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# GEOMETRY OF INTERACTION AND LINEAR COMBINATORY ALGEBRAS


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PRG-RR-01-10



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January 14, 2002

**Abstract**

We present an axiomatic framework for Girard's Geometry of Interaction, based on the notion of linear combinatory algebra. We give a general construction on traced monoidal categories with certain additional structure, sufficient to capture the exponentials of Linear Logic, which produces such algebras (and hence also ordinary combinatory algebras). We illustrate the construction on six standard examples, representing both "particle-style" as well as "wave-style" Geometry of Interaction.

## 1 Introduction

Girard's Geometry of Interaction [Girard 1989, Girard 1990, Girard 1995] is a strikingly original interpretation of Linear Logic [Girard 1987] (and hence, via standard embeddings, of Intuitionistic and Classical Logic), in which cut-elimination is modelled as a dynamical process of information flow. This interpretation was extensively studied in [Danos and Regnier 1993, Danos and Regnier 1995], applied to the analysis of optimal reduction by Gonthier, Asperti and Lévy [Gonthier *et al.* 1992], and to other execution mechanisms in [Mackie 1998], and more recently used in the analysis of Elementary Linear Logic [Baillot and Pedicini 2000]. However, some quite basic questions about the Geometry of Interaction have remained in need of clarification. We focus on two:

- What is the general axiomatic framework for the Geometry of Interaction, as opposed to the specific models considered in the literature?
- In what sense exactly does the Geometry of Interaction yield a model of Linear Logic at all, given that, as already noted by Girard in [Girard 1989], if a proof $\Pi$ reduces to $\Pi'$ under cut-elimination, it is *not* the case in general that $\Pi$ and $\Pi'$ will receive the same interpretation in the Geometry of Interaction.

### 1.1 Axiomatics

As regards the first question, already in [Abramsky and Jagadeesan 1994b] a model was given that clearly exemplified the same general structure as the original Geometry of Interaction interpretation, while being of a significantly different nature. While the "dynamic algebras" of Danos and Regnier do provide some axiomatic generality, they do not apply to the Abramsky-Jagadeesan style models. Conversely, Abramsky

*Research supported in part by EPSRC
†Research supported in part by an NSERC Postdoctoral Fellowship Grant.
‡Research supported in part by an NSERC Operating Grant.

and Jagadeesan did give a categorical axiomatics for their model in [Abramsky and Jagadeesan 1994b], but this did not cover the Girard-style models. A key step was taken with the appearance of the paper on Traced Monoidal Categories [Joyal *et al.* 1996]. The motivation for this paper was quite different, stemming from the axiomatics of categories of tangles (although the authors were aware of possible connections to iteration theories. In fact, similar axiomatics in the symmetric case, motivated by flowcharts and "flownomials" had been developed some years earlier by Stefanescu [Stefanescu 2000].) However, the first author realized, following a stimulating discussion with Gordon Plotkin, that traced monoidal categories provided a common denominator for the axiomatics of both the Girard-style and Abramsky-Jagadeesan-style versions of the Geometry of Interaction, at the basic level of the multiplicatives. This insight was presented in [Abramsky 1996], in which Girard-style GoI was dubbed "particle-style", since it concerns information particles or tokens flowing around a network, while the Abramsky-Jagadeesan style GoI was dubbed "wave-style", since it concerns the evolution of a global information state or "wave". Formally, this distinction is based on whether the tensor product (i.e. the symmetric monoidal structure) in the underlying category is interpreted as a coproduct (particle style) or as a product (wave style). This computational distinction between coproduct and product interpretations of the same underlying network geometry turned out to have been partially anticipated, in a rather different context, in a pioneering paper by E. S. Bainbridge [Bainbridge 1976], as observed by Dusko Pavlovic. These two forms of interpretation, and ways of combining them, have also been studied recently in [Stefanescu 2000]. He uses the terminology "additive" for coproduct-based (i.e. our "particle-style") and "multiplicative" for product-based (i.e. our "wave-style"); this is not suitable for our purposes, because of the clash with Linear Logic terminology.

Relevant to the wave-style interpretation of GoI was the observation, made independently by Hasegawa [Hasegawa 1997] and Hyland, that when the tensor is the categorical product, the trace axioms are equivalent to standard axioms for a fixpoint operator. Dually, with the tensor product being coproduct, the trace axioms are equivalent to standard axioms for the iteration (dagger) operation in iteration theories [Haghverdi 2000a]. On the other hand, the work of Peter Hines [Hines 1997, Hines 1999] is a substantial contribution to the categorical axiomatization of particle-style GoI, and to developing connections between it and the theory of inverse semigroups [Lawson 1998], and also with other models of computation [Hines 2000].

Another contribution of [Abramsky 1996] was to give a range of examples of traced monoidal categories, including *resumptions* (a stateful version of particle-style GoI), and *stochastic relations* (a probabilistic version).

What remained to be clarified following [Abramsky 1996] was how to extend the axiomatics to incorporate the exponentials of Linear Logic. Traced symmetric monoidal categories and the $\mathcal{G}$ construction only account for the GoI interpretation of the *multiplicative* connectives. The technical subtleties of GoI arise in the interpretation of the exponentials. The first main contribution of the present paper is to answer this question in terms of what we call a *Geometry of Interaction situation*, which is a traced symmetric monoidal category equipped with certain additional structure which allows the exponentials to be interpreted in exactly the GoI fashion. This additional structure is fairly simple: the main ingredient is a strong monoidal endofunctor on the category, which will be given by the countable co-power in particle-style models, and by countable power in the wave-style models. This, together with given retractions which allow the arbitrary splitting of an infinite "address space", enable the characteristic GoI interpretation of contraction by copying to be expressed.

We also show that the various models described in [Abramsky 1996] extend to GoI situations, and hence that the full GoI construction and the corresponding interpretation of Multiplicative-Exponential Linear Logic can be carried out for each of them.

## 1.2 Models of what?

As regards the second question, we offer a robust notion, of independent interest, which captures those equational properties which the Geometry of Interaction *does* satisfy. This is the notion of *Linear Combinatory Algebra*, which stands in the same relation to the Hilbert-style axiomatization of the $!, \multimap$ fragment of Linear Logic as standard combinatory algebras do to the implicational fragment of Intuitionistic (or Minimal) logic. The multiplicative part of Linear Combinatory Algebras in fact coincides with the well-known notion of BCI-algebra [Hindley 1997, Troelstra 1992]. The main novelty is that, in addition to the binary operation of application, which corresponds to the inference rule of Modus Ponens, there is also a unary operator (written

! by abuse of notation) corresponding to the Modal inference rule of Necessitation

$$\frac{A}{\Box A}$$

or in Linear Logic terms, seeing ! as an S4 necessitation modality,

$$\frac{A}{!A}.$$

See [Avron 1988, Troelstra 1992] for a Hilbert-style axiomatization of Linear Logic containing this rule.

**Aside** The same idea could be applied to the implicational fragment of Intuitionistic S4 to yield a combinatory equivalent; this might be interesting, for example, as a basis for realizability models of modal set theory, or as a combinatory equivalent of a version of Moggi's computational $\lambda$-calculus [Moggi 1989].

Conceptually, combinatory algebras are algebras of *closed terms*; they are "intensional" in that they satisfy weak equational properties, while still being expressive: for example, the partial recursive functions can be interpreted in any (standard) combinatory algebra. Combinatory algebras are also of interest because of their rôle in realizability semantics, which can be seen as mediating between intensional notions of computation, and extensional mathematical universes.

As regards the Geometry of Interaction, Linear Combinatory Algebras appear to be the right notion, which captures the fact that GoI does model $\lambda$-calculus well in computational terms, while not giving rise to a $\lambda$-model, or even a $\lambda$-algebra [Barendregt 1984]. One basic point is that, corresponding to the interpretation of Intuitionistic Logic in Linear Logic, every Linear Combinatory Algebra gives rise to a (standard) combinatory algebra—or, equivalently, Combinatory Logic can be interpreted in Linear Combinatory Logic.

The main result of the present paper is that *every GoI situation gives rise to a Linear Combinatory Algebra*. We actually proceed via an intermediate step: we show that every GoI situation gives rise to a *Weak Linear Category*; and that every Weak Linear Category gives rise to a Linear Combinatory Algebra. Weak Linear Categories are simply Linear Categories with various equational properties subtracted. They are introduced for reasons of technical convenience, rather than as interesting objects in their own right. A consequence of this result is that our range of examples of GoI situations all give rise to Linear Combinatory Algebras.

This paper provides the general framework and technical background to a series of later developments based on these techniques. For example, the first author and John Longley have shown that one of our examples gives rise via realizability to a fully abstract and universal model of PCF [Abramsky and Longley 2000]; with Marina Lenisa, he has shown that realizability over an LCA of partial involutions gives rise to a model of System $\mathcal{F}$ which is fully complete for ML types [Abramsky and Lenisa 2000]; and he has also shown that a "wave-style" LCA based on continuous functions gives rise via realizability to the Scott model of PCF, while the sub-LCA of sequential functions gives rise to the fully abstract model [Abramsky and Longley 2000]. Also, with Paul-André Mellies he has shown that a concurrent games model, based on ideas closely related to wave-style GoI, gives rise to a fully complete model of Multiplicative-Additive Linear Logic [Abramsky and Melliès 1999]. The second and third authors studied axiomatics of categories for "particle-style" GoI, connecting the framework below to Arbib-Manes' partially additive semantics [Manes and Arbib 1986], as well as to work of Danos and Regnier mentioned earlier. The details are in the second author's PhD thesis [Haghverdi 2000a] and in [Haghverdi 2000b]. In [Haghverdi 2001] there is a full completeness theorem for the multiplicative fragment (MLL + MIX) for such categories, based on Loader-Hyland-Tan techniques [Loader 1994, Tan 1997]. The common framework provided by the present paper allows these and other such results to be understood in a coherent fashion.

The further structure of this paper is as follows. In Section 2, we present some background on Traced Monoidal Categories and related notions. In Section 3, we present Weak Linear Categories and Linear Combinatory Algebras, and in Section 4 we prove our main result, which can be seen as the general form of the GoI construction:

$$\begin{array}{cl}
\boxed{\text{GoI Situation}} & \\
\Downarrow & \text{construct } \mathcal{G}(\mathbb{C}) \\
\boxed{\text{Weak Linear Category}} & \\
\Downarrow & \text{take } \mathcal{G}(\mathbb{C})(I, V) \\
\boxed{\text{Linear Combinatory Algebra}} & \\
\Downarrow & \text{by standardization} \\
\boxed{\text{Combinatory Algebra}} &
\end{array}$$

In Section 5, we show that the examples described in [Abramsky 1996] all extend to GoI situations, and hence give rise to Linear Combinatory Algebras via the GoI construction. Section 6 concludes.

The main ideas of the present paper were introduced by the first author in lectures given in Siena and Edinburgh in April 1997 [Abramsky 1997]. The detailed elaboration of the results, some of which first appeared in the second author's thesis [Haghverdi 2000a], is the joint work of the three authors. The authors visited BRICS in May 1997, where some of this material was developed. We thank Glynn Winskel and Prakash Panangaden for their encouragement and hospitality.

# 2 Traced Monoidal Categories

Traced monoidal categories, introduced in [Joyal *et al.* 1996], provide a convenient framework for discussing iteration and parametrised feedback in computation. They are general enough to include many previous categorical models of iteration [Bloom and EÉsik 1993, Manes and Arbib 1986, Haghverdi 2000a] as well as newer notions arising from linear logic [Abramsky 1996].

In this section we recall the definition of traced symmetric monoidal categories and state some properties of the trace. We also give a normal form theorem for arrows in traced symmetric monoidal categories.

We assume familiarity with the notions of *monoidal category*, *strict monoidal category* and *(strict) monoidal functor* [Mac Lane 1998]. A *symmetric* monoidal category is a monoidal category equipped with a natural isomorphism $\sigma$ with components $\sigma_{A,B} : A \otimes B \longrightarrow B \otimes A$ such that standard coherence diagrams (see [Mac Lane 1998]) commute.

For readability and without loss of generality we consider strict monoidal categories. It is well known that every monoidal category is equivalent to a strict one [Mac Lane 1998].

A monoidal functor $(F, \varphi, \varphi_I) : \mathbb{C} \longrightarrow \mathbb{D}$ between symmetric monoidal categories is *symmetric* if $\varphi_{B,A}\sigma_{FA,FB} = F\sigma_{A,B}\varphi_{A,B}$.

**Definition 2.1** *A* traced symmetric monoidal category *is a symmetric monoidal category* $(\mathbb{C}, \otimes, I, \sigma)$ *with a family of functions* $\mathtt{Tr}^U_{X,Y} : \mathbb{C}(X \otimes U, Y \otimes U) \longrightarrow \mathbb{C}(X, Y)$ *pictured in Figure 1, called a* trace*, subject to the following conditions:*

- **Natural** *in* $X$, $\mathtt{Tr}^U_{X,Y}(f)g = \mathtt{Tr}^U_{X',Y}(f(g \otimes 1_U))$ *where* $f : X \otimes U \longrightarrow Y \otimes U$, $g : X' \longrightarrow X$,
- **Natural** *in* $Y$, $g\mathtt{Tr}^U_{X,Y}(f) = \mathtt{Tr}^U_{X,Y'}((g \otimes 1_U)f)$ *where* $f : X \otimes U \longrightarrow Y \otimes U$, $g : Y \longrightarrow Y'$,
- **Dinatural** *in* $U$, $\mathtt{Tr}^U_{X,Y}((1_Y \otimes g)f) = \mathtt{Tr}^{U'}_{X,Y}(f(1_X \otimes g))$ *where* $f : X \otimes U \longrightarrow Y \otimes U'$, $g : U' \longrightarrow U$,
- **Vanishing (I,II)**, $\mathtt{Tr}^I_{X,Y}(f) = f$ *and* $\mathtt{Tr}^{U\otimes V}_{X,Y}(g) = \mathtt{Tr}^U_{X,Y}(\mathtt{Tr}^V_{X\otimes U,Y\otimes U}(g))$ *for* $f : X \otimes I \longrightarrow Y \otimes I$ *and* $g : X \otimes U \otimes V \longrightarrow Y \otimes U \otimes V$.

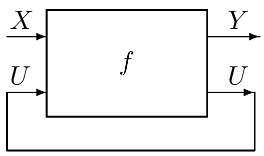



Figure 1: The trace $\mathtt{Tr}^U_{X,Y}(f)$

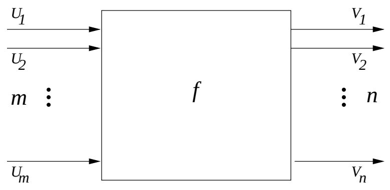



Figure 2: Graphical Representation of $f$

- **Superposing**,
$$g \otimes \mathtt{Tr}^U_{X,Y}(f) = \mathtt{Tr}^U_{W\otimes X, Z\otimes Y}(g \otimes f)$$
*for* $f : X \otimes U \longrightarrow Y \otimes U$ *and* $g : W \longrightarrow Z$ .
- **Yanking**, $\mathtt{Tr}^U_{U,U}(\sigma_{U,U}) = 1_U$.

We think of $\mathtt{Tr}^U_{X,Y}(f)$ as "feedback along $U$", as in Figure 1. The axiomatization of traces given here differs from those given in [Abramsky 1996, Joyal *et al.* 1996]; however, the two versions are equivalent [Haghverdi 2000a].

A monoidal functor $(F, \varphi, \varphi_I) : \mathbb{C} \longrightarrow \mathbb{D}$ between traced symmetric monoidal categories is *traced* if it is symmetric and satisfies

$$\mathtt{Tr}^{FU}_{FA,FB}(\varphi^{-1}_{B,U}(Ff)\varphi_{A,U}) \quad = \quad F(\mathtt{Tr}^U_{A,B}(f))$$

where $A \otimes U \xrightarrow{f} B \otimes U$ and $FA \otimes FU \xrightarrow{\varphi_{A,U}} F(A \otimes U) \xrightarrow{Ff} F(B \otimes U) \xrightarrow{\varphi^{-1}_{B,U}} FB \otimes FU.$

A monoidal natural transformation $m$ from $(F, \varphi, \varphi_I)$ to $(G, \psi, \psi_I)$ is a natural transformation $m : F \Longrightarrow G$ such that $m_{A\otimes B}\varphi_{A,B} = \psi_{A,B}(m_A \otimes m_B)$, and $m_I\varphi_I = \psi_I$. A monoidal *pointwise* natural transformation is a family of maps $m_A : FA \longrightarrow GA$ such that the naturality diagram commutes for morphisms of the form $f : I \longrightarrow A$. That is we have $Gfm_I = m_AFf$ for $f : I \longrightarrow A$, $A$ an object in $\mathbb{C}$.

**Notation**: We introduce the following graphical notation: an arrow $U_1 \otimes \cdots \otimes U_m \xrightarrow{f} V_1 \otimes \cdots \otimes V_n$ is represented as a box as in Figure 2. We sometimes emphasize the I/O interface of a tensor or a trace using a dotted box as in Figure 3. We omit writing the labels on the lines when it is clear. Graphs for the trace axioms are given in Appendix I.

**Example 2.2**

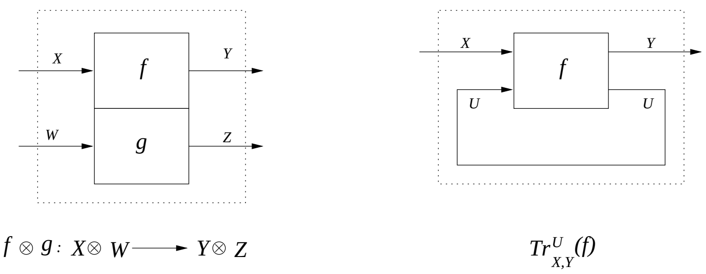



Figure 3: I/O Interfaces

**1.** The category **Rel** is traced. Let $R : X \times U \longrightarrow Y \times U$ be a morphism in **Rel**. Then $\mathtt{Tr}^U_{X,Y}(R) : X \longrightarrow Y$ is defined by: $\mathtt{Tr}^U_{X,Y}(R)(x, y) = \exists u.R(x, u, y, u)$.

**2.** The category **FDVec** is traced. Given a linear transformation $f : V \otimes U \longrightarrow W \otimes U$ where $U, V, W$ are vector spaces with bases $\{u_i\}, \{v_j\}, \{w_k\}$. $\mathtt{Tr}^U_{V,W}(f) : V \longrightarrow W$ is given by

$$\mathtt{Tr}^U_{V,W}(f)(v_i) = \sum_{j,k} a^{kj}_{ij} w_k \qquad \text{where } f(v_i \otimes u_j) = \sum_{k,m} a^{km}_{ij} w_k \otimes u_m.$$

This reduces to the usual trace of $f : U \longrightarrow U$ when $V$ and $W$ are one dimensional.

**3.** Note that both **Rel** and **FDvec** are compact closed categories. More generally, every compact closed category is canonically traced as follows [Joyal *et al.* 1996]: given $f : A \otimes U \longrightarrow B \otimes U$ in a compact closed category $\mathbb{C}$,
$\mathtt{Tr}^U_{A,B}(f) =$

$$A \xrightarrow{1 \otimes \eta_U} A \otimes U \otimes U^* \xrightarrow{f \otimes 1_{U^*}} B \otimes U \otimes U^* \xrightarrow{1 \otimes \sigma} B \otimes U^* \otimes U \xrightarrow{1_U \otimes \epsilon_U} B$$

□

**Remark 2.3** *Following [Joyal* et al. *1996] we will mainly use geometric proofs. Two-dimensional reasoning is valid on the progressive parts of the diagrams because of the results of [Joyal and Street 1991] for symmetric monoidal categories. The reasoning in parts involving trace is deduced from the axioms of trace. With these provisos, geometric reasoning is completely rigorous. Indeed, as remarked in [Joyal* et al. *1996], p. 450, "Algebraic proofs can be constructed from the geometric ones, but algebraic proofs seem only to obfuscate the intuition." Related geometric calculi for diagrammatic reasoning in monoidal and traced monoidal categories are also developed in the works of Blute, Cockett, Seely (see [Blute, Cockett, Seely 2000], and the references there), and arise from their studies of linearly distributive categories and proof-nets.*

*As an example, we will give algebraic and geometric proofs for the next proposition.*

**Proposition 2.4** *(Generalized Yanking)* Let $\mathbb{C}$ be a traced symmetric monoidal category. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be given. Then

$$gf = \mathtt{Tr}^Y_{X,Z}(\sigma_{Y,Z}(f \otimes g)).$$

**Proof**.

$$\begin{aligned} gf &= g1_Y f & (1)\\ &= g\mathtt{Tr}^Y_{Y,Y}(\sigma_{Y,Y})f & (2)\\ &= \mathtt{Tr}^Y_{Y,Z}((g \otimes 1_Y)\sigma_{Y,Y})f & (3)\\ &= \mathtt{Tr}^Y_{X,Z}((g \otimes 1_Y)\sigma_{Y,Y}(f \otimes 1_Y)) & (4)\\ &= \mathtt{Tr}^Y_{X,Z}(\sigma_{Y,Z}(1_Y \otimes g)(f \otimes 1_Y)) & (5)\\ &= \mathtt{Tr}^Y_{X,Z}(\sigma_{Y,Z}(f \otimes g)). & (6) \end{aligned}$$

*Equations 1,2,3,4,5, and 6 respectively correspond to the diagrams in Figure 4. For clarity, we draw $1_Y$ as a straight line, omitting the box.*

□

In particular one can give an equivalent axiomatization of traced symmetric monoidal categories with Yanking replaced by Generalized Yanking [Haghverdi 2000a] (cf. also [Blute, Cockett, Seely 2000]).

We now give a normal form theorem for arrows in a traced symmetric monoidal category.

**Theorem 2.5** Let $\mathbb{C}$ be a traced symmetric monoidal category and $T$ be a set of arrows in $\mathbb{C}$.Then, any expression $E$ built from arrows in $T$ using the tensor product, composition, and trace can be represented as $Tr(\pi F \tau)$ where $F$ consists of a tensor product of arrows in $T$ and $\pi$ and $\tau$ are constructed from symmetry and identity maps using composition and tensor (i.e., $\pi$ and $\tau$ are permutations.)

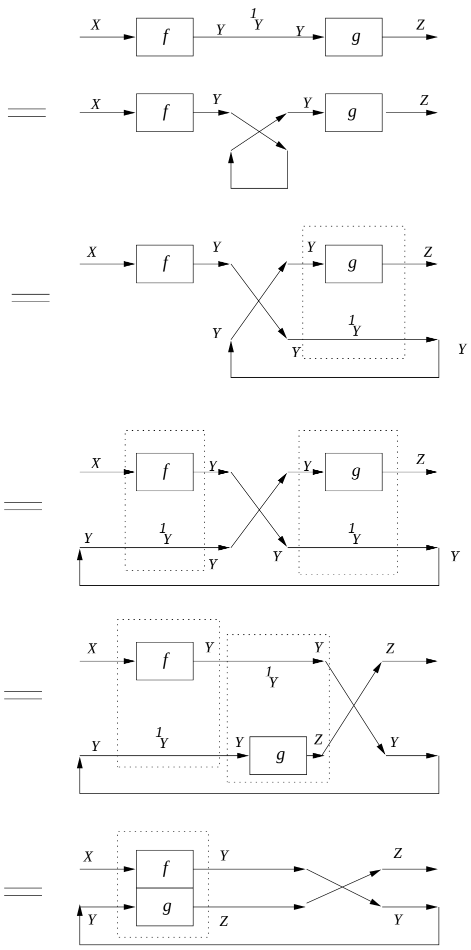



Figure 4: Graphical Proof of Proposition 2.4

**Proof**. By induction on the structure of the expression $E$:

- *Basis step :* Let $E = f : X \longrightarrow Y$ in $T$. Then $f = \mathtt{Tr}^{I}_{X,Y}(f)$.
- Let $E = E_1 \otimes E_2$ with $E_1 : X \longrightarrow Y$ and $E_2 : X' \longrightarrow Y'$. By inductive hypothesis, $E_1 = \mathtt{Tr}^{U}_{X,Y}(\pi_1 F_1 \tau_1)$ and $E_2 = \mathtt{Tr}^{U'}_{X',Y'}(\pi_2 F_2 \tau_2)$. Hence

$$\begin{aligned}
E_1 \otimes E_2 &= \mathtt{Tr}^{U}_{X,Y}(\pi_1 F_1 \tau_1) \otimes \mathtt{Tr}^{U'}_{X',Y'}(\pi_2 F_2 \tau_2) \\
&= \mathtt{Tr}^{U'}_{X\otimes X',Y\otimes Y'}(\mathtt{Tr}^{U}_{X,Y}(\pi_1 F_1 \tau_1) \otimes (\pi_2 F_2 \tau_2)) \text{ using Superposing,} \\
&= \mathtt{Tr}^{U'}_{X\otimes X',Y\otimes Y'}(\mathtt{Tr}^{U}_{X\otimes X'\otimes U',Y\otimes Y'\otimes U'}((1_Y \otimes \sigma_{U,Y'\otimes U'})(\pi_1 F_1 \tau_1 \otimes \pi_2 F_2 \tau_2) \\
&\quad (1_X \otimes \sigma_{X'\otimes U',U}))) \qquad \text{using Superposing,} \\
&= \mathtt{Tr}^{U'\otimes U}_{X\otimes X',Y\otimes Y'}((1_Y \otimes \sigma_{U,Y'\otimes U'})(\pi_1 F_1 \tau_1 \otimes \pi_2 F_2 \tau_2)(1_X \otimes \sigma_{X'\otimes U',U})) \text{ usingVanishingII,} \\
&= \mathtt{Tr}^{U'\otimes U}_{X\otimes X',Y\otimes Y'}(\pi F \tau)
\end{aligned}$$

  where $\pi = (1_Y \otimes \sigma_{U,Y'\otimes U'})(\pi_1 \otimes \pi_2)$, $F = F_1 \otimes F_2$ and $\tau = (\tau_1 \otimes \tau_2)(1_X \otimes \sigma_{X'\otimes U',U})$.
- Let $E = E_2 E_1 : X \longrightarrow Z$ where $E_1 : X \longrightarrow Y$ and $E_2 : Y \longrightarrow Z$.

$$\begin{aligned}
E = E_2 E_1 &= \mathtt{Tr}^{Y}_{X,Z}(\sigma_{Y,Z}(E_1 \otimes E_2)) \text{ by Proposition 2.4,} \\
&= \mathtt{Tr}^{Y}_{X,Z}(\sigma_{Y,Z}\mathtt{Tr}^{U}_{X\otimes Y,Y\otimes Z}(\pi F \tau)) \text{ using previous part,} \\
&= \mathtt{Tr}^{Y}_{X,Z}(\mathtt{Tr}^{U}_{X\otimes Y,Z\otimes Y}((\sigma_{Y,Z} \otimes 1_U)(\pi F \tau))) \text{ using Naturality,} \\
&= \mathtt{Tr}^{Y\otimes U}_{X,Z}(\pi' F \tau) \text{ using Vanishing II.}
\end{aligned}$$

  where $\pi' = (\sigma_{Y,Z} \otimes 1_U)\pi$.
- Let $E = \mathtt{Tr}^{U}_{X,Y}(E_1)$ for $E_1 : X \otimes U \longrightarrow Y \otimes U$.

$$\begin{aligned}
E &= \mathtt{Tr}^{U}_{X,Y}(E_1) \\
&= \mathtt{Tr}^{U}_{X,Y}(\mathtt{Tr}^{U'}_{X\otimes U,Y\otimes U}(\pi F \tau)) \text{ by inductive hypothesis} \\
&= \mathtt{Tr}^{U\otimes U'}_{X,Y}(\pi F \tau) \text{ using Vanishing II.}
\end{aligned}$$

□

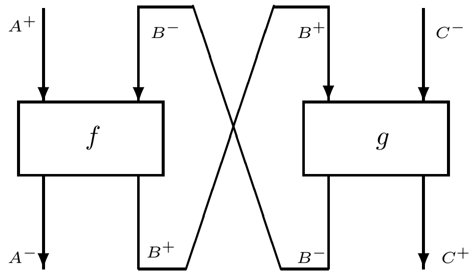
The following construction on traced monoidal categories ([Joyal *et al.* 1996]) isolates the key properties of Girard's GoI. Composition–given by trace–corresponds in this setting to Girard's *execution formula* [Girard 1989].

**Definition 2.6** *(The* Geometry of Interaction *construction) Given a traced monoidal category* $\mathbb{C}$ *we define a new category,* $\mathcal{G}(\mathbb{C})$*, as follows:*

- *Objects: Pairs of objects from* $\mathbb{C}$*, e.g.* $(A^+, A^-)$ *where* $A^+$ *and* $A^-$ *are objects of* $\mathbb{C}$*.*
- *Arrows: An arrow* $f : (A^+, A^-) \longrightarrow (B^+, B^-)$ *in* $\mathcal{G}(\mathbb{C})$ *is* $f : A^+ \otimes B^- \longrightarrow A^- \otimes B^+$ *in* $\mathbb{C}$*.*
- *Identity:* $1_{(A^+,A^-)} = \sigma_{A^+,A^-}$*.*
- *Composition: Composition is given by symmetric feedback. Given* $f : (A^+, A^-) \longrightarrow (B^+, B^-)$ *and* $g : (B^+, B^-) \longrightarrow (C^+, C^-)$*,* $gf : (A^+, A^-) \longrightarrow (C^+, C^-)$ *is given by:*

$$gf = \mathtt{Tr}^{B^-\otimes B^+}_{A^+\otimes C^-,A^-\otimes C^+}(\beta(f \otimes g)\alpha)$$

  *where* $\alpha = (1_{A^+} \otimes 1_{B^-} \otimes \sigma_{C^-,B^+})(1_{A^+} \otimes \sigma_{C^-,B^-} \otimes 1_{B^+})$ *and* $\beta = (1_{A^-} \otimes 1_{C^+} \otimes \sigma_{B^+,B^-})(1_{A^-} \otimes \sigma_{B^+,C^+} \otimes 1_{B^-})(1_{A^-} \otimes 1_{B^+} \otimes \sigma_{B^-,C^+})$*. An informal picture displaying* $gf$ *is given below. For the precise graphical representation of* $gf$ *see Figure 22 in Appendix I.*




- *Tensor:* $(A^+, A^-) \otimes (B^+, B^-) = (A^+ \otimes B^+, A^- \otimes B^-)$ *and for* $(A^+, A^-) \longrightarrow (B^+, B^-)$ *and* $g : (C^+, C^-) \longrightarrow (D^+, D^-)$,

$$f \otimes g = (1_{A^-} \otimes \sigma_{B^+,C^-} \otimes 1_{D^+})(f \otimes g)(1_{A^+} \otimes \sigma_{C^+,B^-} \otimes 1_{D^-})$$

- *Unit:* $(I, I)$.

**Remark 2.7** *We have given a specific definition for* $\alpha$ *and* $\beta$ *above; however, any other permutations* $\alpha : A^+ \otimes C^- \otimes B^- \otimes B^+ \xrightarrow{\cong} A^+ \otimes B^- \otimes B^+ \otimes C^-$ *and* $\beta : A^- \otimes B^+ \otimes B^- \otimes C^+ \xrightarrow{\cong} A^- \otimes C^+ \otimes B^- \otimes B^+$ *will yield the same result for* $gf$*, due to coherence.*

**Proposition 2.8** Let $\mathbb{C}$ be a traced symmetric monoidal category , $\mathcal{G}(\mathbb{C})$ defined as in Definition 2.6 is a compact closed category. Moreover, $F : \mathbb{C} \longrightarrow \mathcal{G}(\mathbb{C})$ with $F(A) = (A, I)$ and $F(f) = f$ is a full and faithful embedding.

**Proof**. (Sketch) For any two objects $(A^+, A^-)$ and $(B^+, B^-)$ in $\mathcal{G}(\mathbb{C})$, we define $\sigma_{(A^+,A^-),(B^+,B^-)} =_{def}$

$$(1_{A^-} \otimes \sigma_{B^+,B^-} \otimes 1_{A^+})(\sigma_{B^+,A^-} \otimes \sigma_{A^+,B^-})(1_{B^+} \otimes \sigma_{A^+,A^-} \otimes 1_{B^-})(\sigma_{A^+,B^+} \otimes \sigma_{B^-,A^-}).$$

The dual of $(A^+, A^-)$ is given by $(A^+, A^-)^* = (A^-, A^+)$ where the unit $\eta : (I, I) \longrightarrow (A^+, A^-) \otimes (A^+, A^-)^* =_{def} \sigma_{A^-,A^+}$ and counit $\epsilon : (A^+, A^-)^* \otimes (A^+, A^-) \longrightarrow (I, I) =_{def} \sigma_{A^-,A^+}$. The internal homs are given by $(A^+, A^-) \multimap (B^+, B^-) = (B^+ \otimes A^-, B^- \otimes A^+)$. $\square$

It is shown in [Joyal *et al.* 1996] that $Int(\mathbb{C})$, which is isomorphic to the above $\mathcal{G}(\mathbb{C})$, is the "free compact closure" of $\mathbb{C}$, in an appropriate bicategorical sense.

**Notation**: Let $A \underset{g}{\overset{f}{\rightleftarrows}} B$ be morphisms in a category $\mathbb{C}$. We write $f : A \lhd B : g$ to mean that $A$ is a retract of $B$ i.e., $gf = 1_A$. We write $A \lhd B$ to mean $f : A \lhd B : g$ for some arrows $f, g$.

**Lemma 2.9**

- Let $A^+ \cong B^+$ and $A^- \cong B^-$ in $\mathbb{C}$, then $(A^+, A^-) \cong (B^+, B^-)$ in $\mathcal{G}(\mathbb{C})$.
- If $f_1 : A^+ \lhd B^+ : g_1$ and $f_2 : A^- \lhd B^- : g_2$ in $\mathbb{C}$, then $\sigma_{B^+,A^-}(f_1 \otimes g_2) : (A^+, A^-) \lhd (B^+, B^-) : \sigma_{A^+,B^-}(g_1 \otimes f_2)$ in $\mathcal{G}(\mathbb{C})$.

# 3 Weak Linear Categories and Linear Combinatory Algebras

In this section we introduce Weak Linear Categories (WLCs) and Linear Combinatory Algebras (LCAs). We show how to construct such algebras from WLCs with reflexive objects. We also show how to get (standard) combinatory algebras from linear ones.

**Definition 3.1** *A* Weak Linear Category (WLC) $(\mathbb{C}, !)$ *consists of the following data:*

- *A symmetric monoidal closed category* $\mathbb{C}$*,*
- *A symmetric monoidal functor* $! : \mathbb{C} \longrightarrow \mathbb{C}$ *(officially,* $! = (!, \varphi, \varphi_I)$*),*

- *The following monoidal pointwise natural transformations:*
  1. $\mathtt{der} : ! \Longrightarrow Id$
  2. $\delta : ! \Longrightarrow !!$
  3. $\mathtt{con} : ! \Longrightarrow ! \otimes !$
  4. $\mathtt{weak} : ! \Longrightarrow \mathcal{K}_I$ . *Here $\mathcal{K}_I$ is the constant $I$ functor.*

We note that in contrast to the usual models of linear logic [Seely 1989, Bierman 1995, Troelstra 1992], in the GoI models below the monoidal transformations $\mathtt{der}, \delta, \mathtt{con}, \mathtt{weak}$ exist but are merely pointwise natural (see Appendix III). Fortunately, pointwise naturality suffices for our main construction of linear combinatory algebras (out of WLCs with a reflexive object) in Theorem 3.5. Similarly, for our purposes we do not require $(!, \mathtt{der}, \delta)$ to form a comonad, nor for $(!A, \mathtt{con}_A, \mathtt{weak}_A)$ to form a comonoid.

**Definition 3.2** *A* reflexive *object in a WLC $(\mathbb{C}, !)$ is an object $V$ in $\mathbb{C}$ with the following retracts:*

- $V \multimap V \lhd V$
- $!V \lhd V$
- $I \lhd V$

Since CCCs are SMCCs, all the usual domain theoretic constructions of reflexive objects in CCCs also yield reflexive objects in the WLC-sense, as follows:

**Proposition 3.3** Let $\mathbb{C}$ be a CCC and $V$ be a reflexive object in $\mathbb{C}$, i.e., $V^V \lhd V$. Then $(\mathbb{C}, Id)$ is a WLC and $V$ is a reflexive object in the WLC-sense.

**Proof**. Any CCC is an SMCC. $Id$ is a symmetric monoidal functor from $\mathbb{C}$ to itself and the following are monoidal natural transformations:

1. $\mathtt{der}_A = 1_A$
2. $\delta_A = 1_A$
3. $\mathtt{con}_A = \langle 1_A, 1_A \rangle$
4. $\mathtt{weak}_A = f : A \longrightarrow T$; the unique map to the terminal object $T$.

It can be easily shown that $V^V \lhd V$ implies $T \lhd V$. Therefore $V \multimap V = V^V \lhd V$, $!V = Id(V) = V \lhd V$ and $I = T \lhd V$ and hence $V$ is a reflexive object in the WLC-sense. $\square$

**Definition 3.4** *A* Linear Combinatory Algebra $(A, \cdot, \mathbf{!})$ *consists of the following data:*

- *An applicative structure $(A, \cdot)$*
- *A unary operator $\mathbf{!} : A \to A$*
- *Distinguished elements $B, C, I, K, W, D, \delta, F$ of $A$,*

*satisfying the following identities (we associate $\cdot$ to the left and write $xy$ for $x \cdot y$, $x\mathbf{!}y = x \cdot (\mathbf{!}(y))$, etc. ) for all variables $x, y, z$ ranging over $A$.*

1. $Bxyz = x(yz)$ *Composition, Cut*
2. $Cxyz = (xz)y$ *Exchange*
3. $Ix = x$ *Identity*
4. $Kx\mathbf{!}y = x$ *Weakening*
5. $Wx\mathbf{!}y = x\mathbf{!}y\mathbf{!}y$ *Contraction*
6. $D\mathbf{!}x = x$ *Dereliction*
7. $\delta\mathbf{!}x = \mathbf{!!}x$ *Comultiplication*
8. $F\mathbf{!}x\mathbf{!}y = \mathbf{!}(xy)$ *Monoidal Functoriality*

The notion of LCA corresponds to a Hilbert style axiomatization of the $\{!, \multimap\}$ fragment of linear logic [Abramsky 1997, Avron 1988, Troelstra 1992]. The *principal types* of the combinators correspond to the axiom schemes which they name. They can be computed by a Hindley-Milner style algorithm [Hindley 1997] from the above equations:

1. $B : (\beta \multimap \gamma) \multimap (\alpha \multimap \beta) \multimap \alpha \multimap \gamma$
2. $C : (\alpha \multimap \beta \multimap \gamma) \multimap (\beta \multimap \alpha \multimap \gamma)$
3. $I : \alpha \multimap \alpha$
4. $K : \alpha \multimap \mathbf{!}\beta \multimap \alpha$
5. $W : (\mathbf{!}\alpha \multimap \mathbf{!}\alpha \multimap \beta) \multimap \mathbf{!}\alpha \multimap \beta$
6. $D : \mathbf{!}\alpha \multimap \alpha$
7. $\delta : \mathbf{!}\alpha \multimap \mathbf{!!}\alpha$
8. $F : \mathbf{!}(\alpha \multimap \beta) \multimap \mathbf{!}\alpha \multimap \mathbf{!}\beta$.

Let $\mathbb{C}$ be a WLC and $V$ be a reflexive object in $\mathbb{C}$ with retracts $r : V \multimap V \lhd V : s$ and $p : !V \lhd V : q$ where $(p, q)$ and $(r, s)$ are the retraction morphisms (see Notation before Lemma 2.9.) We define the following operations on the set $\mathbb{C}(I, V)$:

- Given $f, g \in \mathbb{C}(I, V)$, $f \bullet g = ev(sf \otimes g)$
- Given $f \in \mathbb{C}(I, V)$, $\mathbf{!}f = p!f\varphi_I$ where $\varphi_I : I \longrightarrow !I$ and $! = (!, \varphi, \varphi_I)$.

**Theorem 3.5** Let $(\mathbb{C}, !)$ be a WLC and $V$ be a reflexive object in $\mathbb{C}$ with retracts $r : V \multimap V \lhd V : s$ and $p : !V \lhd V : q$. Then $(\mathbb{C}(I, V), \bullet, \mathbf{!})$ with $\bullet$ and $\mathbf{!}$ defined as above is a linear combinatory algebra.

A detailed proof is given in [Haghverdi 2000a], Chapter 6. This proof only uses monoidal *pointwise* naturality of the transformations $\mathtt{der}, \delta, \mathtt{con}, \mathtt{weak}$. See also Remark 4.3 below.

**Definition 3.6** *A* Standard Combinatory Algebra *consists of a pair $(A, \bullet_s)$ where $A$ is a nonempty set and $\bullet_s$ is a binary operation on $A$ and $B_s, C_s, I_s, K_s$, and $W_s$ are distinguished elements of $A$ satisfying the following identities for all $x, y, z$ variables ranging over $A$:*

1. $B_s \bullet_s x \bullet_s y \bullet_s z = x \bullet_s (y \bullet_s z)$
2. $C_s \bullet_s x \bullet_s y \bullet_s z = (x \bullet_s z) \bullet_s y$
3. $I_s \bullet_s x = x$
4. $K_s \bullet_s x \bullet_s y = x$
5. $W_s \bullet_s x \bullet_s y = x \bullet_s y \bullet_s y$

Note that this is equivalent to the more familiar definition of SK-combinatory algebra; in particular, $S_s$ can be defined from $B_s$, $C_s$, $I_s$ and $W_s$ [Barendregt 1984, Hindley 1997]. Let $(A, \bullet, \mathbf{!})$ be a linear combinatory algebra. We define a binary operation $\bullet_s$ on $A$ as follows: for $\alpha, \beta \in A$, $\alpha \bullet_s \beta =_{def} \alpha \bullet \mathbf{!}\beta$. We define $D' = C(BBI)(BDI)$. Note that

$$D' x \mathbf{!} y = xy.$$

Now consider the following elements of $A$.

1. $B_s =_{def} C \bullet (B \bullet (B \bullet B \bullet B) \bullet (D' \bullet I)) \bullet (C \bullet ((B \bullet B) \bullet F) \bullet \delta)$
2. $C_s =_{def} D' \bullet C$
3. $I_s =_{def} D' \bullet I$
4. $K_s =_{def} D' \bullet K$
5. $W_s =_{def} D' \bullet W$

**Theorem 3.7** Let $(A, \bullet, \mathbf{!})$ be a linear combinatory algebra. Then $(A, \bullet_s)$ with $\bullet_s$ and the elements $B_s, C_s, I_s, K_s, W_s$ as defined above is a combinatory algebra.

We remark that in the case of WLCs coming from CCCs (see Proposition 3.3 above) the associated linear combinatory algebra agrees with the (standard) combinatory algebra structure, since

$$x \bullet_s y = x \bullet \mathbf{!} y = x.y \ .$$

# 4 General GoI Construction

In this section we will present the general form of the GoI construction, extending the $\mathcal{G}$ construction from Definition 2.6 to encompass the exponentials. We then show our main result: that this construction gives rise to linear combinatory algebras (and thus to standard combinatory algebras, using the previous results).

**Definition 4.1** *A* GoI Situation *is a triple* $(\mathbb{C}, T, U)$ *where:*

- $\mathbb{C}$ *is a traced symmetric monoidal category*
- $T : \mathbb{C} \longrightarrow \mathbb{C}$ *is a traced symmetric monoidal functor with the following retractions (which are monoidal natural transformations):*
  1. $e : TT \lhd T : e'$ *(Comultiplication)*
  2. $d : Id \lhd T : d'$ *(Dereliction)*
  3. $c : T \otimes T \lhd T : c'$ *(Contraction)*
  4. $w : \mathcal{K}_I \lhd T : w'$ *(Weakening), where* $\mathcal{K}_I$ *is the constant* $I$ *functor.*
- $U$ *is an object of* $\mathbb{C}$*, called* a reflexive object*, with retractions:*
  1. $j : U \otimes U \lhd U : k$
  2. $I \lhd U$
  3. $u : TU \lhd U : v$

Given a GoI Situation $(\mathbb{C}, T, U)$ with $j : U \otimes U \lhd U : k$ and $u : TU \lhd U : v$, where $(j, k)$ and $(u, v)$ are the associated retract pairs, consider the compact closed category $\mathcal{G}(\mathbb{C})$ from Definition 2.6, with the distinguished objects $I = (I, I)$ and $V = (U, U)$. Note that by definition (since we are in the strict case) $\mathcal{G}(\mathbb{C})(I, V) = \mathbb{C}(U, U)$.

We can define an endofunctor $! : \mathcal{G}(\mathbb{C}) \longrightarrow \mathcal{G}(\mathbb{C})$ as follows: $!(A^+, A^-) = (TA^+, TA^-)$ and given $f : (A^+, A^-) \longrightarrow (B^+, B^-)$,

$$!f \ =_{def} \ TA^+ \otimes TB^- \xrightarrow{\cong} T(A^+ \otimes B^-) \xrightarrow{Tf} T(A^- \otimes B^+) \xrightarrow{\cong} TA^- \otimes TB^+.$$

**Proposition 4.2** Let $(\mathbb{C}, T, U)$ be a GoI Situation satisfying the conditions above. Then:

(i) $(\mathcal{G}(\mathbb{C}), !)$ is a WLC with reflexive object $V = (U, U)$,

(ii) $(\mathcal{G}(\mathbb{C})(I, V), \bullet, \mathbf{!})$ is an LCA, where for any $f, g \in \mathcal{G}(\mathbb{C})(I, V) = \mathbb{C}(U, U)$, $f \bullet g = \mathtt{Tr}^U_{U,U}((1_U \otimes g)(kfj))$, and $\mathbf{!}f = u(Tf)v$.

**Proof**. Note that $\mathcal{G}(\mathbb{C})$ is a compact closed category and hence it is symmetric monoidal closed, see Proposition 2.8.

It can be easily shown that ! is a symmetric monoidal functor. Next we define the following maps:

- $\mathtt{der}_{(A^+,A^-)} : \ !(A^+, A^-) \longrightarrow (A^+, A^-) =_{def} \sigma_{A^+,TA^-}(d'_{A^+} \otimes d_{A^-})$ where $d_A : A \lhd TA : d'_A$,
- $\delta_{(A^+,A^-)} : \ !(A^+, A^-) \longrightarrow !!(A^+, A^-) =_{def} \sigma_{T^2A^+,TA^-}(e'_{A^+} \otimes e_{A^-})$ where $e_A : T^2A \lhd TA : e'_A$,
- $\mathtt{con}_{(A^+,A^-)} : \ !(A^+, A^-) \longrightarrow !(A^+, A^-) \otimes !(A^+, A^-) =_{def} \sigma_{TA^+ \otimes TA^+,TA^-}(c'_{A^+} \otimes c_{A^-})$ where $c_A : TA \otimes TA \lhd TA : c'_A$,
- $\mathtt{weak}_{(A^+,A^-)} : \ !(A^+, A^-) \longrightarrow (I, I) =_{def} \sigma_{I,TA^-}(w'_{A^+} \otimes w_{A^-})$ where $w_A : I \lhd TA : w'_A$.

A direct calculation shows (see Appendix II) that $\mathtt{der}, \delta, \mathtt{con}, \mathtt{weak}$ form monoidal pointwise natural transformations. With the above definitions it is easily seen that $(\mathcal{G}(\mathbb{C}), !)$ is a WLC. Also, it follows (see Lemma 2.9) that $V = (U, U)$ is a reflexive object in $\mathcal{G}(\mathbb{C})$ and hence we have that $(\mathcal{G}(\mathbb{C})(I, V), \bullet, \mathbf{!})$ is an LCA using Theorem 3.5. □

**Remark 4.3** *Let us remark on the necessity of pointwise naturality. As shown in Appendix II, the family of maps* $\mathtt{der}, \delta, \mathtt{con}, \mathtt{weak}$ *are monoidal* pointwise *natural transformations and not natural in general. These maps will be natural transformations iff the underlying monoidal natural transformations used in their definitions are monoidal natural isomorphisms. For example* $\mathtt{der}$ *will be a natural transformation iff* $(d_A, d'_A)$ *is an isomorphism with* $d_A^{-1} = d'_A$ *(a proof of this is sketched at the end of Appendix II). This will in general produce undesirable effects: for example, asking for* $(d, d')$ *to form an isomorphism means that every object in* $\mathbb{C}$ *must be a fixed point of* $T$*. Similarly requiring* $(w, w')$ *to be an isomorphism means* $TA$ *isomorphic to* $I$ *for all* $A$ *and together this means that* $\mathbb{C}$ *must be a trivial one object category. The isomorphisms* $(c, c')$ *and* $(e, e')$ *are on the other hand benign. For example in the case of* **PInj** *below we have chosen* $(c, c')$ *and* $(e, e')$ *that are isomorphisms and hence* $\delta$ *and* $\mathtt{con}$ *are actually natural transformations.*

The diagrammatic definitions of the combinators show vividly the geometric, information flow perspective on computation yielded by GoI, and we shall give them all explicitly. The reductions of the combinators to normal forms is included in Appendix III.

**Notation**: we introduce in Figure 5 an explicit graphical notation for the data in a GoI situation.

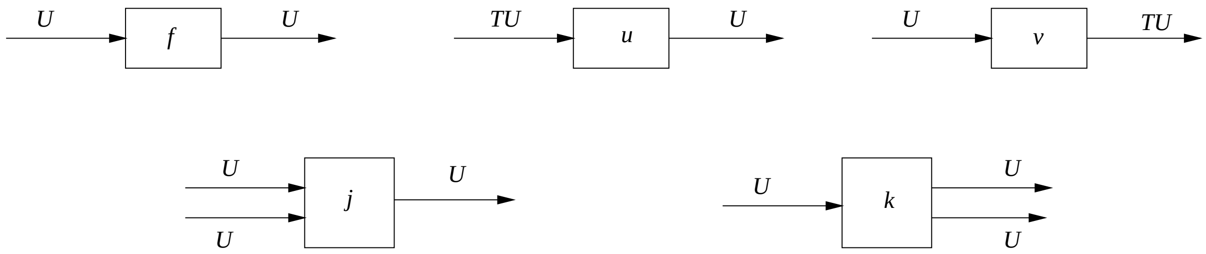



Figure 5: Graphical Representations

The combinators are defined as follows:

1. $I =_{def} \alpha\gamma\beta$, where
   (a) $\alpha = j$
   (b) $\beta = k$
   (c) $\gamma = \sigma_{U,U}$. See Figure 6.

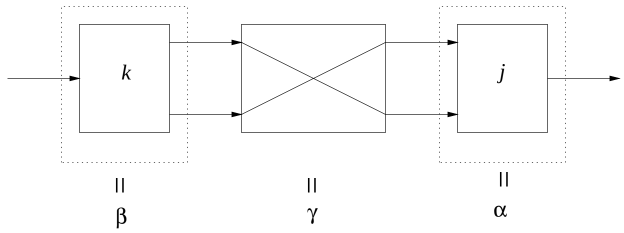



Figure 6: Identity Combinator I

2. $B =_{def} \alpha\gamma\beta$, where
   (a) $\alpha = j(j \otimes 1_U)(j \otimes j \otimes j)$
   (b) $\beta = (k \otimes k \otimes k)(k \otimes 1_U)k$
   (c) $\gamma =$ the permutation $\pi$ in Figure 7.
3. $C =_{def} \alpha\gamma\beta$, where
   (a) $\alpha = j(j \otimes j)(j \otimes 1_U \otimes j \otimes 1_U)$
   (b) $\beta = (k \otimes 1_U \otimes k \otimes 1_U)(k \otimes k)k$
   (c) $\gamma =$ the permutation $\pi$ in Figure 8.

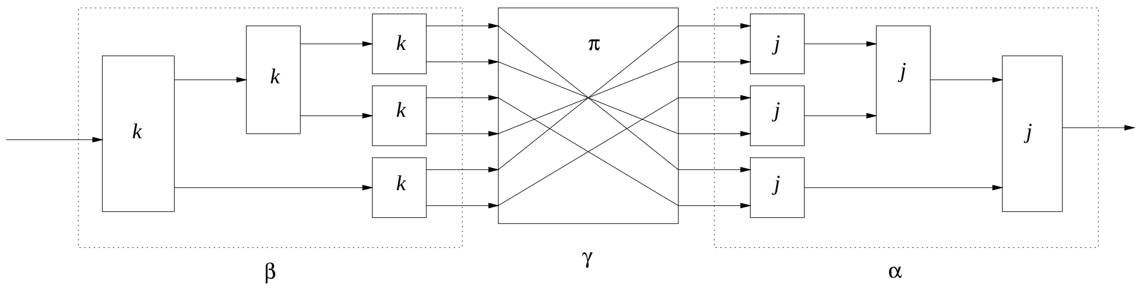



Figure 7: Composition Combinator B

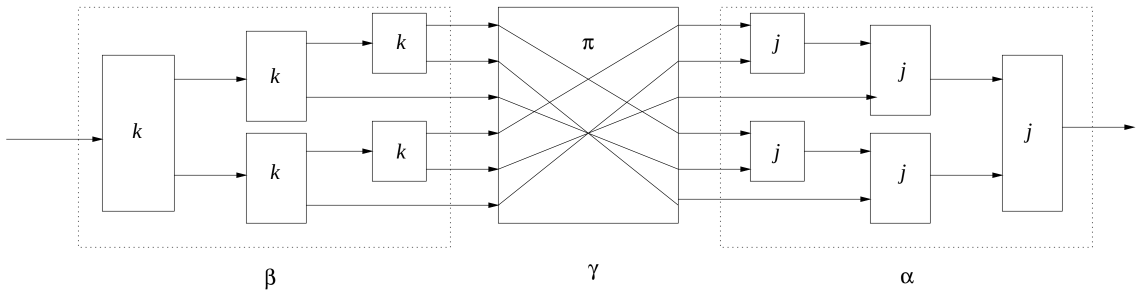



Figure 8: Exchange Combinator C

4. $K =_{def} \alpha\gamma\beta$, where
   (a) $\alpha = j(j \otimes 1)$
   (b) $\beta = (k \otimes 1)k$
   (c) $\gamma = \pi(1_U \otimes f_K \otimes 1_U)$, where $f_K = uw_U w'_U v$ and $\pi$ is the permutation given in Figure 9.

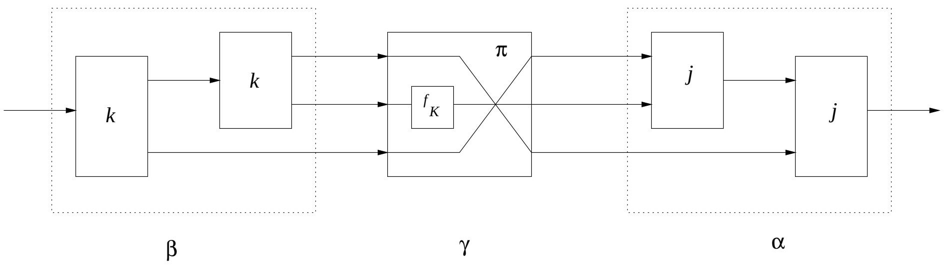



Figure 9: Weakening Combinator K

5. $W =_{def} \alpha\gamma\beta$, where
   (a) $\alpha = j(1_U \otimes j)(j \otimes j \otimes 1_U)$
   (b) $\beta = (k \otimes k \otimes 1_U)(1_U \otimes k)k$
   (c) $\gamma = \pi(1_U \otimes g_W \otimes 1_U \otimes f_W)(1_U \otimes 1_U \otimes 1_U \otimes \sigma)$, where $g_W = (u \otimes u)c'_U v$, $f_W = uc_U(v \otimes v)$, and $\pi$ is the permutation given in Figure 10.
6. $D =_{def} \alpha\gamma\beta$, where
   (a) $\alpha = j(j \otimes j)$
   (b) $\beta = (k \otimes k)k$
   (c) $\gamma = \pi(1_U \otimes g_D \otimes 1_U \otimes f_D)$, where $f_D = ud_U$, $g_D = d'_U v$ and $\pi$ is the permutation given in Figure 11

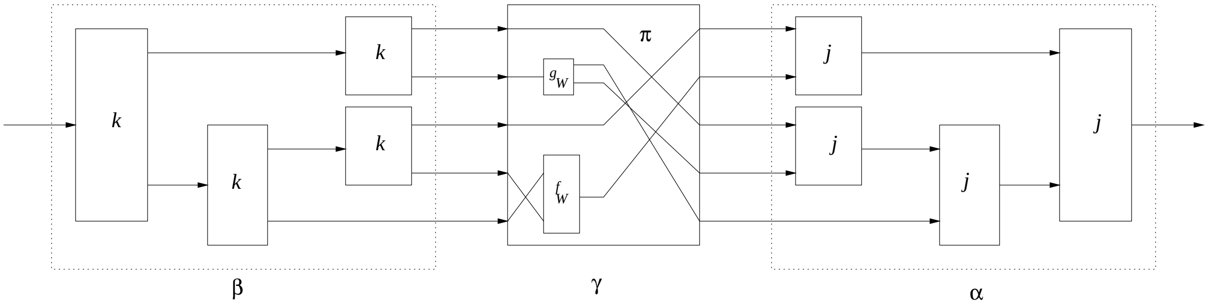



Figure 10: Contraction Combinator W

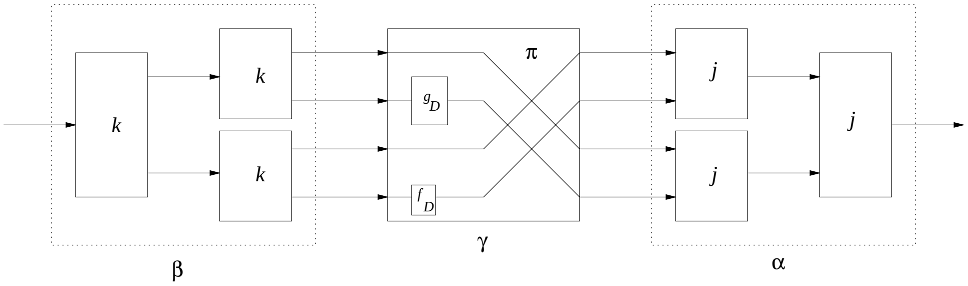



Figure 11: Dereliction Combinator D

7. $\delta =_{def} \alpha\gamma\beta$, where
   (a) $\alpha = j$
   (b) $\beta = k$
   (c) $\gamma = \sigma_{U,U}(f_\delta \otimes g_\delta)$, where $f_\delta = ue_U T(v)v$ and $g_\delta = uT(u)e'_U v$. See Figure 12.
8. $F =_{def} \alpha\gamma\beta$, where
   (a) $\alpha = j(j \otimes 1_U)$
   (b) $\beta = (k \otimes 1_U)k$
   (c) $\gamma = \pi(f_F \otimes g_F)$, where $f_F = uT(j)\psi_{U,U}(v \otimes v)$, $g_F = (u \otimes u)\psi^{-1}_{U,U}T(k)v$, $\pi$ is the permutation given in Figure 13, and $T = (T, \psi, \psi_I)$.

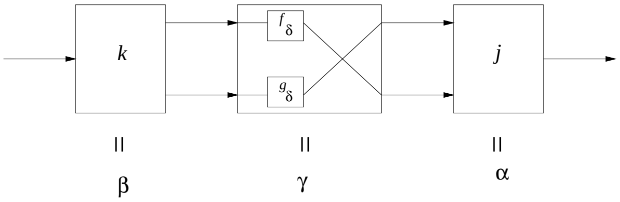



Figure 12: Comultiplication Combinator $\delta$

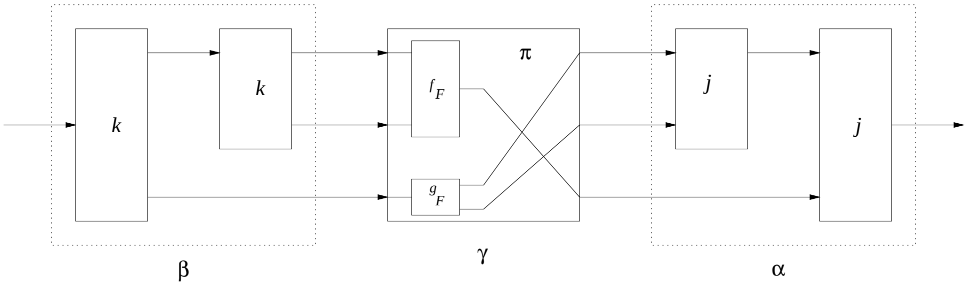



Figure 13: Functoriality Combinator F

# 5 Examples

In this section we illustrate the construction given in previous sections in some specific categories: **PInj** (sets and partial injective functions), **Pfn** (sets and partial functions), $\mathbf{Rel}_+$ (sets and relations), **Res**(the category of Resumptions), **SRel** (measurable spaces and stochastic kernels), and $\omega$-**CPO** ($\omega$-complete partial orders and continuous maps). The rôle of these categories in GoI and the connection of their trace structure to Girard's *execution formula* has been emphasized in [Abramsky 1996, Abramsky 1997, Haghverdi 2000b]. The categories **PInj**, **Pfn**, $\mathbf{Rel}_+$ and **SRel** are all examples of a special subclass of traced symmetric monoidal categories called *traced unique decomposition categories*, studied in [Haghverdi 2000a, Haghverdi 2000b]. These categories generalize Manes-Arbib's *partially additive semantics* [Manes and Arbib 1986] and support a convenient computational framework for many of the "particle-style models" of GoI emphasized in [Abramsky 1996]. Our uniform treatment of the trace formula in **PInj**, **Pfn**, $\mathbf{Rel}_+$ and **SRel** is motivated from this viewpoint.

## 5.1 PInj

Consider the category **PInj** of sets and partial injective functions. **PInj** is a traced symmetric monoidal category. The tensor product is given by the disjoint union of sets, where we identify $A \uplus B = \{1\} \times A \cup \{2\} \times B$ (note that this is not a coproduct in **PInj**). There are the obvious injections $\mathtt{inl}^{A,B} : A \to A \uplus B$ and $\mathtt{inr}^{A,B} : B \to A \uplus B$ given as follows (we omit superscripts) $\mathtt{inl}(a) = (1, a)$ and $\mathtt{inr}(b) = (2, b)$. There are also "quasiprojections" $\rho_1 : A \uplus B \longrightarrow A$ and $\rho_2 : A \uplus B \longrightarrow B$ given by $\rho_1((1, a)) = a$ (where $\rho_1((2, b))$ is undefined ) and by $\rho_2((2, b)) = b$ (where $\rho_2((1, a))$ is undefined.)

Given a morphism $f : X \uplus U \longrightarrow Y \uplus U$, we may consider its four "components" $f_{XY} : X \to Y$, $f_{XU} : X \to U$,$f_{UX} : U \to X$, and $f_{UU} : U \to U$ obtained by pre- and post-composing with injections and quasiprojections: for example, $f_{XY} = X \xrightarrow{\mathtt{inl}} X \uplus U \xrightarrow{f} Y \uplus U \xrightarrow{\rho_1} Y$ ,etc. (See Figure 14); it may be represented as a matrix $f = \begin{bmatrix} f_{XY} & f_{UY} \\ f_{XU} & F_{UU} \end{bmatrix}$.

The trace is given by the following formula

$$\mathtt{Tr}^U_{X,Y}(f) = f_{XY} + \sum_{n \in \omega} f_{UY} f^n_{UU} f_{XU} \ ,$$

which we interpret as follows: a family $\{h_i\}_{i \in I} : X \longrightarrow Y$ is said to be *summable* if the $h_i$'s have pairwise

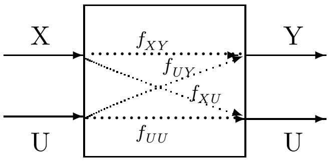



Figure 14: Components of $f : X \uplus U \to Y \uplus U$

disjoint domains and codomains. In that case, we define their sum

$$(\sum_{i\in I} h_i)(x) = \begin{cases} h_j(x), & \text{if } x \in \mathtt{Dom}(h_j) \text{ for some } j \in I; \\ \text{undefined}, & \text{else.} \end{cases}$$

In the next paragraph we show that the above trace formula is well-defined. But we note that from the "particle-style" viewpoint, we may picture the trace formula above as follows: consider the trace in Figure 3, superposed on Figure 14. Imagine a token entering the box through the wire labelled $X$, and either exiting at $Y$ by following $f_{XY}$, or instead following $f_{XU}$, cycling some number of times via $f_{UU} : U \to U$, and again exiting through $Y$ via $f_{UY}$.

Given $f : X \uplus U \longrightarrow Y \uplus U$, it remains to show that the sum appearing in the trace above, i.e. $f_{XY} + \sum_{n\in\omega} f_{UY} f_{UU}^n f_{XU}$, is well-defined. That is, the family of functions appearing in the above sum have pairwise disjoint domains and codomains.

(i) Suppose $\mathtt{Dom}(f_{XY}) \cap \mathtt{Dom}(f_{UY} f_{UU}^j f_{XU}) \neq \emptyset$, for some $j$. If $x \in \mathtt{Dom}(f_{XY}) \cap \mathtt{Dom}(f_{UY} f_{UU}^j f_{XU})$, then $f_{XY}(x) = y$ for some $y \in Y$, hence $f((x,1)) = (y,1)$. On the other hand, $f_{UY} f_{UU}^j f_{XU}(x) = y'$, for some $y' \in Y$; hence $f_{XU}(x) = u$, for some $u \in U$, so $f((x,1)) = (u,2)$, a contradiction.

(ii) Suppose $\mathtt{Dom}(f_{UY} f_{UU}^j f_{XU}) \cap \mathtt{Dom}(f_{UY} f_{UU}^k f_{XU}) \neq \emptyset$, say $x \in \mathtt{Dom}(f_{UY} f_{UU}^j f_{XU}) \cap \mathtt{Dom}(f_{UY} f_{UU}^k f_{XU})$ for $k < j$. Let $f_{XU}(x) = u$ and $f_{UU}^k(u) = u'$; then $u' \in \mathtt{Dom}(f_{UY}) \cap \mathtt{Dom}(f_{UU}^{j-k})$. Hence $f_{UY}(u') = y$, for some $y \in Y$. Since $j - k > 0$, $f_{UU}(u') = u''$, for some $u'' \in U$, we have $f((u',2)) = (y,1)$ and $f((u',2)) = (u'',2)$, a contradiction.

(iii) To show the above family of functions have disjoint codomains, one shows the inverse maps have disjoint domains. The proof is similar to the above cases and is left to the reader.

Finally, the fact that the above trace formula satisfies the axioms for a trace follows from [Haghverdi 2000b], Proposition 2.12.

We will now show that $(\mathbf{PInj}, \mathbb{N}\times -, \mathbb{N})$ is a GoI Situation. However, we first state a lemma that will be used to prove that the functor $T$ is a traced functor as required by a GoI Situation.

**Lemma 5.1** *Let $T : \mathbf{PInj} \longrightarrow \mathbf{PInj}$ be an additive functor, i.e. a functor preserving the additive structure on the homsets. Then, $T$ is traced.*

**Proof**. Let $f : X \otimes U \longrightarrow Y \otimes U$ be a morphism in $\mathbf{PInj}$, then $\mathtt{Tr}_{X,Y}^U(f) = f_{XY} + \sum_{n\in\omega} f_{UY} f_{UU}^n f_{XU}$. Let $T = (T, \psi, \psi_I)$ and $g = \psi_{Y,U}^{-1} T(f) \psi_{X,U}$. Then,

$$\begin{aligned} g_{XY} &= \rho_1^{TY,TU} g \mathtt{inl}^{TX,TU} \\ &= T(\rho_1^{Y,U}) \psi_{Y,U} g \psi_{X,U}^{-1} T(\mathtt{inl}^{X,U}) \\ &= T(\rho_1^{Y,U}) \psi_{Y,U} \psi_{Y,U}^{-1} T(f) \psi_{X,U} \psi_{X,U}^{-1} T(\mathtt{inl}^{X,U}) \\ &= T(\rho_1^{Y,U} f \mathtt{inl}^{X,U}) \\ &= T(f_{XY}). \end{aligned}$$

Similarly, we have $g_{UY} = T(f_{UY}), g_{XU} = T(f_{XU})$, and $g_{UU} = T(f_{UU})$.

$$\begin{aligned} T(\mathtt{Tr}_{X,Y}^U(f)) &= T(f_{XY} + \textstyle\sum_{n\in\omega} f_{UY} f_{UU}^n f_{XU}) \\ &= T(f_{XY}) + \textstyle\sum_{n\in\omega} T(f_{UY}) T(f_{UU})^n T(f_{XU}), \quad T \text{ is additive} \\ &= g_{XY} + \textstyle\sum_{n\in\omega} g_{UY} g_{UU}^n g_{XU} \\ &= \mathtt{Tr}_{TX,TY}^{TU}(\psi_{Y,U}^{-1} T(f) \psi_{X,U}). \end{aligned}$$

Therefore, $T$ is traced. $\square$

**Proposition 5.2** ($\mathbf{PInj}, \mathbb{N}\times -, \mathbb{N}$) is a GoI Situation.

**Proof**. We have previously observed that **PInj** is traced symmetric monoidal category, with the tensor product taken to be the disjoint union of sets with $\emptyset$ as the unit. Clearly $T = \mathbb{N}\times -$, with $T = (T, \psi, \psi_I)$, is a symmetric monoidal functor where $\psi_{X,Y} : \mathbb{N}\times X \uplus \mathbb{N}\times Y \longrightarrow \mathbb{N}\times(X\uplus Y)$ is given by $(1,(n,x)) \mapsto (n,(1,x))$ and $(2,(n,y)) \mapsto (n,(2,y))$ and it has an inverse defined by: $(n,(1,x)) \mapsto (1,(n,x))$ and $(n,(2,y)) \mapsto (2,(n,y))$. Also, $\psi_I : \emptyset \longrightarrow \mathbb{N}\times\emptyset$ given by $1_\emptyset$ is clearly an isomorphism.

We show that $T$ is additive. Let $\{f_i\}_{i\in I}$ be a summable family in $\mathbf{PInj}\,(X,Y)$, then

$$(1_\mathbb{N}\times \textstyle\sum_I f_i)(n,x) = \begin{cases} (n, f_j(x)), & \text{if there exists a } j\in I \text{ such that } x\in \mathtt{Dom}(f_j);\\ \text{undefined}, & \text{else.}\end{cases}$$

but this is exactly the definition of $(\sum_I(1_\mathbb{N}\times f_i))(n,x)$ for all $(n,x)\in\mathbb{N}\times X$. Therefore, $T=\mathbb{N}\times -$ is an additive functor and thus, by Lemma 5.1, it is also traced. In other words, given $f: X\uplus U \longrightarrow Y\uplus U$ we have $1_\mathbb{N}\times \mathtt{Tr}^U_{X,Y}(f) = \mathtt{Tr}^{\mathbb{N}\times U}_{\mathbb{N}\times X,\mathbb{N}\times Y}(\psi^{-1}(1_\mathbb{N}\times f)\psi)$.

We next define the necessary monoidal natural transformations.

- (Comultiplication) $\mathbb{N}\times(\mathbb{N}\times X) \xrightarrow{e_X} \mathbb{N}\times X$ and $\mathbb{N}\times X \xrightarrow{e'_X} \mathbb{N}\times(\mathbb{N}\times X)$

  $\mathbb{N}\times(\mathbb{N}\times X) \xrightarrow{e_X} \mathbb{N}\times X$ is defined by, $e_X(n_1,(n_2,x)) = (\langle n_1,n_2\rangle, x)$. Given $f: X\longrightarrow Y$, $(1_\mathbb{N}\times f)e_X((n_1,(n_2,x)) = (\langle n_1,n_2\rangle, f(x)) = e_Y(1_\mathbb{N}\times(1_\mathbb{N}\times f)(n_1,(n_2,x))$ for all $n_1,n_2\in\mathbb{N}$ and $x\in X$ proving the naturality of $e_X$. $e'_X(n,x) = (n_1,(n_2,x))$ where $\langle n_1,n_2\rangle = n$.
  $e'_X e_X(n_1,(n_2,x)) = e'_X(\langle n_1,n_2\rangle, x) = (n_1,(n_2,x))$ for all $n_1,n_2\in\mathbb{N}$ and $x\in X$.

- (Dereliction) $X \xrightarrow{d_X} \mathbb{N}\times X$ and $\mathbb{N}\times X \xrightarrow{d'_X} X$

  $d_X(x) = (n_0,x)$ for a fixed $n_0\in\mathbb{N}$. Given $f: X\longrightarrow Y$, $(1_\mathbb{N}\times f d_X(x) = (n_0, f(x)) = d_Y f(x)$ for any $x\in X$, proving the naturality of $d_X$.

  $$d'_X(n,x) = \begin{cases} x, & \text{if } n = n_0;\\ \text{undefined}, & \text{else.}\end{cases}$$

  $d'_X d_X(x) = d'_X(n_0,x) = x$ for all $x\in X$.

- (Contraction) $(\mathbb{N}\times X)\uplus(\mathbb{N}\times X) \xrightarrow{c_X} \mathbb{N}\times X$ and $\mathbb{N}\times X \xrightarrow{c'_X} (\mathbb{N}\times X)\uplus(\mathbb{N}\times X)$.

  $$c_X = \begin{cases} (1,(n,x)) \mapsto (2n,x)\\ (2,(n,x)) \mapsto (2n+1,x)\end{cases}$$

  Given $f: X\longrightarrow Y$, $(1_\mathbb{N}\times f)c_X(1,(n,x)) = (2n, f(x)) = c_Y(1_\mathbb{N}\times f\uplus 1_\mathbb{N}\times f)(1,(n,x))$ for all $n\in\mathbb{N}$ and $x\in X$. Similarly $(1_\mathbb{N}\times f)c_X(2,(n,x)) = (2n+1, f(x)) = c_Y(1_\mathbb{N}\times f\uplus 1_\mathbb{N}\times f)(2,(n,x))$ for all $n\in\mathbb{N}$ and $x\in X$, proving the naturality of $c_X$.

  $$c'_X(n,x) = \begin{cases} (1,(n/2,x)), & \text{if } n \text{ is even;}\\ (2,((n-1)/2,x), & \text{if } n \text{ is odd.}\end{cases}$$

  Finally, $c'_X c_X(1,(n,x)) = c'_X(2n,x) = (1,(n,x))$ and $c'_X c_X(2,(n,x)) = c'_X(2n+1,x) = (2,(n,x))$.

- (Weakening) $\emptyset \xrightarrow{w_X} \mathbb{N}\times X$ and $\mathbb{N}\times X \xrightarrow{w'_X} \emptyset$.

  Let $w_X$ and $w'_X$ both be the empty partial function. Clearly for any $f: X\longrightarrow Y$, $(1_\mathbb{N}\times f)w_X = w'_X 1_\emptyset$, proving the naturality of $w_X$. $w'_X w_X = 1_\emptyset$.

Finally, we show that $\mathbb{N}$ is a reflexive object.

- $j : \mathbb{N} \uplus \mathbb{N} \lhd \mathbb{N} : k$ is given as follows: $j : \mathbb{N} \uplus \mathbb{N} \longrightarrow \mathbb{N}, j(1,n) = 2n, j(2,n) = 2n+1$ and $k : \mathbb{N} \longrightarrow \mathbb{N} \uplus \mathbb{N}$,

$$k(n) = \begin{cases} (1, n/2), & \text{if } n \text{ even;} \\ (2, (n-1)/2), & \text{if } n \text{ odd .} \end{cases}$$

  Clearly $kj = 1_{\mathbb{N} \uplus \mathbb{N}}$.
- $\emptyset \lhd \mathbb{N}$ using the empty partial function as the retract morphisms.
- $u : \mathbb{N} \times \mathbb{N} \lhd \mathbb{N} : v$ is defined as: $u(m,n) = \langle m, n \rangle = \frac{(m+n+1)(m+n)}{2} + n$ (Cantor surjective pairing) and $v$ as its inverse, $v(n) = (n_1, n_2)$ with $\langle n_1, n_2 \rangle = n$. Clearly, $vu = 1_{\mathbb{N} \times \mathbb{N}}$.

$\square$

We can generalize the above example to the case of partial functions as follows.

## 5.2 Pfn

Consider the category **Pfn** of sets and partial functions. **Pfn** is a traced symmetric monoidal category with disjoint union (= coproduct) as the tensor product, $I = \emptyset$. Given $f : X \oplus U \longrightarrow Y \oplus U$, (we use $\oplus$ for coproduct),

$$\mathtt{Tr}^U_{X,Y}(f) = f_{XY} + \sum_{n \in \omega} f_{UY} f^n_{UU} f_{XU}.$$

Here, a family $\{h_i\}_{i \in I} : X \longrightarrow Y$ is said to be summable if the $h_i$'s have pairwise disjoint domains. In that case

$$(\sum_{i \in I} h_i)(x) = \begin{cases} h_j(x), & \text{if } x \in \mathtt{Dom}(h_j) \text{ for some } j \in I; \\ \text{undefined}, & \text{else.} \end{cases}$$

The component maps are as in the previous example, with $\mathtt{inl}, \mathtt{inr}, \rho_i$ as before. Clearly $\mathbb{N} \times -$ is an additive and hence a traced functor. Note that the morphisms used in the previous example are partial injective functions, hence partial functions. In view of this we have:

**Proposition 5.3** $(\mathbf{Pfn}, \mathbb{N} \times -, \mathbb{N})$ is a GoI Situation.

## 5.3 $\mathbf{Rel}_+$

The category $\mathbf{Rel}_+$ of sets and binary relations is a traced symmetric monoidal category. Tensor product is again taken to be disjoint union (which will in fact be a biproduct). To keep our discussion in the framework of the previous two examples, we shall declare *any* family of relations $\{R_i\}_{i \in I} \in \mathbf{Rel}_+(X,Y)$ to be summable, with $\sum_{i \in I} R_i = \cup_{i \in I} R_i$. As in the previous examples, we obtain a trace as follows (cf. [Haghverdi 2000b, Abramsky 1996]): given $R : X \oplus U \longrightarrow Y \oplus U$, define

$$\begin{aligned} \mathtt{Tr}^U_{X,Y}(R) &= R_{XY} + \sum_{n \in \omega} R_{UY} R^n_{UU} R_{XU} \\ &= R_{XY} \cup R_{UY} R^*_{UU} R_{XU} \end{aligned}$$

Here, the components of $R$ in the trace are as before, where now $\rho_i$ and $\mathtt{inl}, \mathtt{inr}$ become the projections and injections of the biproduct, and $R^*$ denotes the reflexive, transitive closure of $R$.

As we saw above, $\mathbb{N} \times -$ is an additive and hence a traced functor (by the analog of Lemma 5.1). The morphisms used in the case of **PInj** are partial injective functions and hence in particular they are relations. Thus we have:

**Proposition 5.4** $(\mathbf{Rel}_+, \mathbb{N} \times -, \mathbb{N})$ is a GoI Situation.

## 5.4 Resumptions

### A brief history of resumptions

Our starting point is the classical automata-theoretic notion of *transducers*, *i.e.* structures $(Q, X, Y, q_0, \delta)$ where $Q$ is a set of states, $q_0 \in Q$ the initial state, $X$ the set of inputs, $Y$ the set of outputs, and $\delta : Q \times X \rightharpoonup Y \times Q$ is the transition function (here a partial function). If we supply a sequence of inputs $x_0, \ldots, x_k$ to such a transducer, we obtain the orbit

$$q_0 \xrightarrow{x_0} y_0, q_1 \xrightarrow{x_1} y_1, q_2 \xrightarrow{x_2} \cdots \xrightarrow{x_k} y_k, q_{k+1}$$

where $\delta(q_i, x_i) = y_i, q_{i+1}$, $0 \leq i \leq k$. This generalizes to non-deterministic transducers with transition function $\delta : Q \times X \longrightarrow \mathcal{P}(Y \times Q)$ in an evident fashion (where for any set $X$, $\mathcal{P}(X) = \{U \mid U \subseteq X\}$ denotes the power set of $X$).

A key idea introduced in [Milner 1975] is to give a denotational semantics for concurrent programs as *processes*, which were taken to be extensional versions of transducers. There are two ingredients to this idea:

(i) Instead of modelling programs by functions or relations, to model them by entities with more complex behaviours, taking account of the possible interactions between a program and its environment during the course of a computation.

(ii) Instead of modelling concurrent programs by automata, with all the intensionality this entails, to look for a more extensional description of the *behaviours* of transducers.

To obtain this extensional view of transducers, consider the recursive definition

$$R = X \rightharpoonup (Y \times R).$$

This defines a mathematical space of "resumptions" in which the states of transducers are "unfolded" into their observable behaviours. Milner solved equations such as this over a category of domains in [Milner 1975], but in fact it can be solved in a canonical fashion over **Set**—in modern terminology, the functor $T_{X,Y}$ : **Set** $\longrightarrow$ **Set** defined by $T_{X,Y}(S) = X \rightharpoonup Y \times S$ has a final coalgebra $R \xrightarrow{\cong} T_{X,Y}(R)$. Indeed, Milner defined a notion $\sim$ of behavioural equivalence between transducers, and for any transducer $(Q, X, Y, q_0, \delta)$ a map $h_\delta : Q \longrightarrow R$ which is in fact the final coalgebra homomorphism from the coalgebra

$$\hat{\delta} : Q \longrightarrow T_{X,Y}(Q)$$

to $R$ (where $\hat{\delta}$ is the exponential transpose of $\delta$), and proved that

$$(Q, X, Y, q_0, \delta) \sim (Q', X, Y, q'_0, \delta') \iff h_\delta(q_0) = h_{\delta'}(q'_0).$$

From a modern perspective, we can also make light of a technical problem which figured prominently in [Milner 1975], namely how to model non-determinism. Historically this inspired Plotkin's work on powerdomains [Plotkin 1976], but for the specific application at hand the equation

$$R = X \longrightarrow \mathcal{P}(Y \times R)$$

has a final coalgebra in the category of classes in Peter Aczel's non-well-founded set theory [Aczel 1988], and if we are content to bound the cardinality of subsets by an inaccessible cardinable $\kappa$, then the equation

$$R = X \longrightarrow \mathcal{P}^{<\kappa}(Y \times R)$$

has a final coalgebra in **Set** [Barr 1993]. Moreover, the equivalence induced by this model coincides with strong bisimulation [Aczel 1988].

### The category Res

The category **Res** of resumptions (we will for simplicity confine ourselves to deterministic resumptions) has as objects sets, and as morphisms

$$\mathbf{Res}(X, Y) = X \rightharpoonup (Y \times \mathbf{Res}(X, Y))$$

*i.e.* the space of resumptions parameterized by the sets of "inputs" $X$ and "outputs" $Y$. The composition of resumptions $f \in \mathbf{Res}(X,Y)$ and $g \in \mathbf{Res}(Y,Z)$ is defined (coinductively [Aczel 1988]) by:

$$f;g(x) = \begin{cases} (z, f';g') & f(x) = (y, f'),\ g(y) = (z, g') \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The identity resumption $\mathtt{id}_X \in \mathbf{Res}(X,X)$ is defined by

$$\mathtt{id}_X(x) = (x, \mathtt{id}_X).$$

We can picture this composition as sequential (or "series") composition of transducers.

We can define a monoidal structure on **Res** by

$$X \otimes Y = X + Y \quad \text{(disjoint union of sets)}$$

and if $f \in \mathbf{Res}(X,Y)$, $g \in \mathbf{Res}(X',Y')$, $f \otimes g \in \mathbf{Res}(X \otimes X', Y \otimes Y')$ is defined by:

$$f \otimes g(\mathtt{inl}(x)) = \begin{cases} (\mathtt{inl}(y), f' \otimes g), & f(x) = (y, f') \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$f \otimes g(\mathtt{inr}(x')) = \begin{cases} (\mathtt{inr}(y'), f \otimes g'), & g(x') = (y', g') \\ \text{undefined} & \text{otherwise.} \end{cases}$$

This is (asynchronous) parallel composition of transducers: at each stage, we respond to an input on the $X$ "wire" according to $f$, with output appearing on the $Y$ wire, and to an input on the $X'$ wire according to $g$, with output appearing on the $Y'$ wire.

Let us prove that composition in **Res** is associative. Given $f \in \mathbf{Res}(X,Y), g \in \mathbf{Res}(Y,Z), h \in \mathbf{Res}(Z,W)$ we wish to prove that $(f;g);h = f;(g;h)$. Define

$$R_3 = \mathbf{Res}(X,Y) \times \mathbf{Res}(Y,Z) \times \mathbf{Res}(Z,W)$$

and

$$T_{X,Y}(A) = X \rightharpoonup Y \times A.$$

We define a $T_{X,W}$-coalgebra structure on $R_3$, $\alpha : R_3 \to T_{X,W}(R_3)$ by

$$\alpha(f,g,h)(x) = \begin{cases} (w, (f', g', h')) & \text{if } f(x) = (y, f')\ , g(y) = (z, g'), \text{and } h(z) = (w, h') \\ \text{undefined} & \text{otherwise} \end{cases}$$

We define maps $\beta_1, \beta_2 : R_3 \to \mathbf{Res}(X,W)$ by $\beta_1(f,g,h) = (f;g);h$ and $\beta_2(f,g,h) = f;(g;h)$. It suffices to show that $\beta_1$ and$\beta_2$ are $T_{X,W}$-coalgebra homomorphisms: for then by the final coalgebra property of $\mathbf{Res}(X,W)$ they are equal.

This amounts to showing that

$$\beta_1(f,g,h)(x) = \begin{cases} (w, \beta_1(f', g', h')) & \text{if } \alpha(f,g,h) = (w, (f', g', h')) \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$\beta_2(f,g,h)(x) = \begin{cases} (w, \beta_2(f', g', h')) & \text{if } \alpha(f,g,h) = (w, (f', g', h')) \\ \text{undefined} & \text{otherwise} \end{cases}$$

which is immediate from the definition.

The remaining definitions to make this into a symmetric monoidal structure on **Res** are straightforward, and left to the reader. Note that the associativity and symmetry isomorphisms, like the identities, have just one state; they are "history-free".

Finally, there is a feedback operator: for each $X$, $Y$, $U$ a function

$$\mathtt{Tr}^U_{X,Y} : \mathbf{Res}(X \otimes U, Y \otimes U) \longrightarrow \mathbf{Res}(X,Y)$$

defined by

$$\mathtt{Tr}^U_{X,Y}(f)(x) = \begin{cases} (y, f'), & f(x) = (y, f') \vee \\ & \exists k.\, f(x) = (u_0, f_0), \\ & f_0(u_0) = (u_1, f_1), \\ & \vdots \\ & f_k(u_k) = (y, f') \\ \text{undefined} & \text{otherwise.} \end{cases}$$

One has the same "particle-style" imagery as discussed earlier for **PInj**: imagine a token entering at the $X$ wire, circulating $k$ times around the feedback loop at the $U$ wire, and exiting at $Y$.

It can be verified that this satisfies the axioms for a trace: let us look at the cases of Yanking and Vanishing II. Note that **Pfn** can be embedded in **Res** by an identity-on-objects embedding $r : \mathbf{Pfn} \to \mathbf{Res}$: if $f : X \rightharpoonup Y$, then $r(f) \in \mathbf{Res}(X, Y)$ is defined by

$$r(f)(x) = \begin{cases} (f(x), r(f)), & f(x) \text{ defined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Thus $r(f)$ is the "history-free extension in time" of $f$. Moreover, observe that $r$ is a traced monoidal functor. It immediately follows that Yanking is valid in **Res**, since the symmetry morphism $\sigma_{U,U}$ in **Res** is the image under $\sigma$ of the symmetry in **Pfn**.

To prove Vanishing II, if $g : X \otimes U \otimes V \to Y \otimes U \otimes V$ then let $T_1(g) = \mathtt{Tr}^{U\otimes V}_{X,Y}(g)$ and let $T_2(g) = \mathtt{Tr}^U_{X,Y}(\mathtt{Tr}^V_{X\otimes U,Y\otimes U}(g))$. Unpacking the definitions, we get:

$$T_1(g)(x) = \begin{cases} (y, T_1(g')), & \text{if } g(x) = (y, g') \vee \\ & \exists k \geq 0\, g(x) = (w_0, g_0), \\ & g_0(w_0) = (w_1, g_1), \\ & \vdots \\ & g_k(w_k) = (y, g'), w_i \in U \uplus V, 0 \leq i \leq k \\ \text{undefined} & \text{otherwise.} \end{cases}$$

$$T_2(g)(x) = \begin{cases} (y, T_2(g')), & \text{if } g(x) = (y, g') \vee \\ & \exists k \geq 0\, g(x) = (w_0, g_0), \\ & g_0(w_0) = (w_1, g_1), \\ & \vdots \\ & g_k(w_k) = (y, g'), w_i \in U \uplus V, 0 \leq i \leq k \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The fact that $T_1(g) = T_2(g)$ then follows by standard co-inductive reasoning.

The functor $\mathbb{N} \times -$ can also be extended from **Pfn** to **Res**:

$$\mathbb{N} \times f(n, x) = \begin{cases} ((n, y), \mathbb{N} \times f'), & f(x) = (y, f') \\ \text{undefined} & \text{otherwise} \end{cases}$$

Using these extensions, the GoI situation structure given for **PInj** and **Pfn** can be transferred to **Res**.

### From resumptions to strategies

To interpret the category $\mathcal{G}(\mathbf{Res})$, think of an object $(X^+, X^-)$ as a rudimentary two-person game, in which $X^+$ is the set of moves for Player, and $X^-$ the set of moves for Opponent. A resumption $f : X^- \longrightarrow X^+$ is then a *strategy* for Player. Note that we can represent such a strategy by its set of *plays*:

$$P(f) = \{x_1 y_1 \cdots x_k y_k \mid f(x_1) = (y_1, f_1), \ldots, f_{k-1}(x_k) = (y_k, f_k)\}.$$

One can then show that composition in $\mathcal{G}(\mathbf{Res})$ is given by "parallel composition plus hiding" [Abramsky 1994, Abramsky and Jagadeesan 1994b]:

$$P(f; g) = \{s \upharpoonright X, Z \mid s \in P(f) || P(g)\}$$

$$S||T = \{s \in \mathcal{L}(X,Y,Z) \mid s \upharpoonright X,Y \in S \wedge s \upharpoonright Y,Z \in T\}$$

where $X = X^+ + X^-$, $Y = Y^+ + Y^-$, $Z = Z^+ + Z^-$, and

$$\mathcal{L}(S_1, S_2, S_3) = \{s \in (S_1 + S_2 + S_3)^* \mid s_i \in S_j \wedge s_{i+1} \in S_k \implies |j-k| \le 1\}.$$

The identities are the "copycat" strategies as in [Abramsky and Jagadeesan 1994a]. We can then obtain the simple category of games by applying a specification structure in the sense of [Abramsky *et al.* 1996] to $\mathcal{G}(\mathbf{Res})$, in which the properties over $(X^+, X^-)$ are the prefix-closed subsets of $(X^-X^+)^*$, *i.e.* the "safety properties", which in this context are the game trees.

## 5.5 SRel

We now consider a probabilistic version of **Pfn** or **Rel** as a GoI situation. Consider the category **SRel** of *stochastic relations*, with measurable spaces $(X, \mathcal{F}_X)$ as objects and stochastic kernels as arrows. An arrow $f : (X, \mathcal{F}_X) \longrightarrow (Y, \mathcal{F}_Y)$ is a map $f : X \times \mathcal{F}_Y \longrightarrow [0,1]$ such that $f(\cdot, B) : X \longrightarrow [0,1]$ is a bounded measurable function for fixed $B \in \mathcal{F}_Y$ and $f(x, \cdot) : \mathcal{F}_Y \longrightarrow [0,1]$ is a subprobability measure (i.e., $\sigma$-additive, set function, $f(x, \emptyset) = 0$ and $f(x, Y) \le 1$). The identity morphism $1_X : (X, \mathcal{F}_X) \longrightarrow (X, \mathcal{F}_X)$ is $1_X : X \times \mathcal{F}_X \longrightarrow [0,1]$ and is defined by

$$1_X(x, A) = \delta(x, A) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$$

For $A$ fixed, $\delta(x, A)$ is the characteristic function of $A$ and for $x$ fixed, $\delta(x, A)$ is the Dirac distribution. Finally, composition is defined as follows: given $f : (X, \mathcal{F}_X) \longrightarrow (Y, \mathcal{F}_Y)$ and $g : (Y, \mathcal{F}_Y) \longrightarrow (Z, \mathcal{F}_Z)$, $gf : (X, \mathcal{F}_X) \longrightarrow (Z, \mathcal{F}_Z)$ is given by

$$gf(x, C) = \int_Y g(y, C)\, f(x, dy),$$

where we are using $f(x, \cdot)$ as the measure for the integration and the function being integrated is the measurable function $g(\cdot, C)$.

This category is based on the work of Giry [Giry 1981] and Lawvere [Lawvere 1963]. In fact **SRel** is the Kleisli category of certain functor $\Pi$ over the category **Mes** of measurable spaces and measurable functions. The category **SRel** was defined by Panangaden [Panangaden 1997] with slight modifications on Giry's work. This category first appeared in [Abramsky 1996]. More details on **SRel** and the measure theory background can be found in [Panangaden 1997].

We will sometimes refer to objects in **SRel** by their underlying sets, so we write $X$ for $(X, \mathcal{F}_X)$.

**Proposition 5.5** The category **SRel** has countable coproducts.

**Proof**. Given a family $\{(X_i, \mathcal{F}_{X_i})\}_{i \in I}$ of objects in **SRel**, the coproduct $(X, \mathcal{F}_X)$ is defined as follows. The set $X$ is the disjoint union of the $X_i$. The $\sigma$-field on $X$ is generated by the measurable sets of each component. Thus, a measurable set in $\mathcal{F}_X$ is of the form $\biguplus_{i \in I} A_i$, where $A_i \in \mathcal{F}_{X_i}$ for all $i \in I$. The injections $in_i : X_i \longrightarrow X$ are $in_i(x, \biguplus_{k \in I} A_k) = \delta(x, A_i)$. Given an object $(Y, \mathcal{F}_Y)$ and a family $f_j : X_j \longrightarrow Y$ with $j \in I$, the mediating morphism $f : X \longrightarrow Y$ is defined by $f((x,i), B) = f_i(x, B)$. For further details see [Haghverdi 2000a].

We check the required commutativity

$$\begin{aligned} (f in_j)(x, B) &= \int_X f((x', i), B) in_j(x, d(x', i)) \\ &= \int_X f((x', i), B) \delta(x, d(x', j)) \\ &= \int_{X_j} f_j(x', B) \delta(x, dx'), \text{ the integrals over } X_k \text{ for } k \ne j \text{ are } 0 \\ &= f_j(x, B). \end{aligned}$$

Suppose $g : X \longrightarrow Y$ is another morphism such that $gin_j = f_j$ for all $j \in I$. Then,

$$\begin{aligned}
f((x,j),B) &= f_j(x,B) \\
&= gin_j(x,B) \\
&= \int_X g((x',i),B) in_j(x, d(x',i)) \\
&= \int_X g((x',i),B) \delta(x, d(x',j)) \\
&= \int_{X_j} g_j(x',B)\delta(x,dx') \\
&= g_j(x,B) \\
&= g((x,j),B)
\end{aligned}$$

for all $x \in X$ and $B \in \mathcal{F}_Y$. Thus, $g = f$. $\square$

**SRel** is a symmetric monoidal category with coproduct as the tensor product and $I = (\emptyset, \mathcal{F}_\emptyset)$. As in **PInj**, **Pfn**, and $\mathbf{Rel}_+$, we can define a trace by a general summation formula: given $f : (X \uplus U, \mathcal{F}_{X\uplus U}) \longrightarrow (Y \uplus U, \mathcal{F}_{Y\uplus U})$,

$$\mathtt{Tr}^U_{X,Y}(f) = f_{XY} + \sum_{n\in\omega} f_{UY} f^n_{UU} f_{XU}$$

Here we define $\{f_i\}_{i\in I}$ in $\mathbf{SRel}(X,Y)$ to be summable if

$$\textstyle\sum_i f_i(x,Y) \leq 1 \text{ for all } x \in X.$$

In that case, $(\sum_i f_i)(x,B) = \sum_i f_i(x,B)$ where the latter is the usual sum of real numbers. The component maps in the trace formula are as before: $\mathtt{inl} = in_1, \mathtt{inr} = in_2, \rho_1 : Y \uplus U \longrightarrow Y$ defined by $\rho_1((1,y),B) = \delta(y,B)$ and $\rho_1((2,u),B) = 0$; and dually, $\rho_2 : Y \uplus U \longrightarrow U$, given by $\rho_2((2,u),C) = \delta(u,C)$ and $\rho_2((1,y),C) = 0$.

We define $T : \mathbf{SRel} \longrightarrow \mathbf{SRel}$ as $T(X,\mathcal{F}_X) = (\mathbb{N}\times X, \mathcal{F}_{\mathbb{N}\times X})$ where $\mathcal{F}_{\mathbb{N}\times X}$ is the $\sigma$-field on $X \uplus X \uplus X \cdots$ ($\omega$ copies). For a given $f : (X,\mathcal{F}_X) \longrightarrow (Y,\mathcal{F}_Y)$, $Tf((n,x), \biguplus_{i\in\omega} B_i) = f(x,B_n)$. Let $U = \mathbb{N}^\infty$ with the Baire metric which induces a topology on $\mathbb{N}^\infty$ which corresponds to the product topology obtained from $\mathbb{N}$ with discrete topology. The following proposition is due to P. Panangaden and the second author.

**Proposition 5.6** $(\mathbf{SRel}, T, \mathbb{N}^\infty)$ is a GoI situation.

**Proof**. We show that $T$ as defined above is an additive and hence a traced symmetric monoidal functor (by the analog of Lemma 5.1). First, observe that $T = (T, \psi, \psi_I)$ is a monoidal functor with $\psi_{X,Y} : ((\mathbb{N}\times X) \uplus (\mathbb{N}\times Y), \mathcal{F}_{(\mathbb{N}\times X)\uplus(\mathbb{N}\times Y)}) \longrightarrow (\mathbb{N}\times (X \uplus Y), \mathcal{F}_{\mathbb{N}\times(X\uplus Y)})$ given by

$$\psi_{X,Y}((1,(n,x)), \biguplus_{i\in\omega}(A_i \uplus B_i)) = \delta(x, A_n)$$

$$\psi_{X,Y}((2,(n,y)), \biguplus_{i\in\omega}(A_i \uplus B_i)) = \delta(y, B_n).$$

Also,

$$\psi^{-1}_{X,Y}((n,(1,x)), (\biguplus_{i\in\omega} A_i) \uplus (\biguplus_{i\in\omega} B_i)) = \delta(x, A_n)$$

$$\psi^{-1}_{X,Y}((n,(2,y)), (\biguplus_{i\in\omega} A_i) \uplus (\biguplus_{i\in\omega} B_i)) = \delta(y, B_n).$$

Finally, $\psi_I = 1_I$.

Given a summable family $\{f_i\}_{i\in I} \in \mathbf{SRel}\,(X,Y)$, $\sum_I f_i(x,Y) \leq 1$ since $\{f_i\}$ is summable, and hence $\{Tf_i\}$ is summable.

$$\begin{array}{rcl}(\sum_I Tf_i)((n,x),\biguplus_{i\in\omega}B_i) &=& \sum_I Tf_i((n,x),\biguplus_{i\in\omega}B_i)\\ &=& \sum_I f_i(x,B_n)\\ &=& (\sum_I f_i)(x,B_n)\\ &=& T(\sum_I f_i)((n,x),\biguplus_{i\in\omega}B_i)\end{array}$$

Hence, $T$ is an additive and, by the analog of Lemma 5.1, a traced functor.

Next we consider the monoidal natural transformations.

- (Comultiplication): $e_X : \mathbb{N}\times(\mathbb{N}\times X) \lhd \mathbb{N}\times X : e'_X$
  $e_X((n_1,(n_2,x)),\biguplus_{i\in\omega}A_i) = \delta(x, A_{\langle n_1,n_2\rangle})$ and $e'_X((n,x),\biguplus_i(\biguplus_j A_{ij})) = \delta(x,A_{n_1n_2})$ where $n=\langle n_1,n_2\rangle$.

$$\begin{array}{rcl}e'_Xe_X((n_1,(n_2,x)),\biguplus_{ij}A_{ij}) &=& \displaystyle\int_{\mathbb{N}\times X} e'_X((n,x'),\biguplus_{ij}A_{ij})e_X((n_1,(n_2,x)),d(n,x'))\\ &=& \displaystyle\int_{\mathbb{N}\times X} e'_X((n,x'),\biguplus_{ij}A_{ij})\delta(x,d(\langle n_1,n_2\rangle,x'))\\ &=& e'_X((\langle n_1,n_2\rangle,x),\biguplus_{ij}A_{ij})\\ &=& \delta(x,A_{n_1n_2})\\ &=& 1_{\mathbb{N}\times(\mathbb{N}\times X)}((n_1,(n_2,x)),\biguplus_{ij}A_{ij})\end{array}$$

- (Dereliction) $d_X : X \lhd \mathbb{N}\times X : d'_X$
  $d_X(x,\biguplus_i A_i) = \delta(x,A_{n_0})$ for a fixed $n_0\in\mathbb{N}$ and $d'_X((n,x),A) = \delta(x,A)$, if $n=n_0$ and 0, otherwise.

$$\begin{array}{rcl}d'_Xd_X(x,A) &=& \displaystyle\int_{\mathbb{N}\times X} d'_X((n,x'),A)d_X(x,d(n,x'))\\ &=& \displaystyle\int_{\mathbb{N}\times X} d'_X((n,x'),A)\delta(x,d(n_0,x'))\\ &=& d'_X((n_0,x),A)\\ &=& \delta(x,A)\\ &=& 1_X(x,A).\end{array}$$

- (Contraction) $c_X : \mathbb{N}\times X \uplus \mathbb{N}\times X \lhd \mathbb{N}\times X : c'_X$
  $c_X((1,(n,x)),\biguplus_{i\in\omega}A_i) = \delta(x,A_{2n})$,
  $c_X((2,(n,x)),\biguplus_i A_i) = \delta(x,A_{2n+1})$ and

$$c'_X((n,x),(\biguplus_i A_i)\uplus(\biguplus_i B_i)) = \begin{cases}\delta(x,A_{n/2}), & \text{if } n \text{ is even;}\\ \delta(x,B_{(n-1)/2}), & \text{if } n \text{ is odd.}\end{cases}$$

  $c'_Xc_X((1,(n,x)),(\biguplus_i A_i)\uplus(\biguplus_i B_i))$
  $= \int_{\mathbb{N}\times X} c'_X((n',x'),(\biguplus_i A_i)\uplus(\biguplus_i B_i))c_X((1,(n,x)),d(n',x'))$
  $= \int_{\mathbb{N}\times X} c'_X((n',x'),(\biguplus_i A_i)\uplus(\biguplus_i B_i))\delta(x,d(2n,x'))$
  $= c'_X((2n,x),(\biguplus_i A_i)\uplus(\biguplus_i B_i)))$
  $= \delta(x,A_n)$
  $= 1_{(\mathbb{N}\times X)\uplus(\mathbb{N}\times X)}((1,(n,x)),(\biguplus_i A_i)\uplus(\biguplus_i B_i))$

  Similarly, $c'_Xc_X((2,(n,x)),(\biguplus_i A_i)\uplus(\biguplus_i B_i)) = \delta(x,B_n)$. Therefore, $c'_Xc_X = 1_{(\mathbb{N}\times X)\uplus(\mathbb{N}\times X)}$.

- (Weakening) $w_X : \emptyset \lhd \mathbb{N} \times X : w'_X$. We set $w_X$ to be the empty function and $w'_X((n,x),\emptyset) = 0$. Clearly $w'_X w_X = 1_{(\emptyset,\mathcal{F}_\emptyset)}$.

We prove the naturality and retract property for comultiplication. The proofs of naturality and retract property for $d_X$, $c_X$ and $w_X$ follow in a similar way. Let $f : X \longrightarrow Y$,

$$\begin{aligned}
(Tf)e_X((n_1,(n_2,x)),\biguplus_{i\in\omega} B_i) &= \int_{\mathbb{N}\times X} Tf((n',x'),\biguplus_i B_i)e_X((n_1,(n_2,x)),d(n',x')) \\
&= \int_{\mathbb{N}\times X} Tf((n',x'),\biguplus_i B_i)\delta(x,d(\langle n_1,n_2\rangle,x')) \\
&= Tf((\langle n_1,n_2\rangle,x),\biguplus_i B_i) \\
&= f(x,B_{\langle n_1,n_2\rangle})
\end{aligned}$$

$$\begin{aligned}
e_Y(T(Tf))((n_1,(n_2,x)),\biguplus_i B_i) &= \int_{\mathbb{N}\times(\mathbb{N}\times Y)} e_Y((n'_1,(n'_2,y)),\biguplus_i B_i)T(Tf)((n_1,(n_2,x)),d(n'_1,(n'_2,y)) \\
&= \int_{\mathbb{N}\times(\mathbb{N}\times Y)} e_Y((n'_1,(n'_2,y)),\biguplus_i B_i)f(x,d(n_1,(n_2,y))) \\
&= \int_{\mathbb{N}\times(\mathbb{N}\times Y)} \delta(y,B_{\langle n'_1,n'_2\rangle})f(x,d(n_1,(n_2,y))) \\
&= f(x,B_{\langle n_1,n_2\rangle})
\end{aligned}$$

Therefore, $(Tf)e_X = e_Y(T(Tf))$ for all $f : X \longrightarrow Y$.

$$\begin{aligned}
e'_X e_X((n_1,(n_2,x)),\biguplus_{ij} A_{ij}) &= \int_{\mathbb{N}\times X} e'_X((n,x'),\biguplus_{ij} A_{ij})e_X((n_1,(n_2,x)),d(n,x')) \\
&= \int_{\mathbb{N}\times X} e'_X((n,x'),\biguplus_{ij} A_{ij})\delta(x,d(\langle n_1,n_2\rangle,x')) \\
&= e'_X((\langle n_1,n_2\rangle,x),\biguplus_{ij} A_{ij}) \\
&= \delta(x,A_{n_1 n_2}) \\
&= 1_{\mathbb{N}\times(\mathbb{N}\times X)}((n_1,(n_2,x)),\biguplus_{ij} A_{ij})
\end{aligned}$$

Finally, we show that $\mathbb{N}^\infty$ is a reflexive object. We will denote the objects in **SRel** by their first components in order not to overload the notation.

- $j : \mathbb{N}^\infty \uplus \mathbb{N}^\infty \lhd \mathbb{N}^\infty \ : k$
  Consider the maps $j : \mathbb{N}^\infty \uplus \mathbb{N}^\infty \longrightarrow \mathbb{N}^\infty$ defined by $j((1,\vec{n}),A) = \delta(0.\vec{n},A)$, $j((2,\vec{n}),A) = \delta(1.\vec{n},A)$ and $k : \mathbb{N}^\infty \longrightarrow \mathbb{N}^\infty \uplus \mathbb{N}^\infty$ defined by

$$k(\vec{n},A_1 \uplus A_2) = \begin{cases} \delta(\vec{n}',A_1), & \text{if } \vec{n} = 0.\vec{n}'; \\ \delta(\vec{n}',A_2), & \text{if } \vec{n} = 1.\vec{n}'; \\ 0, & \text{else.} \end{cases}$$

$$\begin{aligned}
kj((1,\vec{n}),A_1 \uplus A_2) &= \int_{\mathbb{N}^\infty} k(\vec{n}',A_1 \uplus A_2)j((1,\vec{n}),d\vec{n}') \\
&= \int_{\mathbb{N}^\infty} k(\vec{n}',A_1 \uplus A_2)\delta(0.\vec{n},d\vec{n}') \\
&= k(0.\vec{n},A_1 \uplus A_2) \\
&= \delta(\vec{n},A_1) \\
&= 1_{\mathbb{N}^\infty \uplus \mathbb{N}^\infty}((1,\vec{n}),A_1 \uplus A_2)
\end{aligned}$$

Similarly, $kj((2,\vec{n}), A_1 \uplus A_2) = \delta(\vec{n}, A_2) = 1_{\mathbb{N}^\infty \uplus \mathbb{N}^\infty}((2,\vec{n}), A_1 \uplus A_2)$.

- $\emptyset \lhd \mathbb{N}^\infty$, as $(\emptyset, \mathcal{F}_\emptyset)$ is the zero object.
- $u : \mathbb{N} \times \mathbb{N}^\infty \longrightarrow \mathbb{N}^\infty : v$
  $u$ is defined by $u((i,\vec{n}), A) = \delta(i.\vec{n}, A)$ and $v : \mathbb{N}^\infty \longrightarrow \mathbb{N} \times \mathbb{N}^\infty$ by $v(i.\vec{n}, \biguplus_{i\in\omega} A_i) = \delta(\vec{n}, A_i)$.

$$\begin{aligned} vu((i,\vec{n}), \biguplus_{i\in\omega} A_i) &= \int_{\mathbb{N}^\infty} v(\vec{n}', \biguplus_{i\in\omega} A_i) u((i,\vec{n}), d\vec{n}') \\ &= \int_{\mathbb{N}^\infty} v(\vec{n}', \biguplus_{i\in\omega} A_i) \delta(i.\vec{n}, d\vec{n}') \\ &= v(i.\vec{n}, \biguplus_{i\in\omega} A_i) \\ &= \delta(\vec{n}, A_i) \\ &= 1_{\mathbb{N}\times\mathbb{N}^\infty}((i,\vec{n}), \biguplus_{i\in\omega} A_i). \end{aligned}$$

□

## 5.6 $\omega$-CPO

Consider the category $\omega$-**CPO** of $\omega$-complete partial orders with bottom element and continuous maps as morphisms. $\omega$-**CPO** is a traced symmetric monoidal category with product as the tensor, $I = \{\perp\}$ and given $f : X \times U \longrightarrow Y \times U$, $\mathtt{Tr}^U_{X,Y}(f) = f_1\langle 1_X, \mathsf{Y}(f_2)\rangle$ where $f_1 = pr_1 f$ and $f_2 = pr_2 f$ and $\mathsf{Y} = \mathsf{Y}_{X,U} : \omega\text{-}\mathbf{CPO}(X \times U, U) \to \omega\text{-}\mathbf{CPO}(C, U)$ is the (parametrized) Tarski least fixed-point operator: given $k : X \times U \longrightarrow U$, $\mathsf{Y}(k)(x) = \mu u.k(x,u) = u$.

**Proposition 5.7** ($\omega$-**CPO**, $(-)^{\mathbb{N}}$, $A^{\mathbb{N}}$) is a GoI Situation where $A$ is any object in $\omega$-**CPO**(and the associated LCA will be nontrivial if $A$ is not $\{\perp\}$).

**Proof**. Note that the functor $T = (-)^{\mathbb{N}}$ determines a symmetric monoidal functor $(T, \psi, \psi_I)$, where we define the isomorphisms $\psi_{A,B}(f,g) = \langle f, g\rangle$, with $\langle f,g\rangle(n) = (f(n), g(n))$ for $n \in \mathbb{N}, f \in A^{\mathbb{N}}, g \in B^{\mathbb{N}}$ and $\psi_I(\perp) = f_\perp$, where $f_\perp$ is the constant-$\perp$ sequence. In order to show that $T$ is traced, it suffices to show that, given $f : X \times U \longrightarrow U$, $T(\mathsf{Y}(f)) = \mathsf{Y}'((Tf)\psi)$, where $\mathsf{Y}' = \mathsf{Y}_{TX,TU}$. Note that, for $g \in TX$,

$$\begin{aligned} T(\mathsf{Y}(f))(g)(n) &= \mathsf{Y}(f)(g(n)), \quad \text{by definition of } T \\ &= \mu u.f(g(n),u) = u, \quad \text{by definition of } \mathsf{Y}. \end{aligned}$$

On the other hand, $\mathsf{Y}'((Tf)\psi)(g) = \mu h.Tf\psi(g,h) = h$, therefore $f(g(n), h(n)) = h(n)$ for all $n \in \mathbb{N}$ and $h$ is the least element in $TU$ with this property. However, $f(g(n),u) = u$, thus $h(n) \le \mathcal{K}_u(n) = u$ where $\mathcal{K}_u$ is the constant-$u$ function in $TU$. Also $u \in U$ is the least element with $f(g(n),u) = u$ and therefore $u \le h(n)$. Hence $h(n) = u$, and $T(\mathsf{Y}(f))g = \mathsf{Y}'(Tf)\psi g$ for all $g \in TX$, which implies $T(\mathsf{Y}(f)) = \mathsf{Y}'(Tf\psi)$.

Now, given $f : X \times U \longrightarrow Y \times U$,

$$\begin{aligned} T(\mathtt{Tr}(f)) &= T(f_1\langle 1_X, \mathsf{Y}(f_2)\rangle) \\ &= T(f_1)T(\langle 1_X, \mathsf{Y}(f_2)\rangle) \\ &= T(f_1)\psi\langle 1_{TX}, \mathsf{Y}'(Tf_2\psi)\rangle \\ &= \mathtt{Tr}(\psi^{-1}Tf\psi) \end{aligned}$$

Hence $T$ is traced. We next discuss the necessary retractions:

1. Define $e_A : (A^{\mathbb{N}})^{\mathbb{N}} \longrightarrow A^{\mathbb{N}}$ by $e_A(f) = g$ for $f \in (A^{\mathbb{N}})^{\mathbb{N}}$ and $g(n) = f(n_1, n_2)$ where $n = \langle n_1, n_2\rangle$. And $e'(A) : A^{\mathbb{N}} \longrightarrow (A^{\mathbb{N}})^{\mathbb{N}}$ by $e'_A(h) = k$ where $k(n_1,n_2) = h(\langle n_1, n_2\rangle)$. Clearly $e'_A e_A = 1$ for all $A$ and $e, e'$ are natural.
2. Define $d_A : A \longrightarrow A^{\mathbb{N}}$ by $d_A(a) = f$ where $f(n) = a$ for all $n \in \mathbb{N}$. And $d'_A : A^{\mathbb{N}} \longrightarrow A$ by $d'_A(f) = f(0)$. Clearly $d, d'$ are natural and $(d, d')$ forms a retract pair.

3. Define $c_A : A^{\mathbb{N}} \times A^{\mathbb{N}} \longrightarrow A^{\mathbb{N}}$ by $c_A(f,g) = h$ where

$$h(n) = \begin{cases} f(n/2), & \text{if } n \text{ is even;} \\ g((n-1)/2), & \text{if } n \text{ is odd.} \end{cases}$$

   And $c'_A : A^{\mathbb{N}} \longrightarrow A^{\mathbb{N}} \times A^{\mathbb{N}}$ by $c'_A(h) = (f,g)$ where $f(n) = h(2n)$ and $g(n) = h(2n+1)$. Clearly $c, c'$ are natural and $(c, c')$ is a retract pair.

4. Define $w_A : I \longrightarrow A^{\mathbb{N}}$ by $w_A(\perp) = f$ where $f(n) =\perp$ for all $n \in \mathbb{N}$ and $w'_A : A^{\mathbb{N}} \longrightarrow I$ by $w'_A(f) =\perp$ for all $f \in A^{\mathbb{N}}$. Clearly $w, w'$ are natural and $(w, w')$ is a retract pair.

Having found the functor $T$, it follows automatically that $U = TA$ is a reflexive object for any object $A$. In fact (i) $U \times U \lhd U$ follows from (3) above, (ii) $TU \lhd U$ from (1) above and finally (iii) $I \lhd U$ from (4) above. Hence $U = TA = A^{\mathbb{N}}$ is a reflexive object, where $A$ is any object in $\omega$-**CPO**.

Finally, observe that if $A$ is $I = \{\perp\}$, then the LCA is trivial since $\omega\text{-}\mathbf{CPO}(U,U) = \omega\text{-}\mathbf{CPO}(TI,TI) = \{id_{TI}\}$. □

**Remark 5.8** *GoI situations (and their associated LCA's) capture the essence of Girard's GoI interpretation: of the examples above (cf. [Abramsky 1996]), $\mathcal{G}(\mathbf{PInj})$ is essentially the original GoI construction in [Girard 1989], while $\mathcal{G}(\omega\text{-}\mathbf{CPO})$ is the "wave-style" model in [Abramsky and Jagadeesan 1994b]. In particular, the fact that our semantics coincides with Girard/Danos/Regnier in the case of partial injective maps has already been observed both in the papers [Abramsky and Jagadeesan 1994a, Abramsky, Jagadeesan, and Malacaria 2000 ] on history-free game semantics, and at greater length in Patrick Baillot's Thèse du Troisième Cycle [Baillot 1995]*

*Moreover, Girard's original operator-theoretic models (in the category of Hilbert spaces* $\mathbf{Hilb_2}$*), as well as Danos-Regnier's* small model *are also captured by particle-style GoI situations with some additional structure (see [Haghverdi 2000b], Section 6).*

We have not explicitly given an interpretation of sequent proofs in Multiplicative-Exponential Linear Logic in GoI situations. However, by showing that GoI situations give rise to Linear Combinatory Algebras, we *have* given an interpretation of proofs in a Hilbert-style system for MELL in GoI situations (the "Curry" part of the Curry-Howard correspondence, so to speak). Moreover, it is standard that sequent proofs in MELL can be translated into proofs in the Hilbert-style system [Avron 1988, Troelstra 1992]. So concrete interpretations of sequent proofs can be extracted quite mechanically from our results. The purpose of this paper is to expose the abstract structure from which the concrete interpretations arise as particular instances.

# 6 Conclusion and Future Work

As we have seen, the abstract framework introduced above captures Girard's Geometry of Interaction interpretation for the full multiplicative-exponential fragment of linear logic. Moreover, from each such GoI situation we obtain a (standard) combinatory algebra. This suggests several intriguing directions for future work.

1. It would be interesting to compare different LCA's: for example, to analyze and classify the fine structure of those combinatory algebras (and their subalgebras) arising from our construction. One candidate for such a classification are realizability toposes generated by partial combinatory algebras (and their associated full subcategories of assemblies and modest sets). These structures have recently enjoyed considerable interest in providing new models of intuitionistic type theories and also fine-grained detail on models of computation, especially computability at higher types [Longley 1995, Longley 1998]. For example, as mentioned in the Introduction, recent work of Abramsky and Longley [Abramsky and Longley 2000] presents a combinatory algebra $\mathcal{A}$ of history-free strategies arising from our LCA construction on **Pfn** above. $\mathcal{A}$ realizes the finite type structure of the strongly stable (= sequentially realizable) functionals, while a sub-PCA of $\mathcal{A}$ gives rise to the PCF-sequential ones. Similarly, [Abramsky and Lenisa 2000] explores the subalgebra of **Pfn** of partial involutions. However many other possibilities remain, for example subalgebras of **SRel**.

2. While the GoI interpretation for the multiplicative-exponential fragment is under control, the extension of our abstract treatment of GoI to cover the additives is open, at least for particle-style semantics. Such a treatment would need to account for Girard's work in [Girard 1995] (for wave-style models, an adequate treatment of the additives is described in [Abramsky and Jagadeesan 1994b]).
3. Finally, related to (1) above, it is known how to interpret Girard's system $\mathcal{F}$ over a combinatory algebra. For example, if one works in $\mathcal{G}(\mathbf{PInj})$, this is essentially given in [Abramsky and Jagadeesan 1994b]. A detailed study of the interpretation of system $\mathcal{F}$ in our other GoI models would be useful (cf. [Girard 1989]). This is especially so for the question of finding fully complete models (arising from a GoI situation) both for system $\mathcal{F}$ as well as for fragments of Linear Logic (starting with MLL + MIX). Such problems are currently being actively pursued by the authors.

# Appendix I: The Trace Axioms

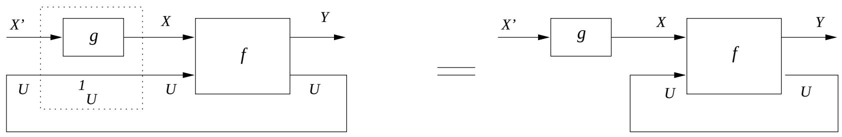



Figure 15: Naturality in $X$

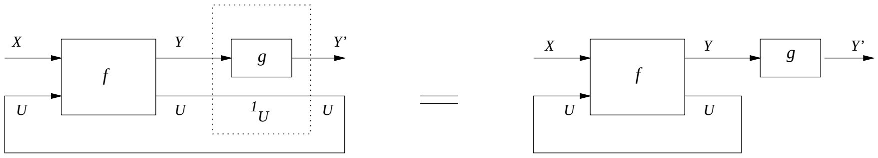



Figure 16: Naturality in $Y$

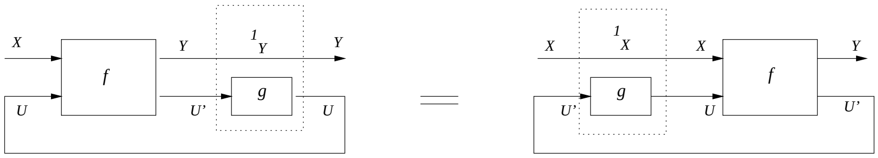



Figure 17: Dinaturality in $U$

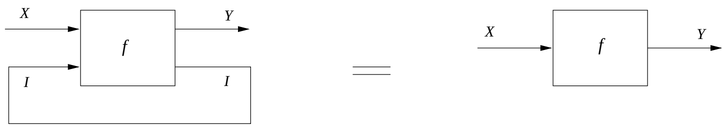



Figure 18: Vanishing I

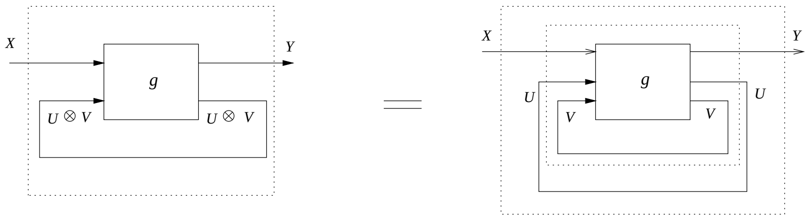



Figure 19: Vanishing II

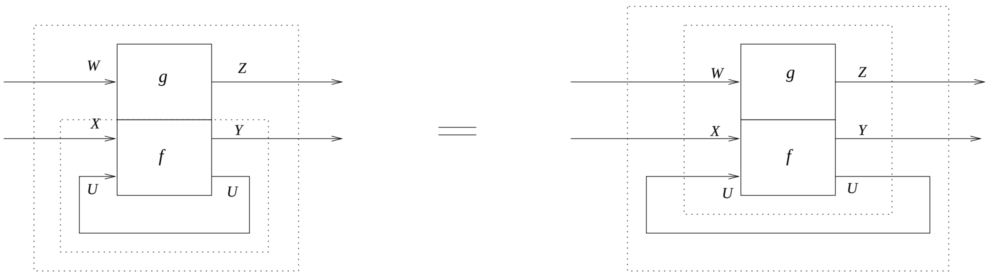



Figure 20: Superposing

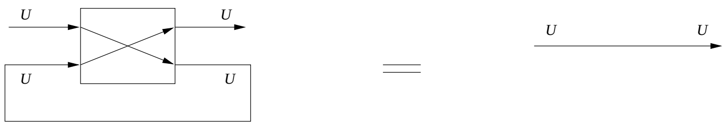



Figure 21: Yanking

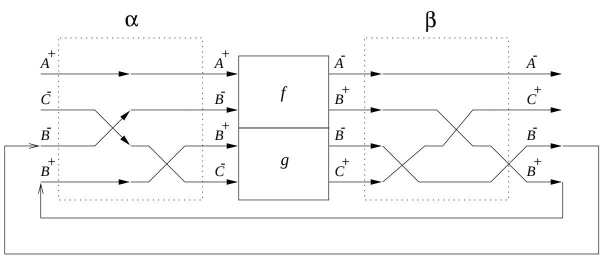



Figure 22: Composition in $\mathcal{G}(\mathbb{C})$

# Appendix II: Pointwise Naturality

**Notation:** To save space and make the text more readable we sometimes abbreviate the object $(A^+, A^-) \in \mathcal{G}(\mathbb{C})$ to $\underline{A}$.

We occasionally use the graphical calculus (cf. Appendix III), but only to simplify the composition of morphisms in $\mathcal{G}(\mathbb{C})$.

Recall the following definition.

**Definition** Let $(F, \varphi, \varphi_I), (G, \psi, \psi_I) : \mathbb{C} \longrightarrow \mathbb{C}$ be monoidal functors. A *monoidal pointwise natural transformation* $m : F \Rightarrow G$ is a family of maps $m_A : FA \longrightarrow GA$ indexed over the objects of $\mathbb{C}$ such that the following diagrams commute for all $f : I \longrightarrow A$:

$$\begin{tikzcd}
FI \arrow[r, "m_I"] \arrow[d, "Ff"'] & GI \arrow[d, "Gf"] \\
FA \arrow[r, "m_A"] & GA
\end{tikzcd}$$

$$\begin{tikzcd}
FA \otimes FB \arrow[r, "\varphi_{A,B}"] \arrow[d, "m_A \otimes m_B"'] & F(A \otimes B) \arrow[d, "m_{A \otimes B}"] \\
GA \otimes GB \arrow[r, "\psi_{A,B}"] & G(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "\varphi_I"] \arrow[dr, "\psi_I"'] & FI \arrow[d, "m_I"] \\
 & GI
\end{tikzcd}$$

Let $T = (T, \psi, \psi_I) : \mathbb{C} \longrightarrow \mathbb{C}$ be a traced symmetric monoidal functor on $\mathbb{C}$ and assume given the following retractions:

- $e : TT \lhd T : e'$, Comultiplication
- $d : Id \lhd T : d'$, Dereliction
- $c : T \otimes T \lhd T : c'$, Contraction
- $w : \mathcal{K}_I \lhd T : w'$, Weakening

Note that the retract morphisms above are monoidal natural transformations. We will prove that the family of maps defined below are monoidal pointwise natural transformations.

**1.** Define $\delta :! \Rightarrow !!$ by

$$\delta_{(A^+,A^-)} :!(A^+, A^-) \longrightarrow !!(A^+, A^-) =_{def} \sigma_{TTA^+,TA^-}(e'_{A^+} \otimes e_{A^-})$$

To prove pointwise naturality we need to show that

$$!!f\delta_{(I,I)} = \delta_{(B^+,B^-)}!f$$

for all $f : (I, I) \longrightarrow (B^+, B^-)$.

$$\begin{aligned}
!!f\delta_{(I,I)} &= (e_I \otimes 1_{T^2B^+})\psi^{-1}_{TI,TB^+}T\psi^{-1}_{I,B^+}T^2fT\psi_{I,B^-}\psi_{TI,TB^-}(e'_I \otimes 1_{T^2B^-}), \\
&\quad \text{using graphical calculus} \\
&= (e_I \otimes 1_{T^2B^+})\psi^{-1}_{TI,TB^+}T\psi^{-1}_{I,B^+}e'_{B^+}Tfe_{B^-}T\psi_{I,B^-}\psi_{TI,TB^-}(e'_I \otimes 1_{T^2B^-}), \\
&\quad e, e' \text{ are natural trans. and } (e, e') \text{ is a retract pair} \\
&= (e_I \otimes 1_{T^2B^+})(e'_I \otimes e'_{B^+})\psi^{-1}_{I,B^+}Tf\psi_{I,B^-}(e_I \otimes e_{B^-})(e'_I \otimes 1_{T^2B^-}), \\
&\quad e, e' \text{ are monoidal} \\
&= (1_{TI} \otimes e'_{B^+})\psi^{-1}_{I,B^+}Tf\psi_{I,B^-}(1_{TI} \otimes e_{B^-}), \\
&\quad e_I = \psi_I\psi_I^{-1}T\psi_I^{-1}, e'_I = T\psi_I\psi_I\psi_I^{-1} \text{ (as } e, e' \text{ are monoidal)} \\
&\quad \text{and } T \text{ is a strong monoidal functor} \\
&= \delta_{(B^+,B^-)}!f, \quad \text{using graphical calculus}
\end{aligned}$$

$\delta$ is monoidal as follows:

$$\begin{aligned}
!\varphi_{\underline{A},\underline{B}}\varphi_{!\underline{A},!\underline{B}}(\delta_{\underline{A}}\otimes\delta_{\underline{B}}) &= \sigma_{T^2(A^+\otimes B^+),TA^-\otimes TB^-}\,(T\psi_{A^+,B^+}\otimes e_{A^-}\otimes e_{B^-}) \\
&\quad (\psi_{TA^+,TB^+}\otimes\psi^{-1}_{TA^-,TB^-})(e'_{A^+}\otimes e'_{B^+}\otimes T\psi^{-1}_{A^-,B^-}) \\
&= \sigma_{T^2(A^+\otimes B^+),TA^-\otimes TB^-}\,(e'_{A^+\otimes B^+}\otimes\psi^{-1}_{A^-,B^-})(\psi_{A^+,B^+}\otimes e_{A^-\otimes B^-}) \\
&= \delta_{\underline{A}\otimes\underline{B}}\varphi_{\underline{A},\underline{B}}
\end{aligned}$$

$$\begin{aligned}
\delta_{(I,I)}\varphi_{(I,I)} &= \sigma_{T^2I,I}(e'_I\otimes\psi_I^{-1})(\psi_I\otimes e_I) \\
&= \sigma_{T^2I,I}(T\psi_I\otimes\psi_I^{-1})(\psi_I\otimes T\psi_I^{-1}) \\
&= !\varphi_{(I,I)}\varphi_{(I,I)}
\end{aligned}$$

**2.** Define $\mathtt{der} :! \Rightarrow Id$ by

$$\mathtt{der}_{(A^+,A^-)} :!(A^+,A^-)\longrightarrow(A^+,A^-) =_{def} \sigma_{A^+,TA^-}(d'_{A^+}\otimes d_{A^-})$$

To prove pointwise naturality we need to show that

$$f\mathtt{der}_{(I,I)} = \mathtt{der}_{(B^+,B^-)}!f$$

for all $f:(I,I)\longrightarrow(B^+,B^-)$.

$$\begin{aligned}
f\mathtt{der}_{(I,I)} &= (d_I\otimes 1_{B^+})f(d'_I\otimes 1_{B^-}), \quad \text{using graphical calculus} \\
&= (d_I\otimes 1_{B^+})d'_{B^+}Tfd_{B^-}(d'_I\otimes 1_{B^-}), \quad d,d' \text{ are natural transformations} \\
&\quad \text{and } (d,d') \text{ is a retract pair} \\
&= (d_I\otimes 1_{B^+})(d'_I\otimes d'_{B^+})\psi^{-1}Tf\psi(d_I\otimes d_{B^-})(d'_I\otimes 1_{B^-}), \quad d,d' \text{ are monoidal} \\
&= (1_{TI}\otimes d'_{B^+})\psi^{-1}Tf\psi(1_{TI}\otimes d_{B^-}), \quad d_I=\psi_I, d'_I=\psi_I^{-1} \text{ (as } d,d' \text{ are monoidal)} \\
&\quad \text{and } T \text{ is a strong monoidal functor} \\
&= \mathtt{der}_{(B^+,B^-)}!f, \quad \text{using graphical calculus}
\end{aligned}$$

$\mathtt{der}$ is monoidal as follows:

$$\begin{aligned}
1_{\underline{A}\otimes\underline{B}}(\mathtt{der}_{\underline{A}}\otimes\mathtt{der}_{\underline{B}}) &= (1_{TA^-}\otimes\sigma_{A^+,TB^-}\otimes 1_{B^+})(\sigma_{A^+,TA^-}\otimes\sigma_{B^+,TB^-}) \\
&\quad (d'_{A^+}\otimes d_{A^-}\otimes d'_{B^+}\otimes d_{B^-})(1_{TA^+}\otimes\sigma_{TB^+,A^-}\otimes 1_{B^-}) \\
&= \sigma_{A^+\otimes B^+,TA^-\otimes TB^-}(d'_{A^+}\otimes d'_{B^+}\otimes d_{A^-}\otimes d_{B^-}) \\
&= \sigma_{A^+\otimes B^+,TA^-\otimes TB^-}(d'_{A^+\otimes B^+}\otimes\psi^{-1}_{A^-,B^-})(\psi_{A^+,B^+}\otimes d_{A^-\otimes B^-}) \\
&= \mathtt{der}_{\underline{A}\otimes\underline{B}}\varphi_{\underline{A},\underline{B}}
\end{aligned}$$

$$\begin{aligned}
\mathtt{der}_{(I,I)}\varphi_{(I,I)} &= \sigma_{I,I}(d'_I\otimes\psi_I^{-1})(\psi_I\otimes d_I) \\
&= \sigma_{I,I} \\
&= 1_{(I,I)}
\end{aligned}$$

**3.** Define $\mathtt{con} :! \Rightarrow !\otimes!$ by

$$\mathtt{con}_{(A^+,A^-)} :!(A^+,A^-)\longrightarrow !(A^+,A^-)\otimes !(A^+,A^-) =_{def} \sigma_{TA^+\otimes TA^+,TA^-}(c'_{A^+}\otimes c_{A^-})$$

To prove pointwise naturality we need to show that

$$\mathtt{con}_{(B^+,B^-)}!f = (!f\otimes !f)\mathtt{con}_{(I,I)}$$

for all $f : (I, I) \longrightarrow (B^+, B^-)$.

$$\begin{array}{rcl}
(!f \otimes !f)\mathtt{con}_{(I,I)} & = & (c_I \otimes 1_{TB^+} \otimes 1_{TB^+})(1_{TI} \otimes \sigma_{TB^+,TI} \otimes 1_{TB^+})(\psi^{-1} \otimes \psi^{-1})(Tf \otimes Tf) \\
& & (\psi \otimes \psi)(1_{TI} \otimes \sigma_{TI,TB^-} \otimes 1_{TB^-})(c'_I \otimes 1_{TB^-} \otimes 1_{TB^-}), \\
& & \text{using graphical calculus} \\
& = & (c_I \otimes 1_{TB^+} \otimes 1_{TB^+})(1_{TI} \otimes \sigma_{TB^+,TI} \otimes 1_{TB^+})(\psi^{-1} \otimes \psi^{-1})c'_{B^+}Tfc_{B^-} \\
& & (\psi \otimes \psi)(1_{TI} \otimes \sigma_{TI,TB^-} \otimes 1_{TB^-})(c'_I \otimes 1_{TB^-} \otimes 1_{TB^-}), \\
& & c, c' \text{ are natural trans. and } (c, c') \text{ is a retract pair} \\
& = & (c_I \otimes 1_{TB^+} \otimes 1_{TB^+})(c'_I \otimes c'_{B^+})\psi^{-1}Tf\psi(c_I \otimes c_{B^-})(c'_I \otimes 1_{TB^-} \otimes 1_{TB^-}) \\
& & c, c' \text{ are monoidal} \\
& = & (1_{TI} \otimes c'_{B^+})\psi^{-1}Tf\psi(1_{TI} \otimes c_{B^-}), \\
& & c_I = \psi_I(\psi_I^{-1} \otimes \psi_I^{-1}), c'_I = (\psi_I \otimes \psi_I)\psi_I^{-1} \\
& & (\text{as } c, c' \text{ are monoidal) and } T \text{ is a strong monoidal functor} \\
& = & \mathtt{con}_{(B^+,B^-)}!f, \quad \text{using graphical calculus}
\end{array}$$

$\mathtt{con}$ is monoidal as follows:

$(\varphi_{\underline{A},\underline{B}} \otimes \varphi_{\underline{A},\underline{B}})(1_{!\underline{A}} \otimes \sigma_{!\underline{A},!\underline{B}} \otimes 1_{!\underline{B}})(\mathtt{con}_{\underline{A}} \otimes \mathtt{con}_{\underline{B}}) =$
$\sigma_{T(A^+ \otimes B^+) \otimes T(A^+ \otimes B^+),TA^- \otimes TB^-}(\psi_{A^+,B^+} \otimes \psi_{A^+,B^+} \otimes c_{A^-} \otimes c_{B^-})$
$(1_{TA^+} \otimes \sigma_{TA^+,TB^+} \otimes 1_{TB^+} \otimes 1_{TA^-} \otimes \sigma_{TB^-,TA^-} \otimes 1_{TB^-})(c'_{A^+} \otimes c'_{B^+} \otimes \psi^{-1}_{A^-,B^-} \otimes \psi^{-1}_{A^-,B^-})$
$= \sigma_{T(A^+ \otimes B^+) \otimes T(A^+ \otimes B^+),TA^- \otimes TB^-}(c'_{A^+ \otimes B^+} \otimes \psi^{-1}_{A^-,B^-})(\psi_{A^+,B^+} \otimes c_{A^- \otimes B^-})$
$= \mathtt{con}_{\underline{A} \otimes \underline{B}}\, \varphi_{\underline{A},\underline{B}}$

$$\begin{array}{rcl}
\mathtt{con}_{\underline{I}}\varphi_{\underline{I}} & = & \sigma_{TI \otimes TI,I}(c'_I \otimes \psi_I^{-1})(\psi_I \otimes c_I) \\
& = & \sigma_{TI \otimes TI,I \otimes I}(\psi_I \otimes \psi_I \otimes \psi_I^{-1} \otimes \psi_I^{-1}) \\
& = & \sigma_{TI \otimes TI,I \otimes I}(1_{TI} \otimes \sigma_{I,TI} \otimes 1_I)(\psi_I \otimes \psi_I^{-1} \otimes \psi_I \otimes \psi_I^{-1})(1_I \otimes \sigma_{I,TI} \otimes 1_{TI}) \\
& = & (1_I \otimes \sigma_{TI,I} \otimes 1_{TI})(\sigma_{TI,I} \otimes \sigma_{TI,I})(\psi_I \otimes \psi_I^{-1} \otimes \psi_I \otimes \psi_I^{-1})(1_I \otimes \sigma_{I,TI} \otimes 1_{TI}) \\
& = & \varphi_{\underline{I}} \otimes \varphi_{\underline{I}}
\end{array}$$

**4.** Define $\mathtt{weak} :! \Rightarrow \mathcal{K}_I$ by

$$\mathtt{weak}_{(A^+,A^-)} :!(A^+, A^-) \longrightarrow (I, I) =_{def} \sigma_{I,TA^-}(w'_{A^+} \otimes w_{A^-})$$

To prove pointwise naturality we need to show that

$$1_{(I,I)}\mathtt{weak}_{(I,I)} = \mathtt{weak}_{(B^+,B^-)}!f$$

for all $f : (I, I) \longrightarrow (B^+, B^-)$.

$$\begin{array}{rcl}
1_{(I,I)}\mathtt{weak}_{(I,I)} & = & \sigma_{I,TI}(1_I \otimes w_I)1_{I \otimes I}(w'_I \otimes 1_I), \quad \text{using graphical calculus} \\
& = & \sigma_{I,TI}(1_I \otimes w_I)w'_{B^+}Tfw_{B^-}(w'_I \otimes 1_I), \quad w, w' \text{ are natural trans.} \\
& & \text{and } (w, w') \text{ is a retract pair} \\
& = & \sigma_{I,TI}(1_I \otimes w_I)(w'_I \otimes w'_{B^+})\psi^{-1}Tf\psi(w_I \otimes w_{B^-})(w'_I \otimes 1_I), \\
& & w, w' \text{ are monoidal} \\
& = & (1_{TI} \otimes w'_{B^+})\psi^{-1}Tf\psi(1_{TI} \otimes w_{B^-}), \quad w_I = \psi_I, w'_I = \psi_I^{-1} \\
& & (\text{as } w, w' \text{ are monoidal) and } T \text{ is strong monoidal, and } \sigma_{I,I} = 1_{I \otimes I} \\
& = & \mathtt{weak}_{(B^+,B^-)}!f, \quad \text{using graphical calculus}
\end{array}$$

`weak` is monoidal as follows:

$$
\begin{aligned}
1_{(I,I)}(\mathtt{weak}_{\underline{A}} \otimes \mathtt{weak}_{\underline{B}}) &= (1_{TA^-} \otimes \sigma_{I,TB^-} \otimes 1_I)(\sigma_{I,TA^-} \otimes \sigma_{I,TB^-}) \\
&\quad (w'_{A^+} \otimes w_{A^-} \otimes w'_{B^+} \otimes w_{B^-})(1_{TA^+} \otimes \sigma_{TB^+,I} \otimes 1_I) \\
&= \sigma_{I\otimes I,TA^-\otimes TB^-}(w'_{A^+} \otimes w'_{B^+} \otimes w_{A^-} \otimes w_{B^-}) \\
&= \sigma_{I,TA^-\otimes TB^-}(w'_{A^+\otimes B^+} \otimes \psi^{-1}_{A^-,B^-})(\psi_{A^+,B^+} \otimes w_{A^-\otimes B^-}) \\
&= \mathtt{weak}_{\underline{A}\otimes\underline{B}}\, \varphi_{\underline{A},\underline{B}}, \quad \text{using graphical calculus and definition of } \varphi
\end{aligned}
$$

$$
\begin{aligned}
\mathtt{weak}_{(I,I)}\varphi_{(I,I)} &= \sigma_{I,I}(w'_I \otimes \psi_I^{-1})(\psi_I \otimes w_I) \\
&= \sigma_{I,I} \\
&= 1_{(I,I)}
\end{aligned}
$$

Finally, let us discuss pointwise naturality versus ordinary naturality of the transformations of WLC's. We examine the case of `der`, the rest being similar.

Observe that `der` $:! \to Id$ is a natural transformation iff the following diagram commutes:

$$
\begin{tikzcd}
TA^+\otimes B^- \arrow[r, "1\otimes d_{B^-}"] \arrow[d, "="'] & TA^+\otimes TB^- \arrow[r, "\psi"] & T(A^+\otimes B^-) \arrow[r, "Tf"] & T(A^-\otimes B^+) \arrow[r, "\psi^{-1}"] \arrow[d, "d'_{A^-\otimes B^+}"'] & TA^-\otimes TB^+ \arrow[r, "1\otimes d'_{B^+}"] \arrow[d, "d'_{A^-}\otimes d'_{B^+}"'] & TA^-\otimes B^+ \arrow[d, "="'] \\
TA^+\otimes B^- \arrow[r, "d'_{A^+}\otimes 1"] & A^+\otimes B^- \arrow[r, "="] \arrow[u, "d_{A^+}\otimes d_{B^-}"] & A^+\otimes B^- \arrow[r, "f"] \arrow[u, "d_{A^+\otimes B^-}"] & A^-\otimes B^+ \arrow[r, "="] & A^-\otimes B^+ \arrow[r, "d_{A^-}\otimes 1"] & TA^-\otimes B^+
\end{tikzcd}
$$

The 3 squares in the middle commute because $d$ is a monoidal natural transformation. The right- and leftmost squares commute iff $d_{A^+}d'_{A^+} = 1$ and $d_{A^-}d'_{A^-} = 1$ (essentially, by setting $B = I$ and using monoidal identities.) We already have that $d'_{A^+}d_{A^+} = 1$ as $(d, d')$ is a retract pair. Since $A^+$ is arbitrary $(d, d')$ must be an isomorphism.

# Appendix III: Normal Forms for Combinators

Recall the equations for the combinators:

$I{\bullet}x = x$.

$B{\bullet}x{\bullet}y{\bullet}z = x{\bullet}(y{\bullet}z)$.

$C{\bullet}x{\bullet}y{\bullet}z = x{\bullet}z{\bullet}y$.

$W{\bullet}x{\bullet}\mathbf{!}y = x{\bullet}\mathbf{!}y{\bullet}\mathbf{!}y$.

$D{\bullet}x{\bullet}\mathbf{!}y = x{\bullet}y$.

$\delta{\bullet}\mathbf{!}x = \mathbf{!!}x$.

$F{\bullet}\mathbf{!}x{\bullet}\mathbf{!}y = \mathbf{!}(x{\bullet}y)$.

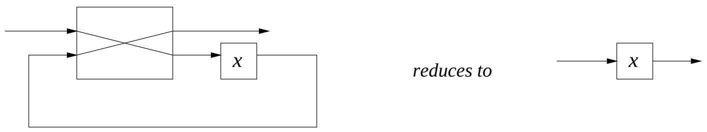



Figure 23: Identity Combinator I

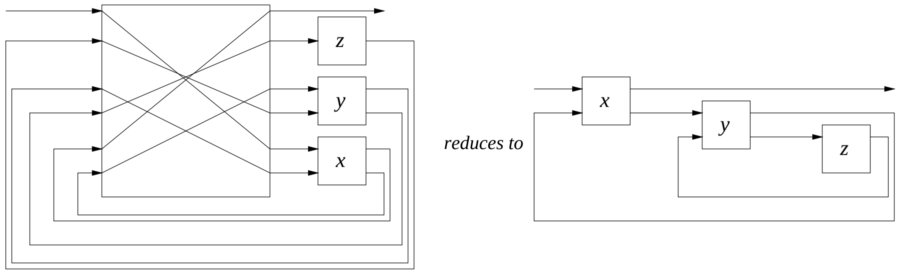



Figure 24: Composition Combinator B

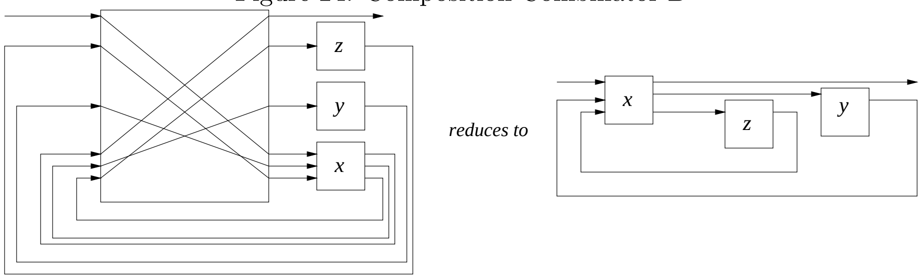



Figure 25: Exchange Combinator C

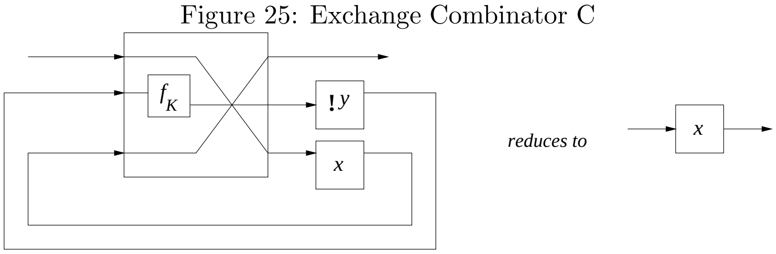



Figure 26: Weakening Combinator K

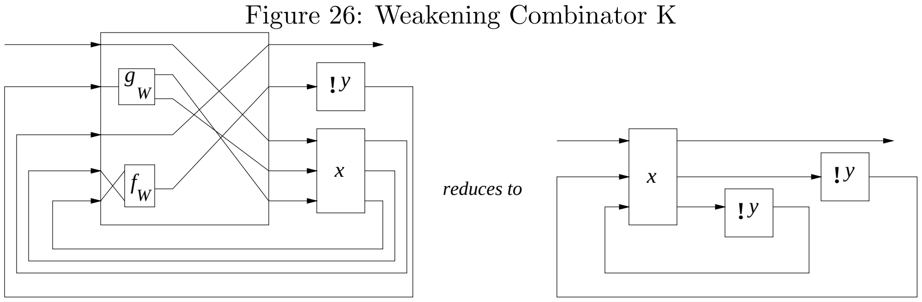



Figure 27: Contraction Combinator W

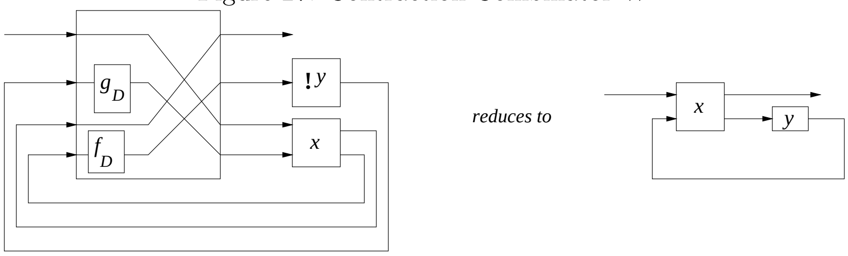



Figure 28: Dereliction Combinator D

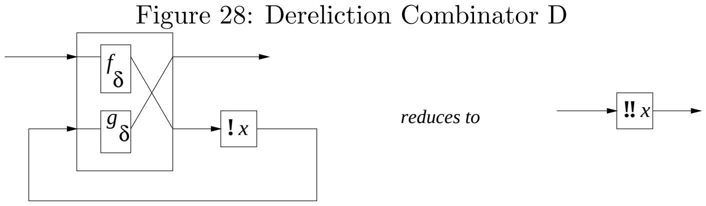



Figure 29: Comultiplication Combinator $\delta$

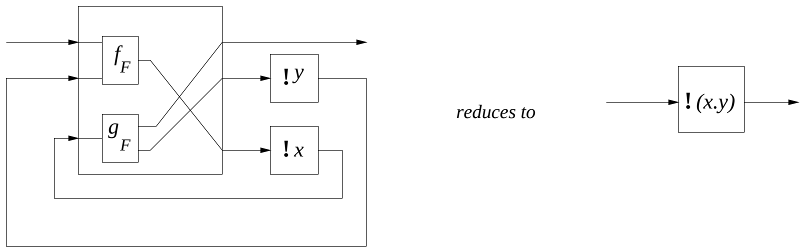



Figure 30: Functoriality Combinator F

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