

Generic Properties of Datatypes

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Generic Programming Summer School
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Outline

- Theorems For Free
- Commuting Datatypes (“Zips”)
- Relators, Fans and Membership
- Properties of Zips
- Conclusion

Parametric Polymorphism

Summary: parametric polymorphism is a verifiable form of (type) genericity.

Common Type = Common Properties

$$\text{length} : \langle \forall \alpha :: \mathbb{N} \leftarrow \text{List}.\alpha \rangle$$

For all types A and B and all functions f of type $A \leftarrow B$,

$$\text{length}_A \circ \text{List}.f = \text{length}_B \text{ .}$$

Let sq denote the function that squares a number.

$$\text{sq} \circ \text{length} : \langle \forall \alpha :: \mathbb{N} \leftarrow \text{List}.\alpha \rangle$$

$$(\text{sq} \circ \text{length}_A) \circ \text{List}.f = \text{sq} \circ \text{length}_B \text{ .}$$

Suppose copycat appends a copy of a list to itself.

$$\text{length} \circ \text{copycat} : \langle \forall \alpha :: \mathbb{N} \leftarrow \text{List}.\alpha \rangle$$

$$(\text{length}_A \circ \text{copycat}_A) \circ \text{List}.f = \text{length}_B \circ \text{copycat}_B \text{ .}$$

Polymorphism

Consider the type expressions defined by the following grammar:

$$\text{Exp} ::= \text{Exp} \times \text{Exp} \mid \text{Exp} \leftarrow \text{Exp} \mid \text{Const} \mid \text{Var} .$$

Here, **Const** denotes a set of constant types, like \mathbb{N} (the natural numbers) and \mathbb{Z} (the integers). **Var** denotes a set of type variables. We use Greek letters to denote type variables.

A term t is said to have *polymorphic* type $\langle \forall \alpha :: T. \alpha \rangle$, where T is a type expression parameterised by type variables α , if t assigns to each type A a value t_A of type $T.A$.

Mapping Relations to Relations

Type expressions are extended to denote functions from relations to relations.

$$R \times S : A \times B \sim C \times D \iff R : A \sim C \wedge S : B \sim D$$

$$((a, b), (c, d)) \in R \times S \iff (a, c) \in R \wedge (b, d) \in S .$$

$$R \leftarrow S : (A \leftarrow B) \sim (C \leftarrow D) \iff R : A \sim C \wedge S : B \sim D$$

$$(f, g) \in R \leftarrow S \iff \langle \forall b, d :: (f.b, g.d) \in R \iff (b, d) \in S \rangle .$$

The constant type A is read as the identity relation id_A on A .

$$(x, y) \in A \iff x = y .$$

Example

$$\mathbf{R \times R \leftarrow R : (A \times A \leftarrow A) \sim (B \times B \leftarrow B) \iff R : A \sim B}$$

$$(f, g) \in \mathbf{R \times R \leftarrow R}$$

$$= \{ \text{definition of } \leftarrow \text{ on relations} \}$$

$$\langle \forall a, b :: (f.a, g.b) \in \mathbf{R \times R} \iff (a, b) \in \mathbf{R} \rangle$$

$$= \{ \text{definition of } \times \text{ on relations} \}$$

$$\langle \forall a, b ::$$

$$(fst.(f.a), fst.(g.b)) \in \mathbf{R} \wedge (snd.(f.a), snd.(g.b)) \in \mathbf{R}$$

$$\iff (a, b) \in \mathbf{R}$$

$$\rangle$$

Example

$$\text{id}_{\text{Bool}} \leftarrow \mathbb{R} \times \mathbb{R} : (\text{Bool} \leftarrow A \times A) \sim (\text{Bool} \leftarrow B \times B) \iff \mathbb{R} : A \sim B$$

$$(f, g) \in \text{id}_{\text{Bool}} \leftarrow \mathbb{R} \times \mathbb{R}$$

$$= \{ \text{definition of } \leftarrow \text{ and } \times \text{ on relations} \}$$

$$\langle \forall a, a', b, b' :: (f.(a, a'), g.(b, b')) \in \text{id}_{\text{Bool}} \iff (a, b) \in \mathbb{R} \wedge (a', b') \in \mathbb{R} \rangle$$

$$= \{ \text{definition of } \text{id}_{\text{Bool}} \}$$

$$\langle \forall a, a', b, b' :: f.(a, a') = g.(b, b') \iff (a, b) \in \mathbb{R} \wedge (a', b') \in \mathbb{R} \rangle$$

Parametric

A term t of polymorphic type $\langle \forall \alpha :: T. \alpha \rangle$ is said to be *parametrically polymorphic* if, for each instantiation of relations R to type variables, $(t_A, t_B) \in T.R$, where R has type $A \sim B$.

$$\text{fst} : \langle \forall \alpha, \beta :: \alpha \leftarrow \alpha \times \beta \rangle$$

Suppose $R : A \sim B$ and $S : C \sim D$.

$$\begin{aligned} & (\text{fst}_{A,C}, \text{fst}_{B,D}) \in R \leftarrow R \times S \\ = & \quad \{ \text{definition of } \leftarrow \text{ and } \times \text{ on relations} \} \\ & \langle \forall a, b, c, d :: (\text{fst}_{A,C}.(a, c), \text{fst}_{B,D}.(b, d)) \in R \iff (a, b) \in R \wedge (c, d) \in S \rangle \\ = & \quad \{ \text{definition of } \text{fst} \} \\ & \langle \forall a, b, c, d :: (a, b) \in R \iff (a, b) \in R \wedge (c, d) \in S \rangle \\ = & \quad \{ \text{calculus} \} \\ & \text{true} \end{aligned}$$

Ad Hoc Polymorphism

Suppose “ $==$ ” denotes a polymorphic “equality” operator. That is,

$$== : \langle \forall \alpha :: \text{Bool} \leftarrow \alpha \times \alpha \rangle$$

$==$ is parametric

$$= \quad \{ \quad \text{definition } (\mathbf{R} \text{ ranges over relations of type } \mathbf{A} \leftarrow \mathbf{B}) \quad \}$$

$$\langle \forall \mathbf{R} :: (==_{\mathbf{A}}, ==_{\mathbf{B}}) \in \text{id}_{\text{Bool}} \leftarrow \mathbf{R} \times \mathbf{R} \rangle$$

$$= \quad \{ \quad \text{definition of } \leftarrow \text{ and } \times \text{ on relations, and of } \text{id}_{\text{Bool}} \quad \}$$

$$\langle \forall \mathbf{R} :: \langle \forall \mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y} :: (\mathbf{u} ==_{\mathbf{A}} \mathbf{v}) = (\mathbf{x} ==_{\mathbf{B}} \mathbf{y}) \iff (\mathbf{u}, \mathbf{x}) \in \mathbf{R} \wedge (\mathbf{v}, \mathbf{y}) \in \mathbf{R} \rangle \rangle$$

$$\Rightarrow \quad \{ \quad \text{take } \mathbf{R} \text{ to be an arbitrary function } \mathbf{f} \quad \}$$

$$\quad \quad \quad \{ \quad \text{so } (\mathbf{u}, \mathbf{x}) \in \mathbf{R} \equiv \mathbf{u} = \mathbf{f}.\mathbf{x} \text{ and } (\mathbf{v}, \mathbf{y}) \in \mathbf{R} \equiv \mathbf{v} = \mathbf{f}.\mathbf{y} \quad \}$$

$$\langle \forall \mathbf{f} :: \langle \forall \mathbf{x}, \mathbf{y} :: (\mathbf{f}.\mathbf{x} ==_{\mathbf{A}} \mathbf{f}.\mathbf{y}) = (\mathbf{x} == \mathbf{y}) \rangle \rangle$$

Conclusion: all functions in the language of terms are injective, or “equality” is not both real equality and parametric.

Commuting Datatypes

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Introductory Examples

Zip (of lists)

$$([\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n], [\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n]) \mapsto [(\mathbf{a}_1, \mathbf{b}_1), (\mathbf{a}_2, \mathbf{b}_2), \dots, (\mathbf{a}_n, \mathbf{b}_n)]$$

$$\text{Pair} \cdot \text{List} \quad \mapsto \quad \text{List} \cdot \text{Pair}$$

Matrix Transposition

$$\text{List} \cdot \text{List} \quad \mapsto \quad \text{List} \cdot \text{List}$$

Broadcast

$$(\mathbf{a}, [\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n]) \mapsto [(\mathbf{a}, \mathbf{b}_1), (\mathbf{a}, \mathbf{b}_2), \dots, (\mathbf{a}, \mathbf{b}_n)]$$

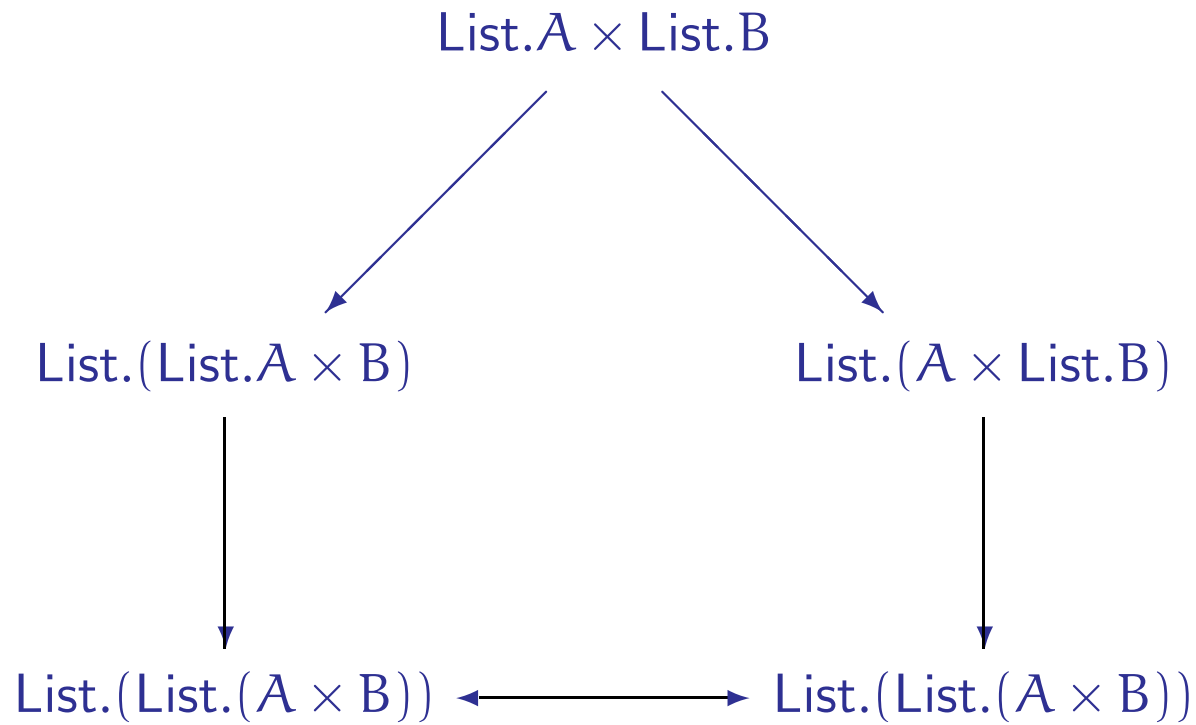
$$\mathbf{A} \times \cdot \text{List} \quad \mapsto \quad \text{List} \cdot \mathbf{A} \times$$

Primitive

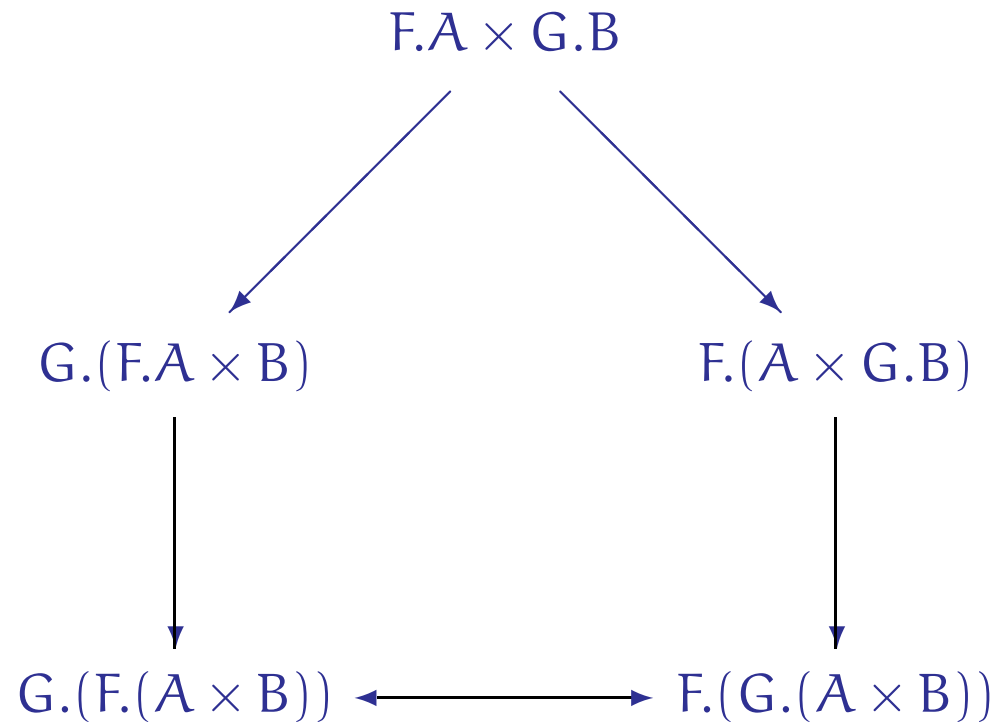
$$(\mathbf{A} + \mathbf{B}) \times (\mathbf{C} + \mathbf{D}) \mapsto (\mathbf{A} \times \mathbf{C}) + (\mathbf{B} \times \mathbf{D})$$

$$\times \cdot + \quad \mapsto \quad + \cdot \times$$

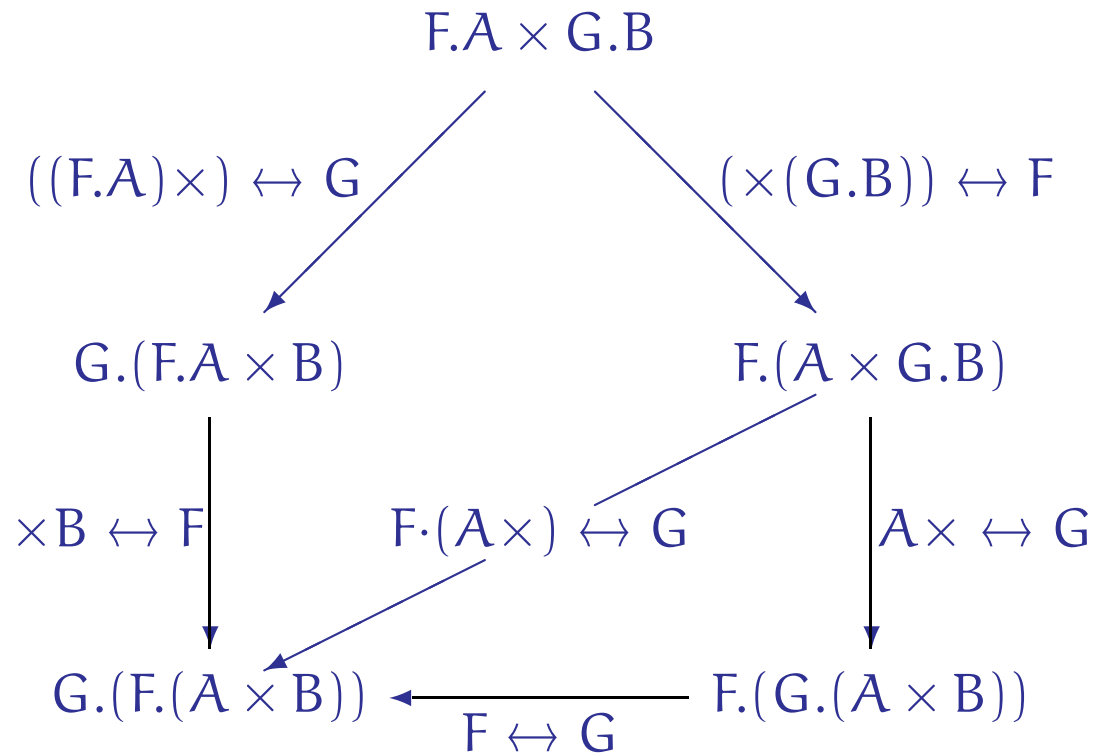
Structure Multiplication ...



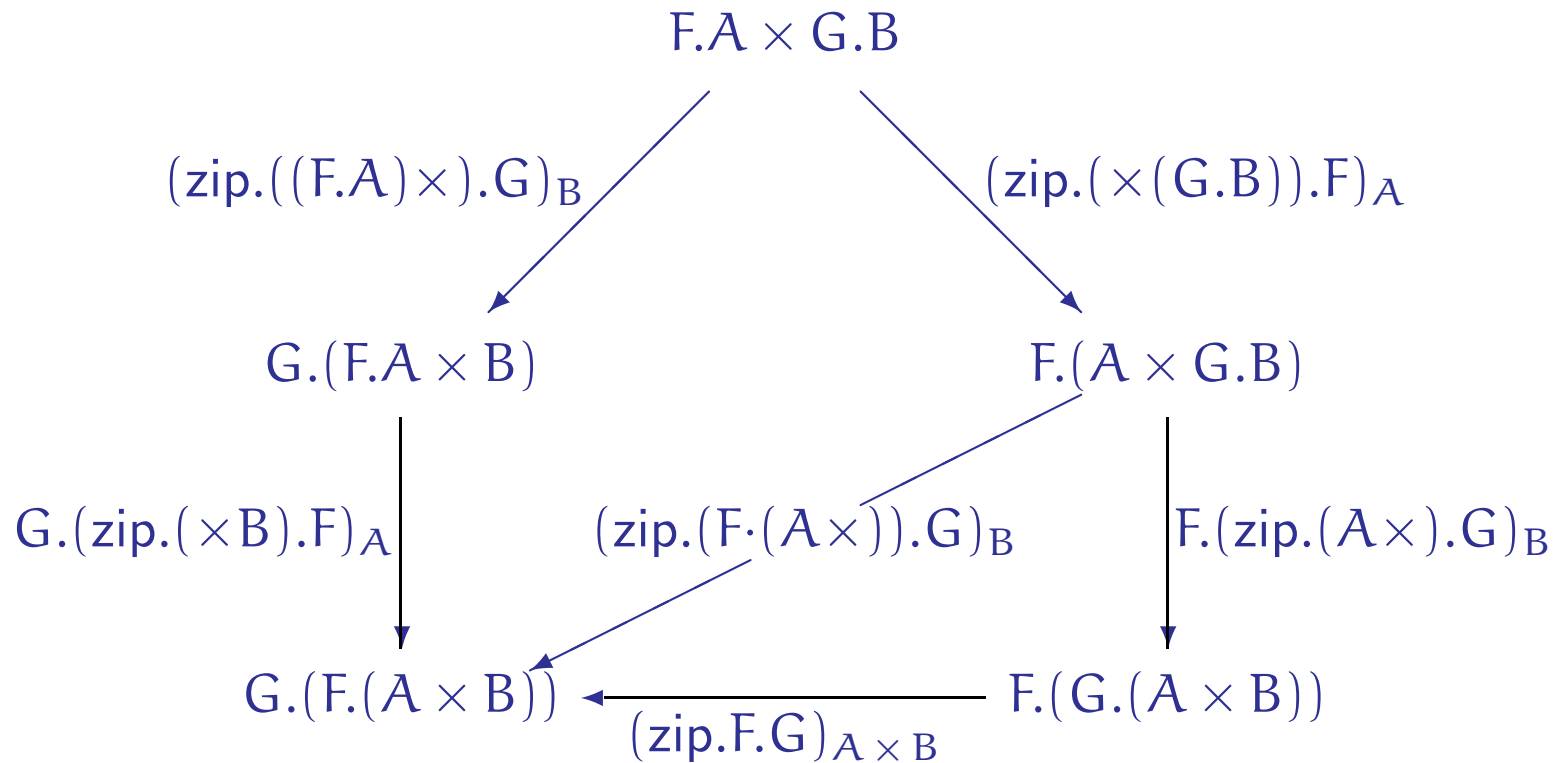
... Generalised ...



... Illustrates Generic Requirements



Multi-Coloured Zips



Broadcasts ...

A broadcast copies a given value across all storage locations of a datatype.

Formally, a family of functions `bcst`, where

$$\text{bcst}_{A,B} : F.(A \times B) \leftarrow F.A \times B$$

is said to be a *broadcast* for datatype `F` iff it is parametrically polymorphic in the parameters `A` and `B` and `bcstA,B` behaves coherently with respect to product in the following sense:

... Respect the Unit of Product ...

The following diagram

$$\begin{array}{ccc}
 F.(A \times \mathbb{1}) & \xleftarrow{bcst_{A, \mathbb{1}}} & F.A \times \mathbb{1} \\
 \searrow (F \cdot rid)_A & & \swarrow (rid \cdot F)_A \\
 & F.A &
 \end{array}$$

(where $rid_A : A \leftarrow A \times \mathbb{1}$ is the obvious natural isomorphism)
 commutes.

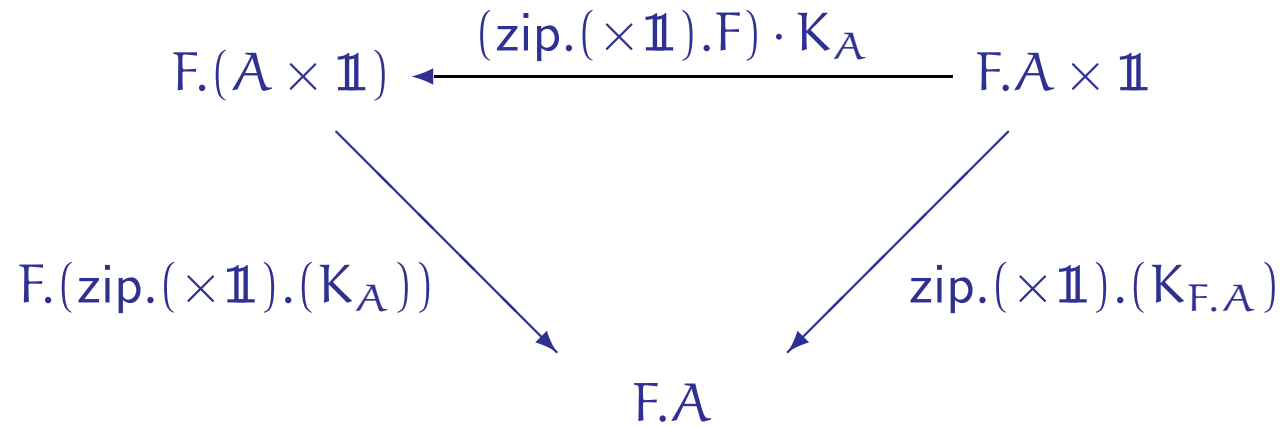
... and Associativity of Product

The following diagram

$$\begin{array}{ccc}
 F.A \times (B \times C) & \xleftarrow{\text{ass}_{F.A, B, C}} & (F.A \times B) \times C \\
 \downarrow \text{bcst}_{A, B \times C} & & \downarrow \text{bcst}_{A, B} \times \text{id}_C \\
 & & F.(A \times B) \times C \\
 & & \downarrow \text{bcst}_{A \times B, C} \\
 F.(A \times (B \times C)) & \xleftarrow{F \cdot \text{ass}_{A, B, C}} & F.((A \times B) \times C)
 \end{array}$$

(where $\text{ass}_{A, B, C} : A \times (B \times C) \leftarrow (A \times B) \times C$ is the obvious natural isomorphism) commutes as well.

Unit of Product is a “zip”



Associativity of Product is a “Zip”

$$\begin{array}{ccc}
 F.A \times (B \times C) & \xleftarrow{(\text{zip}.\times C).\text{((F.A)}\times))_B} & (F.A \times B) \times C \\
 \downarrow (\text{zip}.\times (B \times C)).F)_A & & \downarrow (\text{zip}.\times B).F)_A \times \text{id}_C \\
 & & F.(A \times B) \times C \\
 & \swarrow (\text{zip}.\times C).\text{(F.(A}\times))_B & \downarrow (\text{zip}.\times C).F)_{A \times B} \\
 F.(A \times (B \times C)) & \xleftarrow{F.\text{(zip}.\times C).\text{(A}\times))_B} & F.((A \times B) \times C)
 \end{array}$$

Conclusion

- Commuting Datatypes (“Zips”) are everywhere!
- Generic specification and proof is (potentially) very effective.
- A relational framework is necessary.
- Challenge: give generic specification of “commuting datatypes” from which “zips” can be constructed computationally.

Relators, Fans and Membership

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Allegories

Categorical formulation of (point-free) relation algebra.

Category (objects A, B, C , arrows —"relations"— R, S)

$$R \circ S : A \leftarrow B \iff R : A \leftarrow C \wedge S : C \leftarrow B ,$$

$$\text{id}_A : A \leftarrow A .$$

Arrows of same type are partially ordered by \subseteq .

$$S_1 \circ T_1 \subseteq S_2 \circ T_2 \iff S_1 \subseteq S_2 \wedge T_1 \subseteq T_2 .$$

$$X \subseteq R \wedge X \subseteq S \equiv X \subseteq R \cap S .$$

Converse

$$R^\cup \subseteq S \equiv R \subseteq S^\cup ,$$

$$(R \circ S)^\cup = S^\cup \circ R^\cup ,$$

$$R \circ S \cap T \subseteq (R \cap T \circ S^\cup) \circ S .$$

Relator

Relator: functor that is monotonic and respects converse.

Let \mathcal{A} and \mathcal{B} be allegories. A mapping F from objects of \mathcal{A} to objects of \mathcal{B} and arrows of \mathcal{A} to arrows of \mathcal{B} is a relator iff

$$F.R : F.A \leftarrow F.B \iff R : A \leftarrow B ,$$

$$F.R \circ F.S = F.(R \circ S) \quad \text{for each } R : A \leftarrow B \text{ and } S : B \leftarrow C ,$$

$$F.\text{id}_A = \text{id}_{F.A} \quad \text{for each object } A ,$$

$$F.R \subseteq F.S \iff R \subseteq S \quad \text{for each } R : A \leftarrow B \text{ and } S : A \leftarrow B ,$$

$$(F.R)^\cup = F.(R^\cup) \quad \text{for each } R : A \leftarrow B .$$

Examples: List is an endorelator. \times is a binary relator.

Functions

Relation $R : A \leftarrow B$ is *total* iff

$$\text{id}_B \subseteq R \cup \circ R \text{ ,}$$

and relation R is single-valued or *simple* iff

$$R \circ R \cup \subseteq \text{id}_A \text{ .}$$

A function is a relation that is total and simple.

Relators preserve totality

$$\begin{aligned}
 & (F.R)_{\cup} \circ F.R \\
 = & \{ \text{relators respect converse} \} \\
 & F.(R_{\cup}) \circ F.R \\
 = & \{ \text{relators distribute through composition} \} \\
 & F.(R_{\cup} \circ R) \\
 \supseteq & \{ \text{assume } \text{id}_B \subseteq R_{\cup} \circ R, \text{ relators are monotonic} \} \\
 & F.\text{id}_B \\
 = & \{ \text{relators preserve identities} \} \\
 & \text{id}_{F.B} .
 \end{aligned}$$

Similarly, relators preserve simplicity. Hence relators preserve functions.

Parametricity — point-free

Recall

$$(f, g) \in R \leftarrow S \quad \equiv \quad \langle \forall c, d :: (f.c, g.d) \in R \Leftarrow (c, d) \in S \rangle .$$

Point-free:

$$(f, g) \in R \leftarrow S \quad \equiv \quad f \cup \circ R \circ g \supseteq S .$$

Equivalently, using *shunting* rule:

$$(f, g) \in R \leftarrow S \quad \equiv \quad R \circ g \supseteq f \circ S .$$

Relators are Parametric

Type:

$$F.R : F.A \leftarrow F.B \quad \Leftarrow \quad R : A \leftarrow B \quad .$$

That is,

$$F : \langle \forall \alpha, \beta :: (F.\alpha \leftarrow F.\beta) \leftarrow (\alpha \leftarrow \beta) \rangle \quad .$$

F is parametric iff, for all relations R and S , and all functions f and g ,

$$(F.f, F.g) \in F.R \leftarrow F.S \quad \Leftarrow \quad (f, g) \in R \leftarrow S \quad .$$

Exercise: verify that this is the case using point-free definition of $R \leftarrow S$.

Natural Transformations

Parametricity of reverse function, `rev`, on lists, and of `fork`:

$$\text{List.R} \circ \text{rev}_B \supseteq \text{rev}_A \circ \text{List.R}$$

$$\text{R} \times \text{R} \circ \text{fork}_B \supseteq \text{fork}_A \circ \text{R}$$

In fact,

$$\text{List.R} \circ \text{rev}_B = \text{rev}_A \circ \text{List.R} .$$

But, it is *not* the case that, for all `R`,

$$\text{R} \times \text{R} \circ \text{fork}_B = \text{fork}_A \circ \text{R} .$$

For example,

$$\{(0, 0), (1, 0)\} \times \{(0, 0), (1, 0)\} \circ \text{fork}_B \neq \text{fork}_A \circ \{(0, 0), (1, 0)\} .$$

`fork` is a (lax) *natural transformation*, `rev` is a *proper* natural transformation.

Natural Transformations

$$\theta : F \leftarrow G = F.R \circ \theta_B \supseteq \theta_A \circ G.R \quad \text{for each } R : A \leftarrow B$$

$$\theta : F \hookrightarrow G = F.R \circ \theta_B \subseteq \theta_A \circ G.R \quad \text{for each } R : A \leftarrow B .$$

Facts:

$$(F.f \circ \theta_B = \theta_A \circ G.f \quad \text{for each function } f : A \leftarrow B) \Leftarrow \theta : F \leftarrow G .$$

In a “tabular allegory”,

$$\theta : F \leftarrow G \Leftarrow (F.f \circ \theta_B = \theta_A \circ G.f \quad \text{for each function } f : A \leftarrow B) .$$

In words, $\theta : F \leftarrow G$ iff θ is a (categorical) natural transformation in the underlying category of maps.

Conclusion: we take $\theta : F \leftarrow G$ to be the definition of a *natural transformation* in an allegory.

Division

An allegory is *locally complete* if for each set \mathcal{S} of relations of type $A \leftarrow B$, the union $\bigcup \mathcal{S} : A \leftarrow B$ exists and, furthermore, intersection and composition distribute over arbitrary unions.

$\perp\!\!\!\perp_{A,B}$ is the smallest relation of type $A \leftarrow B$ and $\top\!\!\!\top_{A,B}$ is the largest relation of the same type.

In a *division* allegory, composition distributes through union. That is, there are two *division* operators “\” and “/”, such that, for all $R : A \leftarrow B$, $S : B \leftarrow C$ and $T : A \leftarrow C$,

$$R \circ S \subseteq T \equiv S \subseteq R \backslash T \text{ ,}$$

$$R \circ S \subseteq T \equiv R \subseteq T / S \text{ ,}$$

$$S \subseteq R \backslash T \equiv R \subseteq T / S \text{ .}$$

Domain and Range

The *range* of a relation R is the set of all x such that $(x,y) \in R$ for some y .

Formally, the range operator “ $<$ ” is defined by, for all $R : A \leftarrow B$ and all $X \subseteq \text{id}_A$,

$$R_{<} \subseteq X \equiv R \subseteq X \circ \Pi_{A,B} .$$

The *domain* $R_{>}$ is defined by

$$R_{>} = (R_{\cup})_{<} .$$

Membership

The membership relation of a relator F is a family of relations mem_A , indexed by objects A , such that

$$\text{mem}_A : A \leftarrow F.A \quad , \text{ and}$$

for all A , all $X \subseteq \text{id}_A$ and $Y \subseteq \text{id}_{F.A}$,

$$F.X \supseteq Y \equiv (\text{mem}_A \circ Y)^< \subseteq X \quad .$$

In words, $F.X$ is the largest subset Y of F -structures, each of type $F.A$, such that the data stored in elements is in the set X .

Weakest Liberal Precondition

For all $X \subseteq \text{id}_A$ and $Y \subseteq \text{id}_{F.A}$,

$$(\text{mem}_A \circ Y) \subseteq X$$

$$= \{ \text{definition of range} \}$$

$$\text{mem}_A \circ Y \subseteq X \circ \top$$

$$= \{ \text{division} \}$$

$$Y \subseteq \text{mem}_A \setminus (X \circ \top)$$

$$= \{ Y \subseteq \text{id}_{F.A} \}$$

$$Y \subseteq \text{mem}_A \setminus (X \circ \top) \cap \text{id}_{F.A} .$$

For those familiar with the wp calculus: $\text{mem}_A \setminus (X \circ \top) \cap \text{id}_{F.A}$ is the weakest liberal precondition guaranteeing a state satisfying X after “execution” of mem .

Properties of F structures

For all A , all $X \subseteq \text{id}_A$ and $Y \subseteq \text{id}_{F.A}$,

$$F.X \supseteq Y \quad \equiv \quad \text{mem}_A \setminus (X \circ \Pi) \cap \text{id}_{F.A} \supseteq Y .$$

So,

$$F.X = \text{mem}_A \setminus (X \circ \Pi) \cap \text{id}_{F.A} .$$

Interpreting $X \subseteq \text{id}_A$ as a property of values of type A , $F.X$ is a property of values of type $F.A$. The identity says that a property of an F -structure is characterised by properties of the values stored in the structure (its “members”).

Largest Natural Transformations

Recall: for each object A ,

$$\text{mem}_A : A \leftarrow F.A \ .$$

Membership is parametric: for all R ,

$$R \circ \text{mem} \supseteq \text{mem} \circ F.R \ .$$

Equivalently,

$$\text{mem} : \text{Id} \leftarrow F \ .$$

Also,

$$\text{mem} \backslash \text{id} : F \leftarrow \text{Id} \ .$$

Theorem: The fan of relator F , $\text{mem} \backslash \text{id}$, is the largest natural transformation of type $F \leftarrow \text{Id}$. The membership of relator F is the largest natural transformation of type $\text{Id} \leftarrow F$.

Understanding Natural Transformations

Theorem: Suppose F and G are relators with memberships mem.F and mem.G respectively. Then the largest natural transformation of type $F \leftrightarrow G$ is $\text{mem.F} \setminus \text{mem.G}$.

Interpretation: A natural transformation of type $F \leftrightarrow G$ changes structure only. Stored values may be lost or duplicated, but no computation is performed on them.

A *proper* natural transformation to F from G changes the structure without loss or duplication of stored values.

The Specification of a Generic Zip

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(Lower Order) Naturality

$$\text{zip.F.G} : G \bullet F \leftarrow F \bullet G .$$

A zip is a *proper* natural transformation.

A zip transforms one structure to another without loss or duplication of values.

(Higher Order) Naturality

$\text{zip.F} : (\bullet F) \leftarrow (F \bullet)$.

Categorical Nat Trans (Revision)

A natural transformation is an arrow in the functor category. I.e.,

$$\eta : F \leftarrow G$$

means that the following diagram commutes (for all A, B and $f : A \leftarrow B$)

$$\begin{array}{ccc}
 F.A & \xleftarrow{\eta_A} & G.A \\
 \uparrow F.f & & \uparrow G.f \\
 F.B & \xleftarrow{\eta_B} & G.B
 \end{array}$$

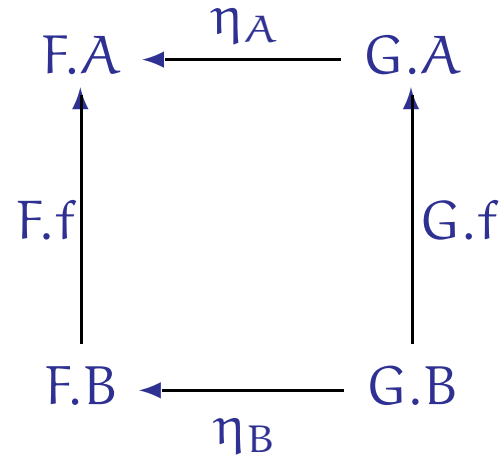
Now, if F is a functor, $(\bullet F)$ and $(F \bullet)$ are endofunctors on the functor category.

$(\bullet F)$ maps functor (object) G to $G \bullet F$ and natural transformation (arrow) η to $\eta \bullet F$, where $(\eta \bullet F)_A = \eta_{F.A}$.

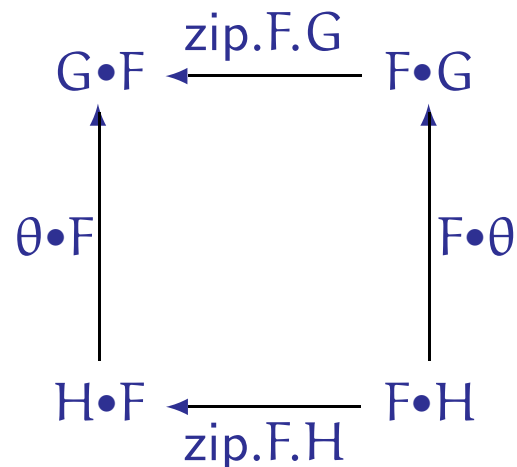
$(F \bullet)$ maps functor (object) G to $F \bullet G$ and natural transformation (arrow) η to $F \bullet \eta$, where $(F \bullet \eta)_A = F.(\eta_A)$.

Categorical NT Revision (Continued)

Diagram defining $\eta : F \leftarrow G$



instantiated for $\text{zip.F} : (\bullet F) \leftarrow (F \bullet)$



where $\theta : G \leftarrow H$ is a natural transformation.

Allegorical Naturality

Recall that parametricity was defined in terms of *relations*.

Recall also that, in the particular case that \mathbf{t} has type $\langle \forall \alpha :: F.\alpha \leftarrow G.\alpha \rangle$, \mathbf{t} is parametric is equivalent to \mathbf{t} is a natural transformation (in the underlying category of maps).

This is a stroke of luck for functional programmers, BUT their luck has run out!

The equality in

$$(\theta \bullet F) \circ \text{zip.F.H} = \text{zip.F.G} \circ (F \bullet \theta)$$

is too severe — because

- θ may be nondeterministic.
- Zips are partial.

Nondeterminism

Take $F := \text{List}$ and $G = H := \times$.

zip.F.H and zip.F.G are both the inverse of conventional zips. They unzip a list of pairs to a pair of lists.

Take $\theta := \text{id} \cup \text{swap}$.

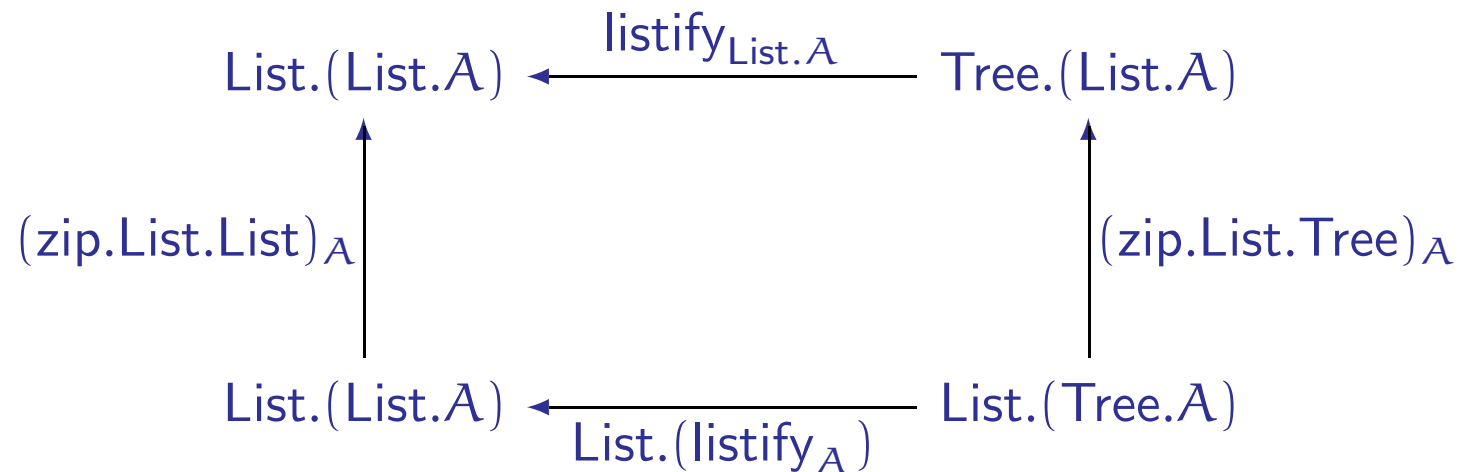
θ nondeterministically swaps the elements of a pair or not.

$(\theta \bullet F) \circ \text{zip.F.H}$ unzips a list of pairs into a pair of lists and swaps the lists or not.

$\text{zip.F.G} \circ (F \bullet \theta)$ first swaps some of the elements of a list of pairs and then unzips it into a pair of lists.

$$(\theta \bullet F) \circ \text{zip.F.H} \quad \subset \quad \text{zip.F.G} \circ (F \bullet \theta) \quad .$$

Partiality



View both paths through the diagram as partial relations of type $\text{List.}(\text{List.A}) \leftarrow \text{List.}(\text{Tree.A})$.

The lower path (via $\text{List.}(\text{List.A})$) includes the upper path (via $\text{Tree.}(\text{List.A})$).

Reason: for the lower path, the sizes of the trees must be the same; for the upper path, the trees must have the same shape.

zip.F is parametric.

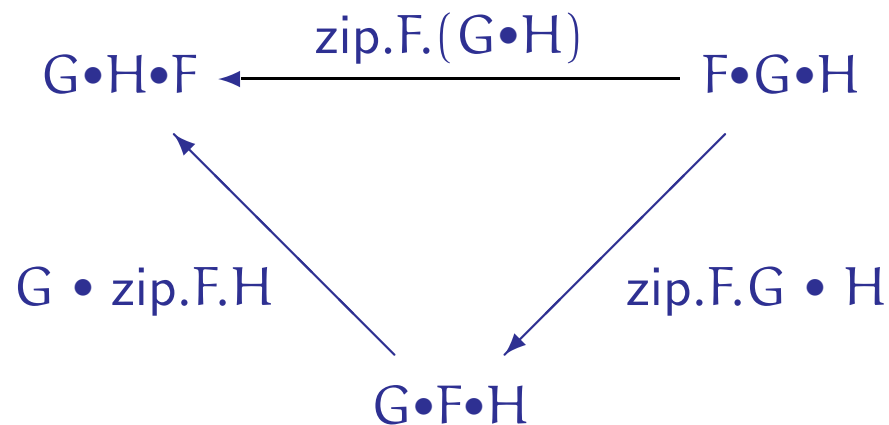
That is, for all $\theta : G \leftrightarrow H$,

$$(\theta \bullet F) \circ \text{zip.F.H} \subseteq \text{zip.F.G} \circ (F \bullet \theta) .$$

Compositionality

Informally, zip.F is a monoid homomorphism.

(Note: more than this: zip.F should respect pointwise extension of relators. For full discussion see Hoogendijk's thesis.)



$$\text{zip.F.}(G \bullet H) = (G \bullet \text{zip.F.H}) \circ (\text{zip.F.G} \bullet H) .$$

$$\text{zip.F.Id} = \text{id} \bullet F .$$

Zips

Definition 1 (Half Zip) Consider a fixed relator F and a pointwise closed class of relators \mathcal{G} . Then the members of the collection zip.F.G , where G ranges over \mathcal{G} , are called *half-zips* iff

- (a) $\text{zip.F.G} : G \bullet F \leftarrow F \bullet G$, for each G in \mathcal{G} ,
- (b) $(\theta \bullet F) \circ \text{zip.F.H} \subseteq \text{zip.F.G} \circ (F \bullet \theta)$ for each $\theta : G \leftrightarrow H$,
- (c) $\text{zip.F.(G \bullet H)} = (G \bullet \text{zip.F.H}) \circ (\text{zip.F.G} \bullet H)$ for all G and H ,
- (d) $\text{zip.F.Id} = \text{id} \bullet F$.

□

Definition 2 (Commuting Relators) The half-zip zip.F.G is said to be a *zip* of (F, G) if there exists a half-zip zip.G.F such that

$$\text{zip.F.G} = (\text{zip.G.F})_{\cup}$$

We say that datatypes F and G *commute* if there exists a zip for (F, G) .

□

Constructing Zips

See Hoogendijk's thesis for how these are calculated:

$$\begin{aligned}
 \text{zip.K}_A.G &= \text{fan.G} \bullet \text{K}_A \quad , \\
 \text{zip.+.G} &= G.\text{inl} \nabla G.\text{inr} \quad , \\
 \text{zip.}\times\text{.G} &= (G.\text{outl} \triangle G.\text{outr}) \cup \quad , \\
 \text{zip.T.G} &= ([\text{id}_G \otimes ; G.\text{in} \circ (\text{zip.}\otimes\text{.G} \bullet \text{Id} \triangle T)]) \quad .
 \end{aligned}$$

where T is the tree relator with pattern relator \otimes .

$$\begin{aligned}
 \text{fan.K}_A &= \text{TT}_{A,-} \\
 \text{fan.}+ &= (\text{id} \nabla \text{id}) \cup \\
 \text{fan.}\times &= \text{id} \triangle \text{id} \\
 \text{fan.T} &= ([\text{id} \otimes ; (\text{fan.}\otimes) \cup]) \cup
 \end{aligned}$$

where T is the tree relator with pattern relator \otimes .