Partial Recursive Functions and Finality

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Outline

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Motivation

- Domain equations

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- Initiality $\Rightarrow$ Primitive Recursion

  Finality $\Rightarrow$ Kleene's μ-Recursion
- So there should be a categorical account of the partial recursive functions

Sets and partial functions

The category **pSet** of sets and partial functions $f : X \rightharpoonup Y$ has:

- A “tensor" functor given by cartesian product on objects, and on partial functions $f : X \rightharpoonup X', g : Y \rightharpoonup Y'$ by:

  $$(f \times g)(x, y) \simeq \langle fx, gy \rangle$$

  The one-point sets $\mathbb{1}$ functions as an identity for cartesian product.
- Distributive binary sums, where, as before

  $$X + Y = (\{0\} \times X) + (\{1\} \times Y)$$

(Remark: **pSet** does have finite binary products:

$$1 = \emptyset \qquad\qquad X \times Y = X + (X \times Y) + Y$$

but they don’t help.)

pSet has a final $(\mathrm{I} + -)$-coalgebra

$$
\begin{array}{ccc}
Y & \xrightarrow{\beta} & \mathbb{1} + Y \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \mathbb{1} + h} \\
\mathbb{N} & \xrightarrow[\alpha^{-1}]{} & \mathbb{1} + \mathbb{N}
\end{array}
$$

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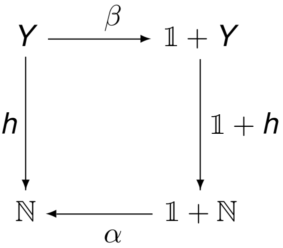

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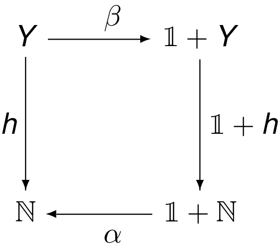

Using Kleene equality we can write this out as an equation:

$$h(y) \simeq \begin{cases} 0 & (\beta(y) \simeq \mathrm{inl}(*)) \\ h(y') + 1 & (\beta(y) \simeq \mathrm{inr}(y')) \\ \text{undefined} & (\beta(y)\uparrow) \end{cases}$$

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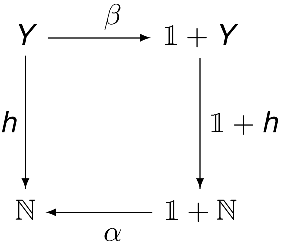

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with, setting $s =_{\mathrm{def}} \mathrm{inr}^{-1} \circ \beta$, unique solution

$$h(y) \simeq_{\mathrm{def}} \mu k \in \mathbb{N}.\ \beta(s^k(y)) \simeq \mathrm{inl}(*)$$

Outline

Natural numbers algebras

- Context a monoidal category **C**
  - Unit: I Tensor Product of objects: $A \otimes B$
  - Tensor product of morphisms:

$$\frac{A \xrightarrow{f} A' \qquad B \xrightarrow{g} B'}{A \otimes B \xrightarrow{f \otimes g} A' \otimes B'}$$

  - Structural isomorphisms:

$$\mathrm{a}_{A,B,C} : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C \qquad \mathrm{l}_A : \mathrm{I} \otimes A \cong A \qquad \mathrm{r}_A : A \otimes \mathrm{I} \cong A$$

    satisfying standard equations.

- and a natural numbers algebra

$$\mathrm{I} \xrightarrow{\mathrm{zero}} \mathrm{N} \xleftarrow{\mathrm{succ}} \mathrm{N}$$

Representing natural numbers functions

- Natural numbers algebra

$$\mathrm{I} \xrightarrow{\text{zero}} \mathrm{N} \xleftarrow{\text{succ}} \mathrm{N}$$

- The morphism $\underline{k} =_{\text{def}} \text{succ}^k \circ \text{zero} : \mathrm{I} \to \mathrm{N}$ represents $k \in \mathbb{N}$.
- The morphism $\underline{f} : \mathrm{N}^n \to \mathrm{N}^m$ *represents* $f : \mathbb{N}^n \to \mathbb{N}^m$ if

$$\underline{f} \circ \langle \underline{k_1}, \ldots, \underline{k_n} \rangle = \underline{f(k_1, \ldots, k_n)} \qquad \text{(for all } k_1, \ldots, k_n \in \mathbb{N})$$

(where we define $\langle c_1, \ldots, c_n \rangle : \mathrm{I} \multimap A_1 \otimes \ldots \otimes A_n$ to be:

$$\mathrm{I} \longrightarrow \mathrm{I} \otimes \ldots \otimes \mathrm{I} \xrightarrow{c_1 \otimes \ldots \otimes c_n} A_1 \otimes \ldots \otimes A_n$$

for $c_i : \mathrm{I} \multimap A_i$.)

Representing natural numbers functions (cntnd)

Successor and zero are representable. The representable functions are closed under *this* composition:

$$\mathbb{N}^l \xrightarrow{f} \mathbb{N}^m \xrightarrow{g} \mathbb{N}^n$$

and *product*:

$$\frac{\mathbb{N}^m \xrightarrow{f} \mathbb{N}^{m'} \qquad \mathbb{N}^n \xrightarrow{g} \mathbb{N}^{n'}}{\mathbb{N}^{m+n} \xrightarrow{f \times g} \mathbb{N}^{m'+n'}}$$

Note: There is no a priori reason why the projections should be representable nor why the representable functions should be closed under ordinary (= cartesian) composition.

Natural numbers objects

For any

$$\mathrm{I} \xrightarrow{a} B \xleftarrow{f} B$$

there is a unique map $h : \mathrm{N} \to B$ such that the following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{I} & \xrightarrow{\mathrm{zero}} & \mathrm{N} & \xrightarrow{\mathrm{succ}} & \mathrm{N} \\
\downarrow{\scriptstyle \mathrm{I}} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\
\mathrm{I} & \xrightarrow[a]{} & B & \xrightarrow[f]{} & B
\end{array}$$

For weak natural numbers object, drop uniqueness

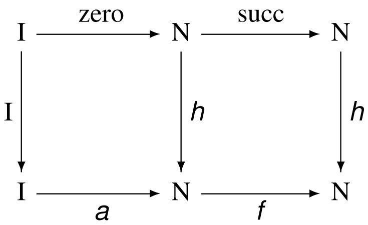
Pure Iteration

$$\begin{array}{ccccc}
\mathrm{I} & \xrightarrow{\mathrm{zero}} & \mathrm{N} & \xrightarrow{\mathrm{succ}} & \mathrm{N} \\
\mathrm{I}\downarrow & & \downarrow h & & \downarrow h \\
\mathrm{I} & \xrightarrow[a]{} & \mathrm{N} & \xrightarrow[f]{} & \mathrm{N}
\end{array}$$

So:

$$h\circ\underline{0} = a \qquad h\circ\underline{k+1} = f\circ h\circ\underline{k}$$

Pure Iteration



So:

$$h \circ \underline{0} = a \qquad h \circ \underline{k+1} = f \circ h \circ \underline{k}$$

Given $a \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$, pure iteration yields $h : \mathbb{N} \to \mathbb{N}$

$$h(0) = a \qquad h(k+1) = f(h(k))$$

Pure Iteration

$$\begin{array}{ccccc} \mathrm{I} & \xrightarrow{\mathrm{zero}} & \mathrm{N} & \xrightarrow{\mathrm{succ}} & \mathrm{N} \\ \downarrow{\scriptstyle \mathrm{I}} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ \mathrm{I} & \xrightarrow[a]{} & \mathrm{N} & \xrightarrow[f]{} & \mathrm{N} \end{array}$$

So:

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$$h(0) = a \qquad h(k+1) = f(h(k))$$

So then the representable functions are closed under pure iteration.

Right Stable Natural numbers objects

(Paré and Román) For any

$$P \xrightarrow{f} B \xleftarrow{g} B$$

there is a unique morphism $h\colon P \otimes \mathrm{N} \to B$ such that the following diagram commutes:

$$\begin{array}{ccccc}
P & \xrightarrow{(P \otimes \mathrm{zero}) \circ r_P^{-1}} & P \otimes \mathrm{N} & \xrightarrow{P \otimes \mathrm{succ}} & P \otimes \mathrm{N} \\
{\scriptstyle P}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\
P & \xrightarrow[f]{} & B & \xrightarrow[g]{} & B
\end{array}$$

For weak right stable natural numbers object, drop uniqueness

A recursion scheme

- Taking $P = \mathrm{N}^n, B = \mathrm{N}^m$ we find that if $f : \mathbb{N}^n \to \mathbb{N}^m$ and $g : \mathbb{N}^m \to \mathbb{N}^m$, are representable then so is $h : \mathbb{N}^{n+1} \to \mathbb{N}^m$ where:

$$\begin{array}{rcl} h(k_1, \ldots, k_n, 0) & = & f(k_1, \ldots, k_n) \\ h(k_1, \ldots, k_n, k+1) & = & g(h(k_1, \ldots, k_n, k)) \end{array}$$

- So the representable functions $\mathbb{N}^n \to \mathbb{N}^m$ on natural numbers are closed under pure iteration with n parameters and m outputs.
- Example
  - Take $m = n$, and $f = g = \mathrm{id}_{\mathbb{N}^n}$.
  - Get that $\langle k_1, \ldots, k_n, k_{n+1} \rangle \mapsto \langle k_1, \ldots, k_n \rangle$ is representable.
  - Composing, find π^n_1 is representable.

A structural map: symmetry

- Define $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N}^2 \to \mathbb{N}^2$ by:

$$f(k) = \langle 0, k \rangle \qquad g(l, k) = \langle l + 1, k \rangle$$

- They are represented by:

$$\mathrm{N} \cong \mathrm{I} \otimes \mathrm{N} \xrightarrow{\underline{0} \otimes \mathrm{id}_{\mathrm{N}}} \mathrm{N} \otimes \mathrm{N} \qquad \mathrm{N} \otimes \mathrm{N} \xrightarrow{\mathrm{succ} \otimes \mathrm{id}_{\mathrm{N}}} \otimes \mathrm{N}$$

- Then the symmetry map $\langle k, l \rangle \mapsto \langle l, k \rangle$ is defined by:

$$\begin{array}{lcl} h(k, 0) & = & f(k) \\ h(k, l + 1) & = & g(h(k, l)) \end{array}$$

- With that, composition, and product one represents the general permutation map:

$$\langle k_1, \ldots, k_n \rangle \mapsto \langle k_{\pi(1)}, \ldots, k_{\pi(n)} \rangle$$

Representing cartesian compositions

- Three successive diagonal maps:

$$\mathbb{N} \xrightarrow{\Delta} \mathbb{N}^2 \qquad \mathbb{N} \xrightarrow{\Delta_n} \mathbb{N}^n \qquad \mathbb{N}^m \xrightarrow{\Delta_{m,n}} \mathbb{N}^{mn}$$

- Cartesian composition

$$\frac{\mathbb{N}^n \xrightarrow{f_i} \mathbb{N} \ \ (i = 1, m) \qquad \mathbb{N}^m \xrightarrow{g} \mathbb{N}}{\mathbb{N}^m \xrightarrow{\Delta_{m,n}} \mathbb{N}^{mn} \xrightarrow{f_1 \times \ldots \times f_m} \mathbb{N}^n \xrightarrow{g} \mathbb{N}}$$

Primitive recursive functions in monoidal categories

Theorem (Gladstone, 1971)

The primitive recursive functions form the least class of functions (with one output) containing zero, successor, and the projections that is closed under composition and pure iteration with one parameter (and one output).

Theorem

All primitive recursive functions are representable in any monoidal category with a weak right stable natural numbers object.

Outline

Categorical context

A monoidal category **C** with:

- binary sums $+$, right distributive over $\otimes$, i.e.:

$$\mathrm{d} : B \otimes A + C \otimes A \cong (B + C) \otimes A$$

- a weak right stable natural numbers object

$$\mathrm{I} \xrightarrow{\mathrm{zero}} \mathrm{N} \xleftarrow{\mathrm{succ}} \mathrm{N}$$

  - such that

$$\alpha =_{\mathrm{def}} [\mathrm{zero}, \mathrm{succ}] : \mathrm{I} + \mathrm{N} \longrightarrow \mathrm{N}$$

    is an isomorphism
  - and the coalgebra

$$\mathrm{N} \xrightarrow{\alpha^{-1}} \mathrm{I} + \mathrm{N}$$

    is weakly final.

Representing functions

- Total functions $\underline{f} : \mathrm{N}^n \to \mathrm{N}^m$ *represents* $f : \mathbb{N}^n \to \mathbb{N}^m$ if:

$$\underline{f} \circ \langle \underline{k_1}, \ldots, \underline{k_n} \rangle = \underline{f(k_1, \ldots, k_n)} \qquad (k_1, \ldots, k_n \in \mathbb{N})$$

- Partial functions $\underline{f} : \mathrm{N}^n \to \mathrm{N}$ *weakly represents* $f : \mathbb{N}^n \rightharpoonup \mathbb{N}$ if:

$$f(k_1, \ldots, k_n) \simeq l \;\; \Rightarrow \;\; \underline{f} \circ \langle \underline{k_1}, \ldots, \underline{k_n} \rangle = \underline{l} \qquad (k_1, \ldots, k_n, l \in \mathbb{N})$$

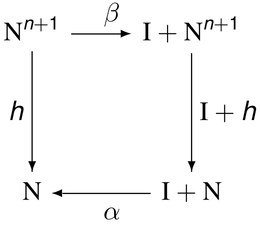
$\underline{f} : \mathrm{N}^n \to \mathrm{N}$ *strongly represents* $f : \mathbb{N}^n \rightharpoonup \mathbb{N}$ if:

$$f(k_1, \ldots, k_n) \simeq l \;\; \Leftrightarrow \;\; \underline{f} \circ \langle \underline{k_1}, \ldots, \underline{k_n} \rangle = \underline{l} \qquad (k_1, \ldots, k_n, l \in \mathbb{N})$$

Kleene's μ-recursion

$$
\begin{array}{ccc}
\mathrm{N}^{n+1} & \xrightarrow{\ \beta\ } & \mathrm{I} + \mathrm{N}^{n+1} \\
\Big\downarrow h & & \Big\downarrow \mathrm{I} + h \\
\mathrm{N} & \xrightarrow[\ \alpha^{-1}\ ]{} & \mathrm{I} + \mathrm{N}
\end{array}
$$

Kleene's μ-recursion



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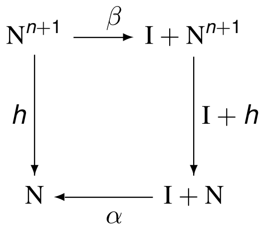

Coalgebra

Let $\underline{P} : \mathrm{N}^{n+1} \to \mathrm{N}$ weakly represent $P : \mathbb{N}^{n+1} \rightharpoonup \mathbb{N}$, and define β so that:

$$\beta \circ \langle \underline{k_1}, \ldots, \underline{k_n}, \underline{k} \rangle = \mathrm{inl} \qquad (\underline{P} \circ \langle \underline{k_1}, \ldots, \underline{k_n}, \underline{k} \rangle = \underline{0})$$
$$\beta \circ \langle \underline{k_1}, \ldots, \underline{k_n}, \underline{k} \rangle = \mathrm{inr} \circ \langle \underline{k_1}, \ldots, \underline{k_n}, \underline{k+1} \rangle \qquad (\underline{P} \circ \langle \underline{k_1}, \ldots, \underline{k_n}, \underline{k} \rangle = \underline{k' + 1})$$

Kleene's μ-recursion

$$\begin{array}{ccc} \mathrm{N}^{n+1} & \xrightarrow{\beta} & \mathrm{I} + \mathrm{N}^{n+1} \\ \Big\downarrow h & & \Big\downarrow \mathrm{I} + h \\ \mathrm{N} & \xleftarrow[\alpha]{} & \mathrm{I} + \mathrm{N} \end{array}$$

Coalgebra

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Morphism

$$\begin{array}{ll} \underline{h} \circ \langle \underline{k_1}, \ldots, \underline{k_n}, \underline{k} \rangle = \underline{0} & (\underline{P} \circ \langle \underline{k_1}, \ldots, \underline{k_n}, \underline{k} \rangle = \underline{0}) \\ \underline{h} \circ \langle \underline{k_1}, \ldots, \underline{k_n}, \underline{k} \rangle = \mathrm{succ} \circ \underline{h} \circ \langle \underline{k_1}, \ldots, \underline{k_n}, \underline{k+1} \rangle & (\underline{P} \circ \langle \underline{k_1}, \ldots, \underline{k_n}, \underline{k} \rangle = \underline{k'+1}) \end{array}$$

Kleene's μ-recursion (cntnd)

Morphism

$$\begin{array}{ll} \underline{h} \circ \langle \underline{\mathbf{k}}, \underline{k} \rangle = \underline{0} & (\text{if } \underline{P} \circ \langle \underline{\mathbf{k}}, \underline{k} \rangle = \underline{0}) \\ \underline{h} \circ \langle \underline{\mathbf{k}}, \underline{k} \rangle = \mathrm{succ} \circ \underline{h} \circ \langle \underline{\mathbf{k}}, \underline{k+1} \rangle & (\text{if } \underline{P} \circ \langle \underline{\mathbf{k}}, \underline{k} \rangle = \underline{k'+1}) \end{array}$$

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Then, by induction on l:

$$\mu k'.\,(P(\mathbf{k},k{+}k') \simeq 0 \wedge \forall k'' < k'.\, P(\mathbf{k},k{+}k'')\downarrow) \;\simeq\; l \quad\Rightarrow\quad h\circ\langle\underline{\mathbf{k}},\underline{k}\rangle \;=\; \underline{l}$$

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Then $\underline{h}$ weakly represents $h : \mathbb{N}^{n+1} \rightharpoonup \mathbb{N}$, where:

$$h(\mathbf{k}, k) \quad \simeq_{\mathrm{def}} \quad \mu k'.\, P(\mathbf{k}, k + k') \simeq 0 \;\wedge\; \forall k'' < k'.\, P(\mathbf{k}, k + k'')\downarrow$$

Kleene's μ-recursion (cntnd)

Morphism

$$\begin{array}{lll} \underline{h}\circ\langle \underline{\mathbf{k}}, \underline{k}\rangle = \underline{0} & & (\text{if } \underline{P}\circ\langle \underline{\mathbf{k}}, \underline{k}\rangle = \underline{0}) \\ \underline{h}\circ\langle \underline{\mathbf{k}}, \underline{k}\rangle = \mathrm{succ}\circ\underline{h}\circ\langle \underline{\mathbf{k}}, \underline{k+1}\rangle & & (\text{if } \underline{P}\circ\langle \underline{\mathbf{k}}, \underline{k}\rangle = \underline{k'+1}) \end{array}$$

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Then $\underline{h} \circ (\mathrm{N}^k \otimes \mathrm{zero})$ weakly represents $g : \mathbb{N}^n \rightharpoonup \mathbb{N}$, where:

$$g(\mathbf{k}) \quad \simeq_{\text{def}} \quad \mu k.\, P(\mathbf{k}, k) \simeq 0 \;\wedge\; \forall k' < k.\, P(\mathbf{k}, k')\downarrow$$

Weak representability theorem

Theorem

Let **C** *be a monoidal category with (right distributive) binary sums and a weak left (or right) natural numbers object* $\mathrm{I} \xrightarrow{\mathrm{zero}} \mathrm{N} \xleftarrow{\mathrm{succ}} \mathrm{N}$ *such that* $[\mathrm{zero}, \mathrm{succ}]$ *is an isomorphism and* $(\mathrm{N}, [\mathrm{zero}, \mathrm{succ}]^{-1})$ *is a weakly final natural numbers coalgebra.*

Then all partial recursive functions are weakly representable in **C**.

Kleisli categories

Let **C** be a cartesian category and let $T : \mathbf{C} \to \mathbf{C}$ be a commutative monad. Then:

- $\mathbf{C}_T$ is a symmetric monoidal category with inherited tensors: $\mathrm{I} = 1$ and $A \otimes B = A \times B$,
- if **C** has (distributive) binary sums, then so does $\mathbf{C}_T$, and
- if $\mathrm{I} \xrightarrow{\text{zero}} \mathrm{N} \xleftarrow{\text{succ}} \mathrm{N}$ is a (weakly) stable natural numbers object in **C**, then it is also one in $\mathbf{C}_T$.

Example If **C** is a cartesian category with distributive binary sums, then $- + 1$ is a commutative monad on **C**.

Representabiity in Kleisli categories

Corollary

Let $T : \mathbf{C} \to \mathbf{C}$ *be a commutative monad on a cartesian category with a distributive binary sum, and let* $1 \xrightarrow{\mathrm{zero}} \mathrm{N} \xleftarrow{\mathrm{succ}} \mathrm{N}$ *be a weakly stable natural numbers object in* $\mathbf{C}$ *(and so in* $\mathbf{C}_T$*) such that* $[\mathrm{zero}, \mathrm{succ}]$ *is an isomorphism and such that* $(\mathrm{N}, [\mathrm{zero}, \mathrm{succ}]^{-1})$ *is a final* $(1 + -)$*-coalgebra in* $\mathbf{C}_T$*.*

Then all partial recursive functions are weakly representable in $\mathbf{C}_T$*.*

Outline

Strong representability theorem

Theorem

Let **C** *be a monoidal category with (right distributive) binary sums and a weak left (or right) natural numbers object* $\mathrm{I} \xrightarrow{\mathrm{zero}} \mathrm{N} \xleftarrow{\mathrm{succ}} \mathrm{N}$ *such that* $[\mathrm{zero}, \mathrm{succ}]$ *is an isomorphism and* $(\mathrm{N}, [\mathrm{zero}, \mathrm{succ}]^{-1})$ *is a weakly final natural numbers coalgebra.*

Then all partial recursive functions with recursive graphs are strongly representable in **C** *(assuming that* $\mathrm{succ} \circ c \neq \mathrm{zero}$*, for all* $c : \mathrm{I} \to \mathrm{N}$*).*

Proof

Suppose that $f : \mathbb{N}^n \to \mathbb{N}$ has recursive graph $P \subseteq \mathbb{N}^{n+1}$.
It suffices to show $g : \mathbb{N}^{n+1} \to \mathbb{N}$ strongly representable, where

$$g(\mathbf{k}, k) \simeq_{\mathrm{def}} \mu k'.\, P(\mathbf{k}, k + k')$$

We have a weak representation $\underline{g} : \mathrm{N}^{n+1} \to \mathrm{N}$ of g, where

$$\begin{array}{ll} \underline{g} \circ \langle \underline{\mathbf{k}}, \underline{k} \rangle = \underline{0} & (\text{if } P(\mathbf{k}, k)) \\ \underline{g} \circ \langle \underline{\mathbf{k}}, \underline{k} \rangle = \mathrm{succ} \circ \underline{g} \circ \langle \underline{\mathbf{k}}, \underline{k+1} \rangle & (\text{otherwise}) \end{array}$$

One then shows by induction on l that:

$$\forall k.\, \underline{g} \circ \langle \underline{\mathbf{k}}, \underline{k} \rangle = \underline{l} \Rightarrow g(\mathbf{k}, k) \simeq l$$

In each case one uses the assumption and the totality of P to obtain the appropriate one of the above two equations for $\underline{g}$.
In the second case one uses the fact that succ has a left inverse.

A counterexample

Theorem

There is a symmetric monoidal category **C** *with distributive binary sums and a natural numbers object* $\mathrm{I} \xrightarrow{\mathrm{zero}} \mathrm{N} \xleftarrow{\mathrm{succ}} \mathrm{N}$ *such that* $[\mathrm{zero}, \mathrm{succ}]$ *is an isomorphism and* $(\mathrm{N}, [\mathrm{zero}, \mathrm{succ}]^{-1})$ *is a final natural numbers coalgebra, and in which* $\mathrm{succ} \circ c \neq \mathrm{zero}$, *for all* $c : \mathrm{I} \to \mathrm{N}$, *but whose only strongly representable partial recursive functions are those with a recursive graph.*

Syntactic category of a logical theory **T**: relations

Let $\varphi(z)$ and $\psi(z)$ be **T**-formulas with only possible free variable z, and let $\gamma(x, y)$ be one with only possible free variables x and y.

- γ is a **T**-*relation from* φ *to* ψ if:

$$\vdash_{\mathbf{T}} \varphi(x) \wedge \gamma(x, y) \Rightarrow \psi(y)$$

- it is **T**-*function from* φ *to* ψ if, in addition:

$$\vdash_{\mathbf{T}} \varphi(x) \wedge \gamma(x, y) \wedge \gamma(x, y') \Rightarrow y = y'$$

- it is *a total* **T**-*function from* φ *to* ψ if, further:

$$\vdash_{\mathbf{T}} \varphi(x) \Rightarrow \exists y.\, \gamma(x, y)$$

Syntactic categories **pC** and **C** of a logical theory **T**

- Objects of **pC**: formulas $\varphi(z)$ whose only possible free variable is z.
- Morphisms of **pC** from φ to ψ: Equivalence classes of **T**-functions from φ to ψ where:

$$\gamma \sim \gamma' \quad \equiv \vdash_{\mathbf{T}} \varphi(x) \Rightarrow (\gamma(x, y) \Leftrightarrow \gamma'(x, y))$$

- Identity and Composition:

$$\mathrm{id}_{\varphi} = [y = x] \qquad [\delta] \circ [\gamma] = [\exists w.\, \gamma(x, w) \wedge \delta(w, y)]$$

And **C** is the subcategory of the total **T**-functions.

The total category **C** when **T** extends **PA**

- **C** is a cartesian category with
  - Products: $1 = (z = 0)$ $\qquad \varphi \times \psi = (\varphi(\pi_1(z)) \wedge \psi(\pi_2(z))$ (using a surjective pairing function)
  - a distributive binary sum:

$$\varphi + \psi = (\exists w.\, z = 2w \wedge \varphi(w)) \vee (\exists w.\, z = 2w + 1 \wedge \psi(w))$$

- and with a stable natural numbers object:

$$1 \xrightarrow{\mathrm{zero}} \mathrm{N} \xleftarrow{\mathrm{succ}} \mathrm{N}$$

  where N is $\top$, zero is $[y = 0]$, and succ is $[y = \mathrm{s}(x)]$, such that

- $\mathrm{succ} \circ c \neq \mathrm{zero}$, for all $c\colon 1 \to \mathrm{N}$ (if **T** is consistent).

The partial category **pC** when **T** extends **PA**

- **pC** is a symmetric monoidal category with
  - Tensors: $\mathrm{I} = 1$ and $\varphi \otimes \psi = \varphi \times \psi$, with structural maps inherited from the total category, and with

$$[\gamma] \otimes [\delta] = [\gamma(\pi_1(x), \pi_1(y)) \wedge \delta(\pi_2(x), \pi_2(y))]$$

  - a distributive binary sum, the same as in the total category:

$$\varphi + \psi = (\exists w.\, z = 2w \wedge \varphi(w)) \vee (\exists w.\, z = 2w + 1 \wedge \psi(w))$$

- and with a stable natural numbers object:

$$\mathrm{I} \xrightarrow{\mathrm{zero}} \mathrm{N} \xleftarrow{\mathrm{succ}} \mathrm{N}$$

  inherited from the total category, such that

- $$[\mathrm{zero}, \mathrm{succ}]^{-1} : \mathrm{N} \to \mathrm{I} + \mathrm{N}$$

  is a final coalgebra.
- $\mathrm{succ} \circ c \neq \mathrm{zero}$, for all $c\colon 1 \to \mathrm{N}$ (if **T** is consistent)

Semi-representability of arithmetic relations

A formula $\chi(x_1, \ldots, x_n)$ *semi-represents* a relation $R \subseteq \mathbb{N}^n$ in an extension **T** of **PA**, if, for all $k_1, \ldots, k_n$, $R(k_1, \ldots, k_n)$ holds if, and only if, $\vdash_{\mathbf{T}} \chi(\underline{k_1}, \ldots, \underline{k_n})$ does.

Theorem (Jockusch and Soare)

There is a complete consistent extension of **PA** *in which the only semi-representable relations are either recursive or non-arithmetical.*

The counterexample

Theorem

There is a symmetric monoidal category **C** *with distributive binary sums and a natural numbers object* $\mathrm{I} \xrightarrow{\mathrm{zero}} \mathrm{N} \xleftarrow{\mathrm{succ}} \mathrm{N}$ *such that* $[\mathrm{zero}, \mathrm{succ}]$ *is an isomorphism and* $(\mathrm{N}, [\mathrm{zero}, \mathrm{succ}]^{-1})$ *is a final natural numbers coalgebra, and in which* $\mathrm{succ} \circ c \neq \mathrm{zero}$*, for all* $c : \mathrm{I} \to \mathrm{N}$*, but whose only strongly representable partial recursive functions are those with a recursive graph.*

Construction of the counterexample

The counterexample is the category **pC** constructed as above, using the Jockusch and Soare theory **JS**.

Suppose $f : \mathbb{N}^n \rightharpoonup \mathbb{N}$ is a partial recursive function strongly representable by $[\gamma] : \mathrm{N}^n \to \mathrm{N}$. Then:

$$f(k_1, \ldots, k_n) \simeq k \quad \equiv \quad [\gamma] \circ \langle \underline{k_1}, \ldots, \underline{k_n} \rangle = \underline{k} \quad \equiv \vdash_{\mathbf{JS}} \gamma(\langle \underline{k_1}, \ldots, \underline{k_n} \rangle, \underline{k})$$

So the graph of f is semi-representable in **JS**.

It is therefore recursive, as it is partial recursive and we are in the Jokusch-Soare theory.

Proof of Jockusch Soare

First some notation,where $\varphi(x)$ is a unary formula:

$$[\varphi]_{\mathbf{T}} = \{k \mid \vdash_{\mathbf{T}} \varphi(\underline{k})\} \qquad [\![\varphi]\!] = \{k \mid \models \varphi(k)\}$$

Define a sequence of theories $\mathbf{T}_n$, with $\mathbf{T}_0 = \mathbf{PA}$, and set **JS** to be their union.

- At even stages add either φ or $\neg\varphi$ for the next sentence φ (in some enumeration) keeping consistency.
- At odd stages consider the next pair $(\varphi(x), \psi(x))$ of unary formulas (in some enumeration).
  1. If $[\varphi]_{\mathbf{T}_i} \not\subseteq [\![\psi]\!]$ do nothing (then φ will not semidefine $[\![\psi]\!]$ in **JS**).
  2. If $[\varphi]_{\mathbf{T}_i} = [\![\psi]\!]$ do nothing ($[\![\psi]\!]$ is recursive)
  3. Otherwise add $\neg\varphi(\underline{k})$ for some $k \in [\![\psi]\!] \backslash [\varphi]$

Strong representability under a recursion-theoretic assumption

Assume we are in our standard categorical context , and that $\underline{0} \neq \underline{1}$. Then every morphism $\underline{f}: \mathrm{N}^n \multimap \mathrm{N}$ strongly represents the partial function $f: \mathbb{N}^n \to \mathbb{N}$ where:

$$f(k_1, \ldots, k_n) \simeq l \quad \Leftrightarrow \quad \underline{f} \circ \langle \underline{k_1}, \ldots, \underline{k_n} \rangle = \underline{l} \qquad (k_1, \ldots, k_n, l \in \mathbb{N})$$

and we have:

Theorem

If all strongly representable functions are partial recursive, then all partial recursive functions are strongly representable.

Examples: Free categories, syntactic categories of partial recursive extensions of **PA.**

Proof

(Visser) Any class of unary partial recursive functions that:

1. contains an upper bound of every partial recursive function and
2. is closed under right composition with all total recursive functions

consists of all unary partial recursive functions.