Categorical coherence in the untyped setting

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The Untyped Setting

Untyped categories

Categories with only one object (i.e. monoids)

– with additional categorical properties.

Properties such as:

Monoidal Tensors, *Cartesian or Compact Closure*,

Duals, *Traces*, *Projections / Injections*, *Enrichment*, &c.

Where might we find such structures?

- Untyped computation (λ calculus & C-monoids)
- Polymorphic types (System F, parametrized types)
- Fractals (e.g. the Cantor space)
- State machines (Pushdown automata / binary stacks)
- Linguistics and models of meaning
- (Infinite-dimensional) quantum mechanics
- Group theory (Thompson's V and F groups)
- Semigroup theory (The polycyclic monoids P_n)
- Crystallography and Tilings
- Modular arithmetic & cryptography

Why study coherence in this setting?

Doesn't MacLane tell us all we need to know about coherence?

Is there anything special about *untyped* categories?

1. They test the limits of various coherence theorems.
2. Untypedness itself is the strictification of a certain categorical property,

   – closely connected to coherence for associativity.

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   – closely connected to coherence for associativity.

A simple example

The **Cantor monoid** $\mathcal{U}$

- **Single object:** $\mathbb{N}$.
- **Arrows:** all bijections on $\mathbb{N}$.

The monoidal structure

We have a tensor $(\_ \star \_) : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$.

$$(f \star g)(n) = \begin{cases} 2.f\left(\frac{n}{2}\right) & n \text{ even}, \\ 2.g\left(\frac{n-1}{2}\right) + 1 & n \text{ odd}. \end{cases}$$

The coherence isomorphisms:

- The **associativity** isomorphism:

$$\tau(n) = \begin{cases} 2n & n \ (mod\ 2) = 0, \\ n+1 & n \ (mod\ 4) = 1, \\ \frac{n-1}{2} & n \ (mod\ 4) = 3. \end{cases}$$

- The **symmetry** isomorphism:

$$\sigma(n) = \begin{cases} n-1 & n \text{ odd}, \\ n+1 & n \text{ even}. \end{cases}$$

MacLane's **pentagon** and **hexagon** conditions are satisfied.

Is it because I is absent?

We can make a genuine monoidal category from $(\mathcal{U}, \star)$.

How to: *adjoin a strict unit*

1. Take the coproduct with the trivial monoid I, giving $\mathcal{U} \coprod I$.
2. Extend $\_ \star \_$ to the coproduct by

$$I \star \_ \; = \; Id_{\mathcal{U} \coprod I} \; = \; \_ \star I$$

3. $(\mathcal{U} \coprod I, \_ \star \_)$ is a genuine monoidal category.

(Construction based on the theory of Saavedra units).

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Some ‘peculiarities’ of the Cantor monoid

Within the Cantor monoid $(\mathcal{U}, \_ \star \_)$

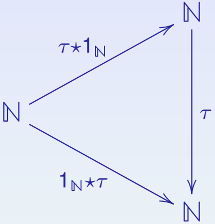
1. Associativity is not strict, even though

$$X \star (Y \star Z) = (X \star Y) \star Z$$

2. Not all canonical (for associativity) diagrams commute.
3. No strictly associative tensor on $\mathcal{U}$ can exist.

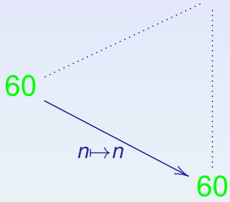
Canonical diagrams that do not commute

This canonical diagram does *not* commute:



Yes, there are two paths you can go by,

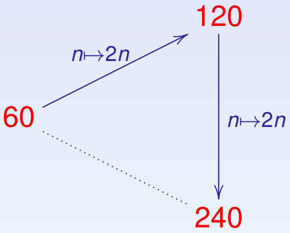
Using a randomly chosen number:



Taking the right hand path, $60 \mapsto 60$

Yes, there are two paths you can go by, but ...

On the left hand path,



Samson is 60, not 240; this diagram does *not* commute!

Not all canonical (for associativity) diagrams commute.

Is there a conflict with MacLane's Theorem?

http://en.wikipedia.org/wiki/Monoidal_category

WIKIPEDIA
The Free Encyclopedia

*"It follows that **any diagram** whose morphisms are built using [canonical isomorphisms], identities and tensor product commutes."*

Tinker, Tailor, Soldier, Sarcasm

Untangling The Web – *N.S.A. guide to internet use*

- *Do not as a rule rely on Wikipedia as your sole source of information.*
- *The best thing about Wikipedia are the external links from entries.*

MacLane, on MacLane's Theorem

Categories for the working mathematician (1^{st} ed.)

(p.158) *Moreover, all diagrams involving [canonical iso.s] must commute.*

(p. 159) *These three [coherence] diagrams imply that "all" such diagrams commute.*

(p. 161) *We can only prove that every "formal" diagram commutes.*

What does his theorem say?

MacLane's coherence theorem for associativity

All diagrams *within the image of a certain functor* are guaranteed to commute.

This **commonly**, but not **always**, means all canonical diagrams.

We are interested in situations where this is **not** the case.

Coherence for associativity — a convention

We will work with monogenic categories

Objects are generated by:

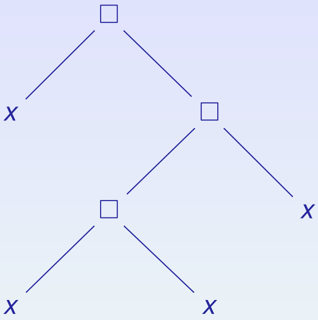
- Some object S,
- A tensor $(\_ \otimes \_)$.

This is not a restriction —

- S should be thought of as a 'variable symbol'.
- We will also rely on naturality.

The source of the functor

This is based on (non-empty) *binary trees*.



- **Leaves** labelled by x,
- **Branchings** labelled by $\square$.

The **rank** of a tree is the number of leaves.

A posetal category of trees

MacLane's category $\mathcal{W}$.

- **(Objects)** All non-empty binary trees.
- **(Arrows)** A **unique** arrow between any two trees *of the same rank*.

— write this as $(v \leftarrow u) \in \mathcal{W}(u, v)$.

Key points:

1. $(\_\Box\_)$ is a monoidal tensor on $\mathcal{W}$.
2. $\mathcal{W}$ is **posetal** — all diagrams over $\mathcal{W}$ commute.

MacLane's *Substitution Functor*

MacLane's theorem relies on a monoidal functor

$$\mathcal{W}Sub : (\mathcal{W}, \Box) \to (\mathcal{C}, \otimes)$$

This is based on a notion of *substitution*.

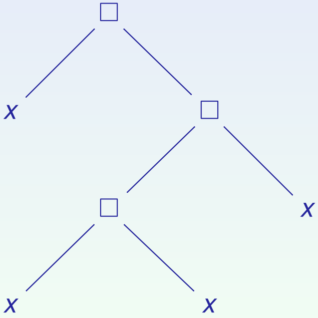
i.e. mapping *formal* symbols to *concrete* objects & arrows.

The functor itself

On objects:

- $\mathcal{W}Sub(x) = S$,
- $\mathcal{W}Sub(u\Box v) = \mathcal{W}Sub(u) \otimes \mathcal{W}Sub(v)$.

An object of $\mathcal{W}$:



An inductively defined functor (I)

On objects:

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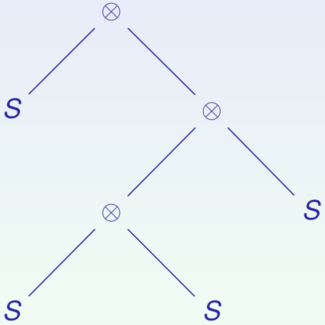
An object of $\mathcal{C}$:



On arrows:

- $\mathcal{W}Sub(u \leftarrow u) = 1_{\_}$.
- $\mathcal{W}Sub(a \Box v \leftarrow a \Box u) = 1_{\_} \otimes \mathcal{W}Sub(v \leftarrow u)$.
- $\mathcal{W}Sub(v \Box b \leftarrow u \Box b) = \mathcal{W}Sub(v \leftarrow u) \otimes 1_{\_}$.
- $\mathcal{W}Sub((a \Box b) \Box c \leftarrow a \Box (b \Box c)) = \tau_{\_,\_,\_}$.

The role of the Pentagon

The Pentagon condition $\Longrightarrow$ $\mathcal{W}Sub$ is a monoidal functor.

An inductively defined functor (II)

On arrows:

- $\mathcal{W}Sub(u \leftarrow u) = 1_{\_}$.
- $\mathcal{W}Sub(a \square v \leftarrow a \square u) = 1_{\_} \otimes \mathcal{W}Sub(v \leftarrow u)$.
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The story so far ...

We have a functor $\mathcal{W}Sub : (\mathcal{W}, \Box) \rightarrow (\mathcal{C}, \otimes)$.

- Every **object** of $\mathcal{C}$ is the image of an object of $\mathcal{W}$
- Every **canonical arrow** of $\mathcal{C}$ is the image of an arrow of $\mathcal{W}$
- Every **diagram** over $\mathcal{W}$ commutes.

As a corollary:

The image of **every diagram** in $(\mathcal{W}, \Box)$ **commutes** in $(\mathcal{C}, \otimes)$.

Question: Are all canonical diagrams in the image of $\mathcal{W}Sub$?

– This is only the case when $\mathcal{W}Sub$ is an *embedding*!

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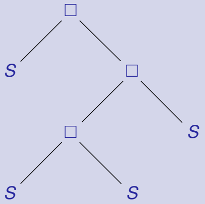
Given a **badly-behaved** category $(\mathcal{C}, \otimes)$, we can *build a **well-behaved** (non-strict) version*.

Think of this as the **Platonic Ideal** of $(\mathcal{C}, \otimes)$.

We (still) assume $\mathcal{C}$ is *monogenic*, with objects generated by $\{S, \_ \otimes \_\}$

Constructing $Plat_{\mathcal{C}}$

Objects are free binary trees



Leaves labelled by $S \in Ob(\mathcal{C})$,

Branchings labelled by $\square$.

There is an **instantiation map** $Inst : Ob(Plat_{\mathcal{C}}) \to Ob(\mathcal{C})$

$$S\square((S\square S)\square S) \mapsto S \otimes ((S \otimes S) \otimes S)$$

This is not just a matter of syntax!

What about arrows?

Homsets are copies of homsets of $\mathcal{C}$

Given trees T_1, T_2,

$$Plat_{\mathcal{C}}(T_1, T_2) = \mathcal{C}(Inst(T_1), Inst(T_2))$$

Composition is inherited from $\mathcal{C}$ in the obvious way.

The tensor $(\_ \square \_) : Plat_{\mathcal{C}} \times Plat_{\mathcal{C}} \to Plat_{\mathcal{C}}$

$$\left. \begin{array}{l} A \xrightarrow{f} X \\ \\ B \xrightarrow{g} Y \end{array} \right\} \quad A \square X \xrightarrow{f \square g} B \square Y$$

The tensor of $Plat_{\mathcal{C}}$ is

- **(Objects)** A free formal pairing, $A \square B$,
- **(Arrows)** Inherited from $(\mathcal{C}, \otimes)$, so $f \square g \stackrel{def.}{=} f \otimes g$.

Some properties of the platonic ideal ...

1. The functor

$$\mathcal{W}Sub : (\mathcal{W}, \Box) \to (Plat_{\mathcal{C}}, \Box)$$

is always **monic.**

2. As a corollary:

All canonical diagrams of $(Plat_{\mathcal{C}}, \Box)$ commute.

3. Instantiation defines an **epic** monoidal functor

$$Inst : (Plat_{\mathcal{C}}, \Box) \to (\mathcal{C}, \otimes)$$

through which McL'.s substitution functor always factors.

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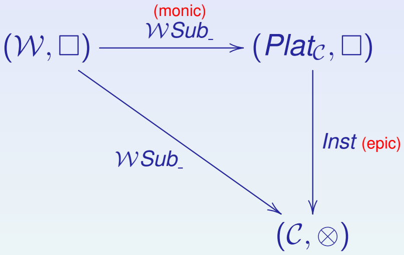
A monic / epic decomposition

MacLane's substitution functor always factors through the platonic ideal:



This gives a monic / epic decomposition of his functor.

The ‘Platonic Ideal’ of an untyped monoidal category

Can we build an **untyped** category over which all canonical diagrams commute?

The simplest possible case:

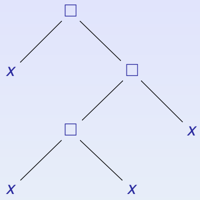
The trivial monoidal category $(\mathcal{I}, \otimes)$.

- **Objects:** $Ob(\mathcal{I}) = \{x\}$.
- **Arrows:** $\mathcal{I}(x, x) = \{1_x\}$.
- **Tensor:**

$$x \otimes x = x \quad , \quad 1_x \otimes 1_x = 1_x$$

What is the platonic ideal of $\mathcal{I}$?

(Objects) All non-empty binary trees:



(Arrows) For all trees T_1, T_2,

$$Plat_{\mathcal{I}}(T_1, T_2) \text{ is a single-element set.}$$

There is a unique arrow between any two trees!

(P.H. 1997) The prototypical self-similar category $(\mathcal{X}, \Box)$

- **Objects:** *All non-empty binary trees.*
- **Arrows:** *A unique arrow between any two objects.*

This monoidal category:

1. was introduced to study **self-similarity** $S \cong S \otimes S$,
2. contains MacLane's $(\mathcal{W}, \Box)$ as a wide subcategory.

The categorical identity $S \cong S \otimes S$

Exhibited by two canonical isomorphisms:

- **(Code)** $\lhd : S \otimes S \to S$
- **(Decode)** $\rhd : S \to S \otimes S$

These are *unique* (up to *unique isomorphism*).

Uniqueness ...

Unique *up to unique isomorphism*

is not the same as

actually unique.

Actual uniqueness implies that S is the unit object.

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Examples of self-similarity

- **(Infinitary examples)**
  The natural numbers $\mathbb{N}$, Separable Hilbert spaces, Infinite matrices, the Cantor set & other fractals, Binary stacks, &c.

- **(Untyped examples)**
  C-monoids, the Cantor monoid $\mathcal{U}$, any untyped monoidal category.

- **(Trivial examples)**
  The unit object I of any monoidal category.

What is **strict self-similarity**?

Can the code / decode maps

$$\lhd : S \otimes S \to S \quad , \quad \rhd : S \to S \otimes S$$

be **strict identities**?

In untyped monoidal categories:

We only have one object, $S = S \otimes S$.

Take the identity as both the **code** and **decode** arrows.

Untyped $\equiv$ Strictly Self-Similar.

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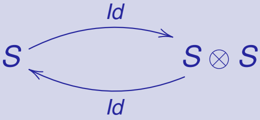

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Untyped $\equiv$ Strictly Self-Similar.

Question: **Does there exist a *strictification* procedure for self-similarity?**

Essential preliminaries

We need a coherence theorem for self-similarity.

and how it relates to associativity.

Coherence for Self-Similarity

(a special case of a much more general theory)

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A straightforward coherence theorem

We base this on the category $(\mathcal{X}, \Box)$

- **Objects** All non-empty binary trees.
- **Arrows** A unique arrow between any two trees.

This category is posetal — all diagrams over $\mathcal{X}$ commute.

We will define a monoidal substitution functor:

$$\mathcal{X}Sub : (\mathcal{X}, \Box) \to (\mathcal{C}, \otimes)$$

The self-similarity substitution functor

An inductive definition of $\mathcal{X}Sub : (\mathcal{X}, \Box) \to (\mathcal{C}, \otimes)$

On objects:

$$\begin{array}{rcl} x & \mapsto & S \\ u\Box v & \mapsto & \mathcal{X}Sub(u) \otimes \mathcal{X}Sub(v) \end{array}$$

On arrows:

$$\begin{array}{rcl} (x \leftarrow x) & \mapsto & 1_S \in \mathcal{C}(S,S) \\ \\ (x \leftarrow x\Box x) & \mapsto & \lhd \in \mathcal{C}(S \otimes S, S) \\ (x\Box x \leftarrow x) & \mapsto & \rhd \in \mathcal{C}(S, S \otimes S) \\ \\ (b\Box v \leftarrow a\Box u) & \mapsto & \mathcal{X}Sub(b \leftarrow a) \otimes \mathcal{X}Sub(v \leftarrow u) \end{array}$$

Interesting properties:

1. $\mathcal{X}Sub : (\mathcal{X}, \Box) \to (\mathcal{C}, \otimes)$ is always functorial.
2. Every arrow built up from

$$\{\lhd \, , \, \rhd \, , \, 1_S \, , \, \_ \otimes \_\}$$

is the image of an arrow in $\mathcal{X}$.
3. The image of every diagram in $\mathcal{X}$ commutes.

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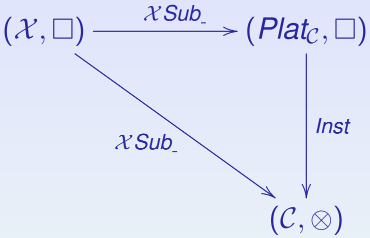
$\mathcal{X}$ *Sub* factors through the Platonic ideal

There is a monic-epic decomposition of $\mathcal{X}$*Sub*.



Every canonical (for self-similarity) diagram in $(Plat_{\mathcal{C}}, \Box)$ commutes.

Relating associativity and self-similarity

A tale of two functors

Comparing the *associativity* and *self-similarity* categories.

MacLane's $(\mathcal{W}, \Box)$

Objects: Binary trees.

Arrows: Unique arrow between two trees *of the same rank*.

The category $(\mathcal{X}, \Box)$

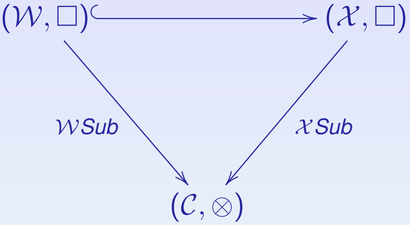
Objects: Binary trees.

Arrows: Unique arrow between *any two trees*.

There is an obvious inclusion $(\mathcal{W}, \Box) \hookrightarrow (\mathcal{X}, \Box)$

Is *associativity* a restriction of *self-similarity*?

Does the following diagram commute?



Does the **associativity** functor

factor through

the **self-similarity** functor?

Proof by contradiction:

Let's assume this is the case.

Special arrows of $(\mathcal{X}, \Box)$

For arbitrary trees u, e, v,

$$
\begin{aligned}
t_{uev} &= ((u\Box e)\Box v \leftarrow u\Box(e\Box v)) \\
l_v &= (v \leftarrow e\Box v) \\
r_u &= (u \leftarrow u\Box e)
\end{aligned}
$$

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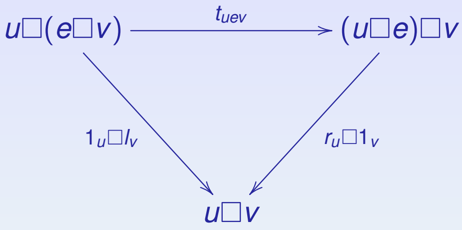
Since all diagrams over X commute:

The following diagram over $(\mathcal{X}, \Box)$ commutes:



Let's apply $\mathcal{X}Sub$ to this diagram.

By Assumption: $t_{uev} \mapsto \tau_{U,E,V}$ (assoc. iso.)

Notation: $u \mapsto U$, $v \mapsto V$, $e \mapsto E$, $l_v \mapsto \lambda_V$, $r_u \mapsto \rho_U$

Since all diagrams over X commute:

The following diagram over $(\mathcal{X}, \Box)$ commutes:

$$\begin{array}{ccc} u\Box(e\Box v) & \xrightarrow{\;t_{uev}\;} & (u\Box e)\Box v \\ & {\scriptstyle 1_u\Box l_v}\searrow \quad \swarrow{\scriptstyle r_u\Box 1_v} & \\ & u\Box v & \end{array}$$

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The following diagram over $(\mathcal{C}, \otimes)$ commutes:

$$\begin{array}{ccc} U \otimes (E \otimes V) & \xrightarrow{\ \tau_{UEV}\ } & (U \otimes E) \otimes V \\ & {\scriptstyle 1_U \otimes \lambda_U} \searrow \qquad \swarrow {\scriptstyle \rho_U \otimes 1_V} & \\ & U \otimes V & \end{array}$$

This is MacLane's **units triangle**
— E is the unit object for $(\mathcal{C}, \otimes)$.

The choice of e was *arbitrary* — every object is the unit object!

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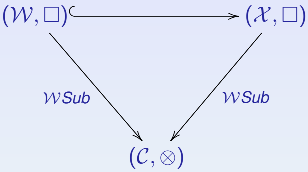
A general result

The following diagram commutes



exactly when $(\mathcal{C}, \otimes)$ is **degenerate** —

i.e. all objects are isomorphic to the unit object.

Generalising Isbell's argument

1. **Strict associativity:** All arrows of $(\mathcal{W}, \Box)$ are mapped to identities of $(\mathcal{C}, \otimes)$
2. **Strict self-similarity:** All arrows of $(\mathcal{X}, \Box)$ are mapped to the identity of $(\mathcal{C}, \otimes)$.

$\mathcal{W}$*Sub* trivially factors through $\mathcal{X}$*Sub*.

The conclusion

Strictly associative untyped monoidal categories are **degenerate**.

Another perspective ...

Another way of looking at things:

One cannot simultaneously *strictify*

(I) Associativity $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$

(II) Self-Similarity $S \cong S \otimes S$

The 'No Simultaneous Strictification' Theorem

A simple consequence:

Strictifying associativity ...

transforms **untyped structures** into *typed structures*.

Strictifying self-similarity ...

transforms **strict associativity** into *lax associativity*.

How to strictify self-similarity

A simple, almost painless, procedure (I)

- Start with a monogenic category $(\mathcal{C}, \otimes)$, generated by a self-similar object

$$S \underset{\lhd}{\overset{\rhd}{\rightleftarrows}} S \otimes S$$

- Construct its platonic ideal $(Plat_{\mathcal{C}}, \Box)$

- Use the (monic) self-similarity substitution functor

$$\mathcal{X}Sub : (\mathcal{X}, \Box) \to (Plat_{\mathcal{C}}, \Box)$$

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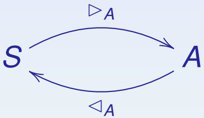
A simple,almost painless, procedure (II)

- The image of $\mathcal{X}Sub$ is a wide subcategory of $(\mathcal{P}lat_{\mathcal{C}}, \Box)$.

  It contains, for all objects A,

  a unique pair of inverse arrows



- Use these to define an **endofunctor** $\Phi : Plat_{\mathcal{C}} \to Plat_{\mathcal{C}}$.

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- Use these to define an **endofunctor** $\Phi : Plat_{\mathcal{C}} \rightarrow Plat_{\mathcal{C}}$.

The **type-erasing** endofunctor

- **Objects**

$$\Phi(A) = S \quad , \quad \text{for all objects } A$$

- **Arrows**

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ \uparrow{\scriptstyle \triangleright_A} & & \downarrow{\scriptstyle \triangleleft_B} \\ S & \xrightarrow[\quad \Phi(f) \quad]{} & S \end{array}$$

- **Functoriality** is trivial ...

A natural tensor on $\mathcal{C}(S, S)$

As a final step:

Define a tensor $(\_ \star \_)$ on $\mathcal{C}(S, S)$ by

$$\begin{array}{ccc} S \otimes S & \xrightarrow{t \otimes u} & S \otimes S \\ \uparrow \rhd & & \downarrow \lhd \\ S & \xrightarrow[t \star u]{} & S \end{array}$$

$(C(S, S), \_ \star \_)$ is an untyped monoidal category!

Type-erasing as a monoidal functor

- Recall, $Plat_{\mathcal{C}}(S,S) \cong \mathcal{C}(S,S)$.
- Up to this obvious isomorphism,

$$\Phi : (Plat_{\mathcal{C}}, \Box) \to (\mathcal{C}(S,S), \star)$$

is a monoidal functor.

What we have ...

A monoidal functor from $Plat_{\mathcal{C}}$

to an untyped monoidal category.

— every canonical (for self-similarity) arrow is mapped to 1_S.

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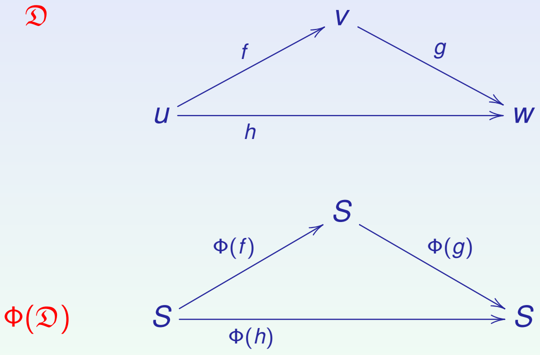
A useful property

Basic Category Theory

diagram $\mathfrak{D}$ commutes $\Rightarrow$ diagram $\Phi(\mathfrak{D})$ commutes.



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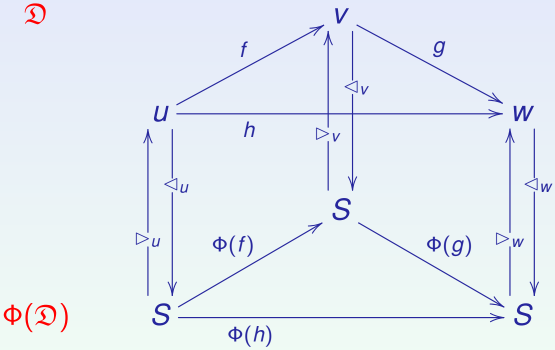
As above, so below

In this case ...

diagram $\mathfrak{D}$ commutes $\Leftrightarrow$ diagram $\Phi(\mathfrak{D})$ commutes.



To arrive where we started . . .

A monogenic category:

- **The generating object:** natural numbers $\mathbb{N}$.
- **The arrows** bijective functions.
- **The tensor** disjoint union $A \uplus B = A \times \{0\} \cup B \times \{1\}$.

The self-similar structure:

$$\mathbb{N} \underset{c^{-1}}{\overset{c}{\leftrightarrows}} \mathbb{N} \uplus \mathbb{N}$$

Based on the familiar **Cantor pairing** $c(n, i) = 2n + i$.

Let us strictify this self-similar structure.

The end is where we started from

The Cantor monoid:

The object	The natural numbers $\mathbb{N}$
The arrows	All bijections $\mathbb{N} \to \mathbb{N}$
The tensor	$(f \star g)(n) = \begin{cases} 2.f\left(\frac{n}{2}\right) & n \text{ even}, \\ 2.g\left(\frac{n-1}{2}\right)+1 & n \text{ odd}. \end{cases}$
The associativity isomorphism	$\tau(n) = \begin{cases} 2n & n\ (mod\ 2) = 0, \\ n+1 & n\ (mod\ 4) = 1, \\ \frac{n-3}{2} & n\ (mod\ 4) = 3. \end{cases}$
The symmetry isomorphism	$\sigma(n) = \begin{cases} n+1 & n \text{ even}, \\ n-1 & n \text{ odd}. \end{cases}$