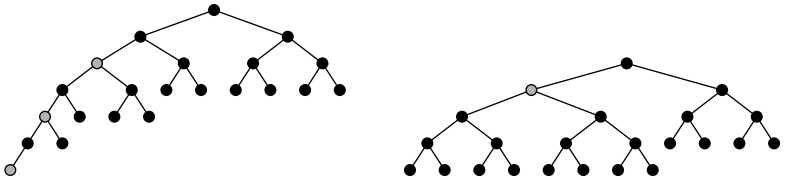
$$()_2, (1)_2, (2)_2, (11)_2, (12)_2, (21)_2, (22)_2, (111)_2, (112)_2 \ldots$$

Incrementing a 1-2 number:

$$\begin{array}{lcl} s1 + 1 & = & s2 \\ s2 + 1 & = & (s+1)\,1\ . \end{array}$$

Approach #1: corollaries

The trees corresponding to $(12^{\{n\}})_{\text{1-2}}$ are *skinny trees* (a skinny tree is a tree of smallest possible size for a given height); the trees corresponding to $(21^{\{n\}})_{\text{1-2}}$ are *left-complete trees*.



$(1222)_{\text{1-2}}$ $(2111)_{\text{1-2}}$

Approach #1: improving *top-down*

The analogy to the 1-2 number system can be exploited to give a better implementation of *top-down* for the special case that the elements appear in ascending order. The digits become containers for pennants:

$$\begin{array}{lcl} \textbf{data}\ \mathit{Digit}\ a & = & \mathit{One}\ a\ (\mathit{RBTree}\ a) \\ & | & \mathit{Two}\ a\ (\mathit{RBTree}\ a)\ a\ (\mathit{RBTree}\ a)\ . \end{array}$$

Inserting an element corresponds to incrementing a 1-2 number.

$$\begin{array}{lcl} \mathit{incr}\ (\mathit{One}\ a\ t)\ [] & = & [\mathit{One}\ a\ t] \\ \mathit{incr}\ (\mathit{One}\ a_1\ t_1)\ (\mathit{One}\ a_2\ t_2 : ps) & = & \mathit{Two}\ a_1\ t_1\ a_2\ t_2 : ps \\ \mathit{incr}\ (\mathit{One}\ a_1\ t_1)\ (\mathit{Two}\ a_2\ t_2\ a_3\ t_3 : ps) & & \\ \quad = \ \mathit{One}\ a_1\ t_1 : \mathit{incr}\ (\mathit{One}\ a_2\ (N\ B\ t_2\ a_3\ t_3))\ ps & & \end{array}$$

Approach #1: improving *top-down*

$$
\begin{array}{lcl}
\mathit{bottom\text{-}up} & :: & [a] \to \mathit{RBTree}\ a \\
\mathit{bottom\text{-}up} & = & \mathit{linkAll} \cdot \mathit{foldr}\ \mathit{add}\ [\,] \\
\mathit{add}\ a\ \mathit{ps} & = & \mathit{incr}\ (\mathit{One}\ a\ E)\ \mathit{ps} \\
\mathit{linkAll} & = & \mathit{foldl}\ \mathit{link}\ E \\
\mathit{link}\ l\ (\mathit{One}\ a\ t) & = & N\ B\ l\ a\ t \\
\mathit{link}\ l\ (\mathit{Two}\ a_1\ t_1\ a_2\ t_2) & = & N\ B\ (N\ R\ l\ a_1\ t_1)\ a_2\ t_2
\end{array}
$$

If as is ordered, we have $\mathit{top\text{-}down}\ as = \mathit{bottom\text{-}up}\ as$.

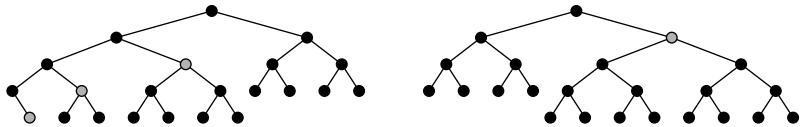
☞ A standard amortization argument shows that *bottom-up* runs in linear time.

Approach #1: less height, please!

Interestingly, it is easy to modify *bottom-up* so that it produces *optimal search-trees*. A simple *rotation to the right* suffices:

$$
\mathit{link'}\ l\ (\mathit{Two}\ a_1\ t_1\ a_2\ t_2) \quad = \quad N\ B\ l\ a_1\ (N\ R\ t_1\ a_2\ t_2)\ .
$$

Here are the shallow variants of the trees above.

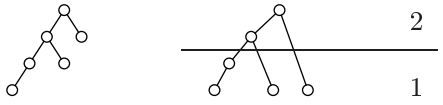


$(1222)_{1\text{-}2}$ $(2111)_{1\text{-}2}$

Approach #2: coloring binary search-trees

Problem: given an arbitrary binary search-tree, is there a way of coloring the nodes such that a red-black tree emerges?

The first test case.

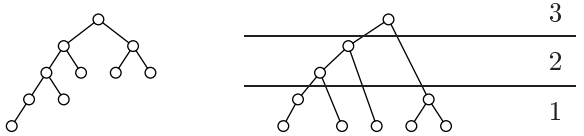


On the right-hand side the vertical position of a node corresponds to its height rather than its distance from the root. We assign a *level number* to each node: a node of height h receives the level number $\lceil h\,/\,2 \rceil$.

☞ A node is colored red iff it has a parent with the same level number.

Approach #2: coloring binary search-trees

The second test case.



☞ The right son of the root must be 'lifted' to the second level.

Approach #2: coloring binary search-trees

We assume that trees are given as elements of the data type

$$\begin{array}{lcl} \textbf{type}\ \mathit{Level} & = & \mathit{Int} \\ \textbf{data}\ \mathit{Tree}\ a & = & \mathit{Empty} \mid \mathit{Node}\ \mathit{Level}\ (\mathit{Tree}\ a)\ a\ (\mathit{Tree}\ a)\ , \end{array}$$

in which $\lceil \mathit{height}\ (\mathit{Node}\ h\ l\ a\ r) \,/\, 2 \rceil = h$.

Approach #2: coloring binary search-trees

$$\begin{array}{lcl} \mathit{rbtree} & :: & \mathit{Tree}\ a \to \mathit{RBTree}\ a \\ \mathit{rbtree}\ \mathit{Empty} & = & E \\ \mathit{rbtree}\ (\mathit{Node}\ h\ l\ a\ r) & = & N\ B\ (\mathit{rbtree}'\ h\ l)\ a\ (\mathit{rbtree}'\ h\ r) \\ \mathit{rbtree}' & :: & \mathit{Level} \to \mathit{Tree}\ a \to \mathit{RBTree}\ a \\ \mathit{rbtree}'\ \mathit{hp}\ \mathit{Empty} & & \\ \quad \mid \mathit{hp} == 1 & = & E \\ \quad \mid \mathit{otherwise} & = & \mathit{error}\ \texttt{"not a red-black tree"} \\ \mathit{rbtree}'\ \mathit{hp}\ (\mathit{Node}\ h\ l\ a\ r) & = & N\ \mathit{color}\ (\mathit{rbtree}'\ h'\ l)\ a\ (\mathit{rbtree}'\ h'\ r) \\ \quad \textbf{where} & & \\ \quad h' & = & h\ `\mathit{max}`\ (\mathit{hp} - 1) \\ \quad \mathit{color} \mid \mathit{hp} == h & = & R \\ \qquad \mid \mathit{otherwise} & = & B \end{array}$$

Properties and cross connections

top-down **and** *bottom-up*: construct trees with a minimal number of red nodes among all trees of that size; depending on the number of elements they produce both trees of minimum and of maximum height.

bottom-up′: constructs trees of minimum height (optimal search-trees); relates red-black trees to *left-complete trees*.

rbtree: generalizes an algorithm for converting *AVL trees* into red-black trees; relates red-black trees to *half-balanced trees*; produces trees of minimum black height; applied to left-complete trees it constructs trees with a maximal number of red nodes among all trees of that size.

Résumé

- Solving the problem of constructing red-black trees gives many insights into their structure (and relates red-black trees to left-complete trees, half-balanced trees and AVL trees).
- Number systems are great for analysing and designing data structures and algorithms.
- Functional programming languages are an excellent tool for presenting algorithms and data structures.