# Deadlock-Free Processes and Non-Zero Vectors

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# Plugging Components in a Compact Closed Category

- A component has a number of typed interaction points (ports).
- Duality describes pluggability: $A$ and $A^*$.
- Compact closure distributes ports between the left and the right of a morphism.
- 1992–1997: interaction categories (classical concurrent processes): Abramsky, with Gay and Nagarajan.
- 2004–present: categorical quantum mechanics: Abramsky, Coecke and many others.

# Specification Structures

Let $\mathcal{C}$ be a category. A *specification structure* $S$ over $\mathcal{C}$ is defined by the following data:

- for each object $A$ of $\mathcal{C}$, a set $P_S A$ of "properties over $A$".
- for each pair of objects $A$, $B$ of $\mathcal{C}$, a "Hoare triple" relation $S_{A,B} \subseteq P_S A \times \mathcal{C}(A, B) \times P_S B$.

We write $\phi\{f\}\psi$ for $S_{A,B}(\phi, f, \psi)$. This relation is required to satisfy the following axioms, for $f : A \to B$, $g : B \to C$, $\phi \in P_S A$, $\psi \in P_S B$ and $\theta \in P_S C$:

$$\phi\{\mathrm{id}_A\}\phi \tag{1}$$

$$\phi\{f\}\psi, \psi\{g\}\theta \Longrightarrow \phi\{f\,;g\}\theta. \tag{2}$$

Given $\mathcal{C}$ and $S$ as above, we can define a new category $\mathcal{C}_S$. An object of $\mathcal{C}_S$ is a pair $(A, \phi)$ with $A \in \mathrm{ob}\,\mathcal{C}$ and $\phi \in P_S A$. A $\mathcal{C}_S$-morphism $f : (A, \phi) \to (B, \psi)$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that $\phi\{f\}\psi$. Composition and identities are inherited from $\mathcal{C}$; the axioms (1) and (2) ensure that $\mathcal{C}_S$ is a category.

# Examples

1. $\mathcal{C} = \mathbf{Set}$, $P_S X = X$, $a\{f\}b \iff f(a) = b$.
   Then $\mathcal{C}_S$ is the category of pointed sets. We think of it as a setting for compositional analysis of functions that have an additional property: mapping a distinguished element to a distinguished element.
2. $\mathcal{C} = \mathbf{Rel}$, $P_S X = \{*\}$,
   $*\{R\}* \iff \forall x \in A, y, z \in B.\, xRy \wedge xRz \implies y = z$.
   Then $\mathcal{C}_S$ is the category of sets and partial functions. We think of it as a setting for compositional analysis of relations that have an additional property: being a partial function.

# Specification Structures over Structured Categories

Suppose that $\mathcal{C}$ is a monoidal category. If we have a specification structure $S$ over $\mathcal{C}$, and if we want to define a monoidal structure on $\mathcal{C}_S$, then we must define an action

$$\otimes_{A,B} : P_S A \times P_S B \to P_S(A \otimes B)$$

and an element $I_S \in P_S I$ satisfying, for $f : A \to B$, $f' : A' \to B'$ and properties $\phi$, $\phi'$, $\psi$, $\psi'$, $\theta$ over suitable objects:

$$\begin{gathered}
\phi\{f\}\psi, \phi'\{f'\}\psi' \Longrightarrow (\phi \otimes \phi')\{f \otimes f'\}(\psi \otimes \psi') \\
((\phi \otimes \psi) \otimes \theta)\{\mathsf{assoc}_{A,B,C}\}(\phi \otimes (\psi \otimes \theta)) \\
(I_S \otimes \phi)\{\mathsf{unitl}_A\}\phi \\
(\phi \otimes I_S)\{\mathsf{unitr}_A\}\phi.
\end{gathered}$$

# A Specification Structure for Deadlock-Free Processes

- All processes are deadlock-free (non-terminating), and their types in the specification structure contain enough information to support deadlock-free composition.
- The base category is compact closed, but the deadlock-free category is not compact closed; it is $*$-autonomous.
- The definition of the specification structure starts by defining, for processes $p$ and $q$ of type $A$, the relation

$$p \bowtie q$$

  to mean that $p$ and $q$ can interact in $A$ without deadlock.
- This relation is used, through a sequence of formal definitions, to generate a specification structure.

# A Specification Structure for Non-Zero Vectors

- Take the base category to be **FDHilb**, and for vectors $u$ and $v$ define

$$u \bowtie v \iff \langle u \mid v \rangle \neq 0.$$

- Working through the same sequence of formal definitions yields a specification structure and a new category, $\mathbf{FDHilb}_{\mathrm{NZ}}$. All vectors are non-zero and morphisms preserve non-zero-ness.
- Non-zero vectors in **FDHilb** represent quantum states, and types in $\mathbf{FDHilb}_{\mathrm{NZ}}$ do not distinguish between vectors that are scalar multiples. So the type structure of $\mathbf{FDHilb}_{\mathrm{NZ}}$ might have some interpretation as a quantum logic.

# Example 1

- $\{|0\rangle\}^*$ is the type of vectors that have a non-zero component along $|0\rangle$. It is convenient to write this type as $|0\rangle$.
- For example: $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) : |0\rangle$.
- The Pauli map X can be given the type $|0\rangle \multimap |1\rangle$, so the application $\mathsf{X}\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ is well-typed and the result has type $|1\rangle$.
- Interpretation:
  1. $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ is non-zero because it has a non-zero component along $|0\rangle$.
  2. X maps vectors with a non-zero component along $|0\rangle$ to vectors with a non-zero component along $|1\rangle$.
  3. Therefore $\mathsf{X}\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ is non-zero because it has a non-zero component along $|1\rangle$.

# Example 2

Let $\pi_0$ be one of the projection operators corresponding to a standard basis measurement of a single qubit: $\pi_0|0\rangle = |0\rangle$, $\pi_0|1\rangle = \mathbf{0}$.

We can give $\pi_0$ the type

$$\pi_0 : |0\rangle \to |0\rangle$$

If we have

$$|\psi\rangle : |0\rangle$$

for some quantum state $|\psi\rangle$, then we can conclude

$$\pi_0|\psi\rangle : |0\rangle$$

and interpret this as the statement that measuring $|\psi\rangle$ has a non-zero probability of producing the result 0.

# Example 3

Let

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

so we have

$$|\psi\rangle : |0\rangle$$

and

$$|\psi\rangle : |1\rangle.$$

We can conclude

$$\pi_0|\psi\rangle : |0\rangle$$

and

$$\pi_1|\psi\rangle : |1\rangle$$

and interpret this as the statement that measuring $|\psi\rangle$ has a non-zero probability of producing either 0 or 1.

# Definitions (1)

## Definition

If $A$ is an object of **FDHilb**, let $\mathrm{NZ}(A) = \{v \in A \mid v \neq \mathbf{0}\}$ be its set of non-zero vectors.

## Definition

The non-orthogonality relation $\bowtie$ on $\mathrm{NZ}(A)$ is defined by

$$u \bowtie v \quad \Longleftrightarrow \quad \langle u \mid v \rangle \neq 0.$$

## Definition

Let $U \subseteq \mathrm{NZ}(A)$ for some object $A$.

$$\begin{array}{rcl} v \bowtie U & \Longleftrightarrow & \forall u \in U.\ v \bowtie u \\ U^* & = & \{v \in \mathrm{NZ}(A) \mid v \bowtie U\}. \end{array}$$

# Definitions (2)

## Definition

1. $P_{\mathrm{NZ}}A = \{U \subseteq \mathrm{NZ}(A) \mid U^{**} = U\}$
2. $v \models U \iff v \in U$.
3. $(-)^* : P_{\mathrm{NZ}}A \to P_{\mathrm{NZ}}A$ has already been defined.
4. 
$$\begin{array}{rcl} U \otimes V & = & \{u \otimes v \mid u \in U, v \in V\}^{**} \\ U \multimap V & = & (U \otimes V^*)^* \\ I_{\mathrm{NZ}} & = & \mathbb{C} \setminus \{0\}. \end{array}$$
5. $U\{f\}V \iff |f\rangle \models U \multimap V$.

## Theorem

NZ *is a specification structure over* **FDHilb**.

# Conclusion and Open Questions

- I have a refinement of **FDHilb** in which vectors are non-zero, constructed as a specification structure.
- This is a new connection between Classical Samson and Quantum Samson.
- I have proved that the $*$-autonomous structure of **FDHilb** lifts to $\mathbf{FDHilb}_{\mathrm{NZ}}$, but I don't know whether $\mathbf{FDHilb}_{\mathrm{NZ}}$ is compact closed.
- Does this construction really have any interesting application to quantum information?
- I invite anyone who is interested, to collaborate with me to finish this work.