Axiomatizing Fully Complete Models for ML Polymorphic Types*

Samson Abramsky[1] and Marina Lenisa[2]

[1] Division of Informatics
University of Edinburgh, UK.
samson@dcs.ed.ac.uk
[2] Dipartimento di Matematica e Informatica
Università di Udine, Italy.
lenisa@dimi.uniud.it

Abstract. We present *axioms* on models of system F, which are sufficient to show *full completeness* for *ML-polymorphic types.* These axioms are given for *hyperdoctrine* models, which arise as *adjoint models*, i.e. *co-Kleisli categories* of *linear categories.* Our axiomatization consists of two crucial steps. First, we axiomatize the fact that every relevant morphism in the model generates, under *decomposition*, a possibly *infinite* typed Böhm tree. Then, we introduce an axiom which rules out infinite trees from the model. Finally, we discuss the necessity of the axioms.

Introduction

In this paper we address the problem of *full completeness* (*universality*) for system F. A *categorical* model of a type theory (or logic) is said to be *fully-complete* ([AJ94a]) if, for all *types* (*formulae*) A, B, each morphism $f : [\![A]\!] \to [\![B]\!]$, from the interpretation of A into the interpretation of B, is the denotation of a *proof-term* witnessing the entailment $A \vdash B$. The notion of full-completeness is the counterpart of the notion of *full abstraction* for programming languages, in the sense that, if the term language is executable, then a fully-complete model is (up-to a possible quotient) fully-abstract. Besides full completeness, one can ask the question whether the theory induced by a model $\mathcal{M}$ coincides precisely with the syntactical theory or whether more equations are satisfied in $\mathcal{M}$. A model $\mathcal{M}$ is called *faithful* if it realizes exactly the syntactical theory.

The importance of fully (and faithfully) complete, and fully-abstract *denotational* models is that they characterize the space of proofs/programs in a *compositional*, *syntax-independent* way.

Recently, Game Semantics has been used to define fully-complete models for various fragments of Linear Logic ([AJ94a,AM99]), and to give fully-abstract models for many programming languages, including PCF [AJM96,HO96,Nic94], richer functional languages [McC96], and languages with non-functional features such as reference types and non-local control constructs [AM97,Lai97].

* Work partially supported by Linear FMRX-CT98-0170.

Once many concrete fully-complete and fully-abstract models have been studied, the problem of abstracting and axiomatizing the key properties of these constructions arises naturally. This line of research originated with [Abr97], where axioms, sufficient to prove full abstraction for PCF and full completeness for the simply typed λ-calculus, are given.

The axioms for PCF are abstracted from the key lemmas in the proof of full abstraction of the game model of [AJM96]. This proof makes essential use of the underlying *linear* structure of the game category. Consequently, the axiomatization in [Abr97] applies to models of PCF which arise as co-Kleisli categories of some *linear category* [See87,Bie95]. These kind of models, given by a linear category and a cartesian closed category, together with a *monoidal adjunction* between the two categories, are called *adjoint models*, following [Bie95,BW96].

The problem of *full completeness* for second order (polymorphic) λ-calculus, i.e. Girard's system F ([Gir72]), is a very important problem, which has been extensively studied. In [HRR90], the category of Partial Equivalence Relations (PER) over the open term model of the untyped λ-calculus has been proved to be fully (and faithfully) complete for *algebraic types*, a small subclass of *ML-types*. ML-types are universal closures of simple types, i.e. types of the form $\forall X_1. \dots X_n.T$, where T is $\forall$-free and $FV(T) \subseteq \{X_1, \dots, X_n\}$. A fully-complete model for the whole system F has been provided in [BC88], but this model is syntactical in nature being defined as a quotient on terms, and therefore it is not compositional and not sufficiently abstract. More recently, in [Hug99], a fully and faithfully complete game model for system F has been given. But, although this is a game model, it still has a somewhat syntactical flavor—and the construction of the model is extremely complex.

Summarizing the situation, the previous work on the full completeness problem for system F has produced semantically satisfactory models only for algebraic types. In this paper, we present a set of axioms on models of system F, sufficient to guarantee full completeness for ML-types. This axiomatization is put to use in [AL99,AL00], in order to provide a concrete denotational model fully-complete for the whole class of ML-types. The axioms presented in this paper are given on the models of system F originated from Lawvere ([Law70]) which are called *hyperdoctrines* (see also [Pit88]). As in [Abr97], our axiomatization works in the context of *adjoint models* and, although the full completeness result applies to *intuitionistic* types, it makes essential use of the *linear* decomposition of these types. Our axiomatization consists of two crucial steps. First, we axiomatize the fact that every morphism $f : 1 \to [\![T]\!]$, where T is an ML-type, generates, under *decomposition*, a possibly *infinite* typed Böhm tree. Then, we introduce an axiom which rules out infinite trees from the model.

The abstract work carried out in this paper has interesting concrete model-theoretic consequences, in that it enables a clean conceptual structure to be given to the proof of full completeness of the concrete model studied in [AL99,AL00].

The model construction in [AL99,AL00] is based on the technique of *linear realizability*, which is used to define hyperdoctrine adjoint models. This technique, which is described in [AL99,AL00], is to construct a PER category over a *Linear Combinatory Algebra*.

The proof of full completeness of the PER model in [AL99] consists in showing that this model satisfies the axioms presented in this paper. We feel that the axiomatic technique presented in this paper is both interesting in itself, and illuminates the concrete detailed proof of full completeness of the PER model.

The paper is organized as follows. In Section 1, we recall the syntax ML-types, and we present a result by Statman about theories of the simply typed λ-calculus with *typical ambiguity.* In Section 2, we carry out a *linear* analysis of the notion of $2\lambda\times$*-hyperdoctrine*, by introducing the notion of *adjoint hyperdoctrine.* In Section 3, we present our set of axioms for full completeness at ML-types. Final remarks and directions for future work appear in Section 4.

The authors are thankful to Furio Honsell, Radha Jagadeesan, Jim Laird, John Longley, Simone Martini and Alex Simpson for useful discussions on some of the issues of the paper.

1 System F and ML Polymorphism

First, we recall the syntax of *ML-types.* Then, we recall a crucial result on the simply typed λ-calculus concerning theories which satisfy *Typical Ambiguity*, namely Statman's *Typical Ambiguity Theorem.* A theory is said to satisfy Typical Ambiguity if two terms are equated if and only if they are equated for all possible substitutions of type variables. Statman's *Typical Ambiguity Theorem* asserts that there is exactly one consistent theory satisfying Typical Ambiguity on the simply typed λ-calculus with infinite type variables: this is the $\beta\eta$-theory. An immediate consequence of this result is that the only consistent theory on the fragment of system F consisting of *ML-types* is precisely the $\beta\eta$-theory.

We assume the reader familiar with System F (see e.g. [AL91]).

We introduce now the class of *ML-polymorphic types*, which correspond to the limited kind of polymorphism allowed in the language ML.

Definition 1 (ML-types). *The class* ML-Type *of ML-types is defined by:*

$$\textit{ML-Type} = \{\forall \boldsymbol{X}.T \mid T \in \ \textit{SimType} \ \wedge \ FV(T) \subseteq \boldsymbol{X}\} \ ,$$

where $\boldsymbol{X}$ is an abbreviation for $X_1, \ldots, X_n$, for some $n \geq 0$, and SimType *is the set of* simple types *over an infinite set of type variables, i.e. the set of types built inductively from the set of type variables only using the arrow type constructor.*

Terms of ML-types have essentially the same "combinatorics" as typically ambiguous terms of the simply typed λ-calculus. In fact, any theory on ML-terms induces a theory satisfying Typical Ambiguity.

The following is a result about simply typed λ-calculus with an *infinite* set of type variables first proved in [Sta88].

Theorem 1 (Statman's Typical Ambiguity). *Let T be a type of the simply typed λ-calculus with infinitely many type variables s.t. $FV(T) \subseteq \{X_1, \ldots, X_n\}$. If $\not\vdash M =_{\beta\eta} N \ : \ T$, then, there exist types $S_1, \ldots, S_n$, and $Y \in TVar$, and a term L s.t. $\vdash L[\boldsymbol{S}/\boldsymbol{X}] : T[\boldsymbol{S}/\boldsymbol{X}] \rightarrow Bool_Y$, where $Bool_Y = Y \rightarrow Y \rightarrow Y$, s.t. $\vdash (LM)[\boldsymbol{S}/\boldsymbol{X}] =_{\beta\eta} true \ : \ Bool_Y \ \wedge \ \vdash (LN)[\boldsymbol{S}/\boldsymbol{X}] =_{\beta\eta} false \ : \ Bool_Y$, where $true = \lambda x : Y.y : Y.x$ and $false = \lambda x : Y.y : Y.y$.*

Corollary 1. *i) The maximal consistent theory on the simply typed λ-calculus with infinitely many type variables satisfying Typical Ambiguity is the $\beta\eta$-theory. ii) The maximal consistent theory on the fragment of system F consisting of ML-types is the $\beta\eta$-theory.*

Corollary 1ii) implies that any non-trivial fully-complete model for ML-types of system F is necessarily *faithful* at ML-types, i.e. it realizes exactly the $\beta\eta$-theory at ML-types.

2 Models of System F

We focus on *hyperdoctrine models* of system F. First, we recall the notion of $2\lambda\times$-hyperdoctrine (see [Pit88]). This essentially corresponds to the notion of *external model* (see [AL91]). Then, we give the formal definition of full and faithful complete hyperdoctrine model. Finally, we carry out a *linear* analysis of the notion of $2\lambda\times$-hyperdoctrine. This will allow us to express conditions which guarantee full completeness of the model w.r.t. ML-types. In particular, we introduce a categorical notion of *adjoint hyperdoctrine*. Adjoint hyperdoctrines arise as *co-Kleisli* indexed categories of *linear* indexed categories.

In what follows, we assume that all indexed categories which we consider are strict (see e.g. [AL91,Cro93] for more details on indexed categories).

Definition 2 ($2\lambda\times$-hyperdoctrine, [Law70,Pit88]). *A* $2\lambda\times$-hyperdoctrine *is a triple $(\mathcal{C}, \boldsymbol{G}, \forall)$, where:*

- *$\mathcal{C}$ is the* base category*, it has finite products, and it consists of a distinguished object $\mathcal{U}$ which generates all other objects using the product operation $\times$. We will denote by $\mathcal{U}^m$, for $m \geq 0$, the objects of $\mathcal{C}$.*
- *$\boldsymbol{G} : \mathcal{C}^{op} \to$ CCCat is a $\mathcal{C}$-indexed cartesian closed category such that: for all $\mathcal{U}^m$, the underlying collection of objects of the cartesian closed fibre category $\boldsymbol{G}(\mathcal{U}^m)$ is indexed by the collection of morphisms from $\mathcal{U}^m$ to $\mathcal{U}$ in $\mathcal{C}$, i.e. the objects of $\boldsymbol{G}(\mathcal{U}^m)$ are the morphisms in $Hom_{\mathcal{C}}(\mathcal{U}^m,\mathcal{U})$, and, for any morphism $f : \mathcal{U}^m \to \mathcal{U}^n$ in $\mathcal{C}^{op}$, the cartesian closed functor $\boldsymbol{G}(f) : \boldsymbol{G}(\mathcal{U}^n) \to \boldsymbol{G}(\mathcal{U}^m)$, called* reindexing functor *and denoted by f^*, is such that, for any object $h : \mathcal{U}^n \to \mathcal{U}$, $f^*(h) = f; h$;*
- *For each object $\mathcal{U}^m$ of $\mathcal{C}$, there are functors $\forall_m : \boldsymbol{G}(\mathcal{U}^m \times \mathcal{U}) \to \boldsymbol{G}(\mathcal{U}^m)$ s.t.*
  - *$\forall_m$ is right adjoint to the functor $\pi_m^* : \boldsymbol{G}(\mathcal{U}^m) \to \boldsymbol{G}(\mathcal{U}^m \times \mathcal{U})$, where $\pi_m : \mathcal{U}^m \times \mathcal{U} \to \mathcal{U}^m$ is the projection in $\mathcal{C}$;*
  - *$\forall_m$ satisfies the* Beck-Chevalley condition.

Any $2\lambda\times$-hyperdoctrine can be endowed with a notion of interpretation $[\![\ ]\!]$ for the language of system F.
Types with free variables in $X_1, \ldots, X_m$ are interpreted by morphisms from $\mathcal{U}^m$ to $\mathcal{U}$ in the category $\mathcal{C}$, i.e. by objects of $\mathbf{G}(\mathcal{U}^m)$: $[\![X_1, \ldots, X_m \vdash T]\!] : \mathcal{U}^m \to \mathcal{U}$.
Well-typed terms, i.e. $X_1, \ldots, X_m; x_1 : T_1, \ldots, x_n : T_n \vdash M : T$, are interpreted by morphisms in the category $\mathbf{G}(\mathcal{U}^m)$:
$[\![X_1, \ldots, X_m; x_1 : T_1, \ldots, x_n : T_n \vdash M : T]\!] : [\![\boldsymbol{X} \vdash T_1]\!] \times \ldots \times [\![\boldsymbol{X} \vdash T_n]\!] \to [\![\boldsymbol{X} \vdash T]\!]$.
See e.g. [Cro93] for more details.

Definition 3 (Full and Faithful Completeness). *Let* $\mathcal{M} = (\mathcal{C}, \boldsymbol{G}, \forall, [\![\ ]\!])$ *be a* $2\lambda\times$*-hyperdoctrine.* $\mathcal{M}$ *is* fully and faithfully complete *w.r.t. the class of closed types* $\mathcal{T}$ *if, for all* $T \in \mathcal{T}$,

$\forall f \in Hom_{\boldsymbol{G}(1)}(1, [\![\vdash T]\!])$. $\exists(!)\beta\eta$*-normal form* M. $\vdash M : T \ \wedge\ f = [\![\vdash M : T]\!]$.

Before presenting the notion of adjoint hyperdoctrine, we recall some definitions:

Definition 4 (Linear Category, [See87,Bie95]). *A* linear category *is a symmetric monoidal closed category* $(\mathcal{L}, I, \otimes, \multimap)$ *with*

- *a symmetric monoidal comonad* $(!, der, \delta, \phi, \phi')$ *on* $\mathcal{L}$*;*
- *monoidal natural transformations with components* $weak_A :!A \to I$ *and* $con_A :!A \to !A \otimes !A$ *such that*
  - *each* $(!A, weak_A, con_A)$ *is a commutative comonoid,*
  - $weak_A$ *and* con_A *are !-coalgebra maps from* $(!A, \delta_A)$ *to* (I, ϕ'_I)*, and from* $(!A, \delta_A)$ *to* $(!A \otimes !A, \delta_A \otimes \delta_A; \phi_{!A,!A})$*, respectively.*
  - *all coalgebra maps between free !-coalgebras preserve the canonical structure.*

Definition 5 (Adjoint Model, [BW96]). *An* adjoint model *is specified by*

1. *a symmetric monoidal closed category* $(\mathcal{L}, I, \otimes, \multimap)$*;*
2. *a cartesian closed category* $(\mathcal{C}, 1, \times, \rightarrow)$*;*
3. *a symmetric monoidal adjunction from* $\mathcal{C}$ *to* $\mathcal{L}$*.*

Now we give the *indexed version* of the notion of adjoint model:

Definition 6 (Indexed Adjoint Model). *An* indexed adjoint model *is specified by*

1. *a symmetric monoidal closed indexed category* $\boldsymbol{L} : \mathcal{C}^{op} \rightarrow$ SMCCat*, where* SMCCat *is the category of symmetric monoidal closed categories and strict monoidal closed functors;*
2. *a cartesian closed indexed category* $\boldsymbol{G} : \mathcal{C}^{op} \rightarrow$ CCCat*, where* CCCat *is the category of cartesian closed categories and strict cartesian closed functors;*
3. *a symmetric monoidal indexed adjunction from* $\boldsymbol{G}$ *to* $\boldsymbol{L}$*.*

In the following definition, we capture those $2\lambda\times$-hyperdoctrines which arise from a co-Kleisli construction over an *indexed linear category.* This definition is inspired to [See90].

Definition 7 (Adjoint Hyperdoctrine).

An adjoint hyperdoctrine *is a quadruple* $(\mathcal{C}, \boldsymbol{L}, \boldsymbol{G}, \forall)$*, where:*

- $\mathcal{C}$ *is the* base category*, it has finite products, which consists of a distinguished object* $\mathcal{U}$ *which generates all other objects using the product operation* $\times$*. We will denote by* $\mathcal{U}^m$*, for* $m \geq 0$*, the objects of* $\mathcal{C}$*.*
- $\boldsymbol{L} : \mathcal{C}^{op} \rightarrow$ LCat *is a* $\mathcal{C}$*-indexed linear category such that: for all* $\mathcal{U}^m$*, the underlying collection of objects of the linear fibre category* $\boldsymbol{L}(\mathcal{U}^m)$ *is indexed by the collection of morphisms from* $\mathcal{U}^m$ *to* $\mathcal{U}$ *in* $\mathcal{C}$*.*
- $\boldsymbol{G} : \mathcal{C}^{op} \rightarrow$ CCCat *is the* $\mathcal{C}$*-indexed cartesian closed co-Kleisli category of* $\boldsymbol{L}$*.*
- *For each object* $\mathcal{U}^m$ *of* $\mathcal{C}$*, there are functors* $\forall_m : \boldsymbol{G}(\mathcal{U}^m \times \mathcal{U}) \rightarrow \boldsymbol{G}(\mathcal{U}^m)$ *s.t.*
  - $\forall_m : \boldsymbol{G}(\mathcal{U}^m \times \mathcal{U}) \rightarrow \boldsymbol{G}(\mathcal{U}^m)$ *is right adjoint to the functor* $\boldsymbol{G}(\pi_m) : \boldsymbol{G}(\mathcal{U}^m) \rightarrow \boldsymbol{G}(\mathcal{U}^m \times \mathcal{U})$*, where* $\pi_m : \mathcal{U}^m \times \mathcal{U} \rightarrow \mathcal{U}^m$ *is the projection in* $\mathcal{C}$*;*

- $\forall_m : \boldsymbol{G}(\mathcal{U}^m \times \mathcal{U}) \to \boldsymbol{G}(\mathcal{U}^m)$ *satisfies the* Beck-Chevalley condition.

An adjoint hyperdoctrine is, in particular, an indexed adjoint model, and it gives rise to a $2\lambda\times$-hyperdoctrine:

Theorem 2. *Let* $(\mathcal{C}, \boldsymbol{L}, \boldsymbol{G}, \forall)$ *be an adjoint hyperdoctrine. Then*
i) the categories $\boldsymbol{L}$ *and* $\boldsymbol{G}$ *form an indexed adjoint model;*
ii) $(\mathcal{C}, \boldsymbol{G}, \forall)$ *is an hyperdoctrine.*

Remark. In the definition of adjoint hyperdoctrine, we require the indexed categories $\mathbf{L}$ and $\mathbf{G}$ to form an adjoint model, but we assume the existence of a family of functors $\forall_m$ only on the fibre categories of $\mathbf{G}$. Therefore, we have a model of linear *first order* types, but not of linear *higher order* types, and our definition does not capture models of L/NL system F, i.e. system F with both linear and intuitionistic types. But our notion of model is sufficient for dealing with ML-types, and for expressing axioms for full completeness at ML-types.

3 Axiomatizing Models Fully Complete for ML Types

We isolate sufficient conditions on adjoint hyperdoctrine models for system F, in order to guarantee full completeness at ML-polymorphic types. These conditions amount to the six axioms of Subsection 3.1. Our axiomatization of full completeness for ML polymorphism is in the line of the work in [Abr97], where an axiomatic approach to full abstraction/full completeness for PCF/simply typed λ-calculus is presented. These axiomatizations are inspired by the proof of full abstraction of the Game Semantics model for PCF of [AJM96]. Our axiomatization of full completeness for ML-types consists of two parts:
1) Axioms for ensuring the Decomposition Theorem. This theorem allows to recover the *top-level structure* of the (possibly infinite) Böhm tree denoted by morphisms from the terminal object into the interpretation of an ML-type in the fibre category $\mathbf{G}(1)$. The axioms for the Decomposition Theorem (Axioms 1–5 of Section 3.1) make essential use of the linear category underlying an adjoint hyperdoctrine. These axioms (apart from the axioms 1 and 3), are expressed by requiring some canonical maps between suitable spaces of morphisms in the fibre categories $\mathbf{L}(\boldsymbol{U})$ to be isomorphisms.
2) A *Finiteness Axiom*, which allows to rule out infinite Böhm trees from the model.

Notice that, by definition of interpretation function on a hyperdoctrine (see e.g. [Cro93]), morphisms f in $\mathbf{G}(1)$ from the terminal object of $\mathbf{G}(1)$ into $[\![\vdash T]\!]$, where $\forall \boldsymbol{X}.T_1 \to \ldots \to T_n \to X_k$ is an ML-type, are λ-definable if and only if morphisms of $\mathbf{G}(\boldsymbol{\mathcal{U}})$ from $\times_{i=1}^n [\![\boldsymbol{X} \vdash T_i]\!]$ into $[\![\boldsymbol{X} \vdash X_k]\!]$ are λ-definable.
Namely, $f = [\![\vdash \Lambda \boldsymbol{X}.\lambda \boldsymbol{x} : \boldsymbol{T}.x_i M_1 \ldots M_{q_i} : \forall \boldsymbol{X}.\boldsymbol{T} \to X_k]\!]$ if and only if $\boldsymbol{\Lambda}^{-1}(\widehat{f}) = [\![\boldsymbol{X}; \boldsymbol{x} : \boldsymbol{T} \vdash x_i M_1 \ldots M_{q_i} : X_k]\!]$, where $\widehat{\ }$ denotes the inverse of the bijection given by the adjunction between $\forall$ and π^* in Definition 2. Therefore, from now on we focus on the space of morphisms of $\mathbf{G}(\boldsymbol{\mathcal{U}})$ from $\times_{i=1}^n [\![\boldsymbol{X} \vdash T_i]\!]$ into $[\![\boldsymbol{X} \vdash X_k]\!]$, where $T_1, \ldots, T_n$ are simple types.

We start by presenting the main result of this section, i.e. the Decomposition Theorem. The proof of this theorem follows from the Strong Decomposition Theorem 4, which is proved in Section 3.1.

If a morphism f of $\mathbf{G}(\boldsymbol{\mathcal{U}})$ from $\times_{i=1}^{n}[\![\boldsymbol{X} \vdash T_i]\!]$ into $[\![\boldsymbol{X} \vdash X_k]\!]$ is λ-definable, then $f = [\![\boldsymbol{X};\boldsymbol{x} : \boldsymbol{T} \vdash x_i M_1 \ldots M_{q_i} : X_k]\!]$, for some $\boldsymbol{X};\boldsymbol{x} : \boldsymbol{T} \vdash M_1 : U_{i1}, \ldots, \boldsymbol{X};\boldsymbol{x} : \boldsymbol{T} \vdash M_{q_i} : U_{iq_i}$. I.e., making evident the *top-level* structure of the Böhm tree:

$$f = [\![\boldsymbol{X};\boldsymbol{x} : \boldsymbol{T} \vdash x_i : T_i]\!] \bullet [\![\boldsymbol{X};\boldsymbol{x} : \boldsymbol{T} \vdash M_1 : U_{i1}]\!] \bullet \ldots \bullet [\![\boldsymbol{X};\boldsymbol{x} : \boldsymbol{T} \vdash M_{q_i} : U_{iq_i}]\!] \ ,$$

where $\bullet$ denotes the application in the model.

The Decomposition Theorem allows recovering the top-level structure of the Böhm tree corresponding to f in the following sense:

Theorem 3 (Decomposition). *Let $(\mathcal{C}, \boldsymbol{L}, \boldsymbol{G}, \forall)$ be an adjoint hyperdoctrine satisfying Axioms 1–5 of Section 3.1. Let $T = T_1 \to \ldots \to T_n \to X_k$ be a simple type with $FV(T) \subseteq \{X_1, \ldots, X_n\}$, where, for all $i = 1, \ldots, n$, $T_i = U_{i1} \to \ldots \to U_{iq_i} \to X_i$. Then, for all $f \in Hom_{\boldsymbol{G}(\boldsymbol{\mathcal{U}})}(\times_{i=1}^{n}[\![\boldsymbol{X} \vdash T_i]\!], [\![\boldsymbol{X} \vdash X_k]\!])$, there exist $i \in \{1, \ldots, n\}$ and $g_j \in Hom_{\boldsymbol{G}(\boldsymbol{\mathcal{U}})}(\times_{i=1}^{n}[\![\boldsymbol{X} \vdash T_i]\!], [\![\boldsymbol{X} \vdash U_{ij}]\!])$, for all $j = 1, \ldots, q_i$, such that $f = [\![\boldsymbol{X};\boldsymbol{x} : \boldsymbol{T} \vdash x_i : T_i]\!] \bullet g_1 \ldots \bullet g_{q_i}$.*

Since the g's appearing in the Decomposition Theorem still live (up-to-uncurrying) in a space of morphisms denoting a simple type, we can keep on iterating the decomposition, expanding in turn these g's, thus getting a possibly infinite tree from f. If the Decomposition Theorem holds, in order to get the full completeness result, we are left only to rule out morphisms generating trees whose height is infinite, which would correspond to infinite typed Böhm trees. This is expressed in the *Finiteness Axiom* 6 below.

3.1 The Axioms

The first axiom expresses the fact that the type $\forall \boldsymbol{X}.X_k$ is empty.

Axiom 1 (Base) $\qquad Hom_{\boldsymbol{L}(\boldsymbol{\mathcal{U}})}(1, \pi_k) = \emptyset$,
where 1 *is the terminal object in* $\boldsymbol{G}(\boldsymbol{\mathcal{U}})$, *and* $\pi_k : \boldsymbol{\mathcal{U}} \to \boldsymbol{\mathcal{U}}$ *denotes the k-th projection in* $\boldsymbol{G}(\boldsymbol{\mathcal{U}})$, *i.e.* $\pi_k = weak_1 \otimes \ldots \otimes weak_{k-1} \otimes der_k \otimes weak_{k+1} \otimes \ldots \otimes weak_n$.

The following axiom allows extracting *one* copy of the type of the head variable, corresponding to the *first* use of this variable. the property expressed by this axiom is *truly* linear. In fact, in order to state it, we are implicitly using the canonical morphism $!A \to A \otimes !A$ to capture the idea of a "first occurrence".

Axiom 2 (Linearization of Head Occurrence)

$$case_i\{\sigma_i\}_{i=1,\ldots,n} \ : \ \coprod_{i=1}^{n} Hom_{\boldsymbol{L}(\boldsymbol{\mathcal{U}})}(h_i, (\boldsymbol{h} \multimap \pi_k)) \ \simeq \ Hom_{\boldsymbol{L}(\boldsymbol{\mathcal{U}})}(\boldsymbol{h}, \pi_k) \ ,$$

where

- $\coprod$ *denotes coproduct in* Set;
- $\boldsymbol{h} = \otimes_{i=1}^{n} !h_i$ *and* $\forall i \in \{1, \ldots, n\}$. $h_i = \otimes_{j=1}^{q_i} !l_{ij} \multimap \pi_{p_i}$;
- $\sigma_i : Hom_{\boldsymbol{L}(\boldsymbol{\mathcal{U}})}(h_i, (\boldsymbol{h} \multimap \pi_k)) \to Hom_{\boldsymbol{L}(\boldsymbol{\mathcal{U}})}(\boldsymbol{h}, \pi_k)$ *is the following canonical morphism:* $\Lambda^{-1}; Hom_{\boldsymbol{L}(\boldsymbol{\mathcal{U}})}(\pi_i; \tau, id_{\pi_k}); Hom_{\boldsymbol{L}(\boldsymbol{\mathcal{U}})}(con_{\boldsymbol{h}}, id_{\pi_k})$, *where* $\tau : I \otimes \ldots \otimes I \otimes h_i \otimes I \ldots \otimes I \simeq h_i$.

The following axiom reflects a form of coherence of the type of the head variable w.r.t. the global type of the term. I.e., if $\boldsymbol{T} \to X_k$ is the type of a term, then the type of the head variable must be of the shape $\boldsymbol{U} \to X_i$, with $k = i$.

Axiom 3 (Type Coherence) $\qquad Hom_{\boldsymbol{L}(\boldsymbol{\mathcal{U}})}(l \multimap \pi_i, h \multimap \pi_k) = \emptyset$, *if* $i \neq k$.

The following axiom expresses the fact that the only thing that we can do with a *linear* functional parameter is applying it to an argument *which does not itself depend on the parameter.* Note that, again, linearity is essential here. For example, if copying were allowed, then the argument could itself contain further occurrences of the parameter.

Axiom 4 (Linear Function Extensionality)
$Hom_{\boldsymbol{L}(\boldsymbol{\mathcal{U}})}((\cdot), id_{\pi_k}) : \ Hom_{\boldsymbol{L}(\boldsymbol{\mathcal{U}})}(h, l) \ \simeq \ Hom_{\boldsymbol{L}(\boldsymbol{\mathcal{U}})}(l \multimap \pi_k, h \multimap \pi_k)$.

The following axiom expresses the fact that morphisms from $!f$ to $!g$ in the fibre category $\mathbf{L}(U)$ have *uniform* behavior in all threads.

Axiom 5 (Uniformity of Threads)
$Hom_{\boldsymbol{L}(\boldsymbol{\mathcal{U}})}(id_{!h}, der_l) : \ Hom_{\boldsymbol{L}(\boldsymbol{\mathcal{U}})}(!h, !l) \ \simeq \ Hom_{\boldsymbol{L}(\boldsymbol{\mathcal{U}})}(!h, l) \ : \boldsymbol{\lambda} f \in Hom_{\boldsymbol{L}(\boldsymbol{\mathcal{U}})}(!h, l).(f)^{\dagger}_{h,l}$, *where* $(\ )^{\dagger}_{h,l} : Hom_{\boldsymbol{L}(\boldsymbol{\mathcal{U}})}(!h, l) \to Hom_{\boldsymbol{L}(\boldsymbol{\mathcal{U}})}(!h, !l)$ *is the canonical morphism given by the comonad* $!$.

The final axiom in our axiomatization guarantees that the tree generated via repeated applications of the Decomposition Theorem 4 to morphisms in $Hom_{\mathbf{L}(\boldsymbol{\mathcal{U}})}(\otimes_{i=1}^{n} !h_i, \pi_k)$, where $h_i = \otimes_{j=1}^{q_i} !l_{ij} \multimap \pi_{p_1}$, for $j = 1, \ldots, q_i$, is *finite.*

Axiom 6 (Finiteness) *There exists a* size function

$$\mathcal{H} : \bigcup \{ Hom_{\boldsymbol{L}(\boldsymbol{\mathcal{U}})}(\otimes_{i=1}^{n} !h_i, \pi_k) \mid k \in \boldsymbol{N}, \ h_i = \otimes_{j=1}^{q_i} !l_{ij} \multimap \pi_{p_i} \} \longrightarrow \boldsymbol{N} ,$$

such that $\forall j \in \{1, \ldots, q_i\}. \ \mathcal{H}(g_j) < \mathcal{H}(f)$, *where the* g_j*'s are defined in the Decomposition Theorem 4.*

Notice that Axioms 1–5 actually give a stronger form of decomposition than Theorem 3. Namely, the decomposition is *unique*, and it holds for all morphisms in $\mathrm{Hom}_{\mathbf{L}(\boldsymbol{\mathcal{U}})}(\boldsymbol{h}, \pi_k)$, where $\boldsymbol{h} = \otimes_{i=1}^{n} !h_i$ and, for all $i = 1, \ldots, n$, $h_i = \otimes_{j=1}^{q_i} !l_{ij} \multimap \pi_{p_i}$, and not just for the morphisms which are definable.

Theorem 4 (Strong Decomposition). *Let* $(\mathcal{C}, \boldsymbol{G}, \boldsymbol{L}, \forall)$ *be an adjoint hyperdoctrine satisfying Axioms 1–5 of Section 3.1. Let* $f \in Hom_{\boldsymbol{L}(\boldsymbol{\mathcal{U}})}(\boldsymbol{h}, \pi_k)$, *where* $\boldsymbol{h} = \otimes_{i=1}^{n} !h_i$ *and, for all* $i = 1, \ldots, n$, $h_i = \otimes_{j=1}^{q_i} !l_{ij} \multimap \pi_{p_i}$. *Then there exist a* unique i *and unique* $g_1, \ldots, g_{q_i}$ *such that, for all* $j = 1, \ldots, q_i$, $g_j \in Hom_{\boldsymbol{L}(\boldsymbol{\mathcal{U}})}(\boldsymbol{h}, l_{ij})$, *and* $f = con_{\boldsymbol{h}}; (\pi_k \otimes \langle g_1, \ldots, g_{q_i} \rangle^{\dagger}); Ap$.

Proof. If $\boldsymbol{h} = 1$, then, by Axiom 1, we have immediately the thesis. Otherwise, by Axiom 2, there exists a unique $i \in \{1, \ldots, n\}$ and a unique $f' \in \mathrm{Hom}_{\mathbf{L}(\boldsymbol{\mathcal{U}})}(h_i, \boldsymbol{h} \multimap \pi_k)$ such that $f = \mathrm{con}_{\boldsymbol{h}}; \pi_i \otimes id_{\boldsymbol{h}}; \Lambda^{-1}(f')$. By Axiom 3, $\pi_i = \pi_k$. By Axiom 4, there exists a unique $g \in \mathrm{Hom}(\boldsymbol{h}, \boldsymbol{l_i})$, where $\boldsymbol{l_i} = \otimes_{j=1}^{q_i} !l_{ij}$, such that $f' = g \multimap \pi_k = \Lambda((id \otimes g); Ap)$. Then $f = \mathrm{con}_{\boldsymbol{h}}; \pi_k \otimes g; Ap$. Finally, by Axiom 5, and by the universal property of the product, we obtain $g = \langle g_1, \ldots, g_{q_i} \rangle^{\dagger}$. □

Finally, we can show the main result:

Theorem 5 (Axiomatic Full Completeness). *Let* $\mathcal{M}$ *be an adjoint hyperdoctrine. If* $\mathcal{M}$ *satisfies Axioms 1–6, then* $\mathcal{M}$ *is fully and faithfully complete at ML-types.*

Proof. Let $\forall \boldsymbol{X}.T = T_1 \to \ldots \to T_n \to X_k$ be an ML-type, and let $f \in \mathrm{Hom}_{\mathbf{L}(1)}(!I, [\![\vdash \forall \boldsymbol{X}.T]\!])$. Using the Decomposition Theorem, one can easily prove, by induction on the measure provided by Axiom 6 that there exists $\boldsymbol{X}; \boldsymbol{x} : \boldsymbol{T} \vdash x_i M_1 \ldots M_{q_i} : X_k$ such that $\overline{\boldsymbol{\Lambda}(f)} = [\![\boldsymbol{X}; \boldsymbol{x} : \boldsymbol{T} \vdash x_i M_1 \ldots M_{q_i} : X_k]\!]$, where $\overline{}$ is the bijection given by the adjiunction between $\forall$ and π^* in Definition 2. Then $f = [\![\vdash \Lambda \boldsymbol{X}.\lambda \boldsymbol{x} : \boldsymbol{T}.x_i M_1 \ldots M_{q_i} : \forall \boldsymbol{X}.\boldsymbol{T} \to X_k]\!]$. □

4 Final Remarks and Directions for Future Work

As we remarked earlier, the Strong Decomposition Theorem 4 is stronger than the Decomposition Theorem 3 in two respects. First of all, it provides a decomposition for *all* morphisms of a certain shape, and not just for those whose domains and codomains are *denotations* of types. Secondly, it guarantees the *uniqueness* of the decomposition. Correspondingly, the axioms could be weakened either by considering only spaces of morphisms whose domains and codomains are *denotations* of types, instead of generic objects of appropriate "top-level" shape, or by substituting the isomorphisms requirements by weaker conditions, which only ensure the existence of *a* decomposition. More precisely, the first kind of restriction is obtained by taking, e.g. in the *Linearization of Head Occurrence*, $\boldsymbol{h}$ to be $\otimes_{i=1}^{n} ![\![T_i]\!]$, where each T_i is a simple type with free variables in $\boldsymbol{X}$. In order to guarantee the existence of *a* decomposition, it is sufficient to ask that the canonical morphisms in Axioms 2 and 4 and the morphism $\boldsymbol{\lambda} f \in \mathrm{Hom}(!h, l).(f)^\dagger$ in Axiom 5 are *surjective* maps. By weakening the axioms in either or both of these ways, we still get a set of sufficient conditions for full completeness. Notice that, if we take the weaker form of the axioms which imply only the existence of a decomposition, we need indeed Statman's result to conclude faithfulness. The use of the Typical Ambiguity Theorem, on the other hand, is not necessary, when strong decomposition is available.

In the concrete model of PERs over the LCA of partial involutions of [AL99], we succeeded in proving the weak variant of the axioms obtained by restricting the spaces of morphisms. But we conjecture that the full strong form holds.

Finally, notice that all the Axioms 1–6 in the strong form presented in Section 3 are *consistent*, since they are satisfied in the underlying category of the adjoint hyperdoctrine induced by the linear term model. Moreover, Axiom 1 is trivially *necessary.* The question of the necessity of the Axioms 2–6 in their weak or strong form remains open.

In this paper, we have presented axioms for full completeness at ML-types. A natural question arises: what happens beyond ML-types. Here is a partial answer. Already at the type Nat $\to$ Nat, where *Nat* is the type of Church's numerals, i.e. $\forall X.(X \to X) \to X \to X$, the PER model over the linear term combinatory algebra is not fully-complete. In fact, all recursive functions can be encoded in the type Nat $\to$ Nat. A similar problem arises also if we consider the LCA of partial involutions studied in [AL99]. PER models as they are defined in [AL99], do not seem to give full-completeness beyond ML-types. In principle, one could give axioms for full completeness w.r.t. larger fragments of system F, but, at the moment, the real challenge is that of isolating a fragment of system F which properly includes the ML-types, while still admitting "good" fully-complete models.

References

[Abr97] S.Abramsky. Axioms for Definability and Full Completeness, in *Proof, Language and Interaction: Essays in Honour of Robin Milner*, G. Plotkin, C. Stirling and M. Tofte, eds., MIT Press, 2000, 55–75.

[AJ94a] S.Abramsky, R.Jagadeesan. Games and Full Completeness for Multiplicative Linear Logic, *J. of Symbolic Logic* **59**(2), 1994, 543–574.

[AJM96] S.Abramsky, R.Jagadeesan, P.Malacaria. Full Abstraction for PCF, 1996, *Inf. and Comp.*, to appear.

[AL91] A.Asperti, G.Longo. *Categories, Types ad Structures*, Foundations of Computing Series, The MIT Press, 1991.

[AL99] S.Abramsky, M.Lenisa. Fully Complete Models for ML Polymorphic Types, Report **ECS-LFCS-99-414**, University of Edinburgh, October 1999.

[AL00] S.Abramsky, M.Lenisa. A Fully Complete PER Model for ML Polymorphic Types, *CSL'2000* Conf. Proc., to appear.

[AM97] S.Abramsky, G.McCusker. Full abstraction for idealized Algol with passive expressions, *TCS* **227**, 1999, 3–42.

[AM99] S.Abramsky, P.Mellies. *Concurrent Games and Full Completeness*, *LICS'99.*

[BC88] V.Breazu-Tannen, T.Coquand. Extensional models for polymorphism, *TCS* **59**, 1988, 85–114.

[BW96] N.Benton, P.Wadler. Linear Logic, Monads and the Lambda Calculus, *LICS'96* Conf. Proc., 1996.

[Bie95] G.Bierman. What is a categorical Model of Intuitionistic Linear Logic?, *TLCA'95* Conf. Proc., LNCS, 1995.

[Cro93] R.Crole, *Categories for Types*, Cambridge University Press, 1993.

[Gir72] J.Y.Girard. Interprétation functionelle et élimunation des coupures de l'arithmètique d'ordre supérieur, Thèse d'Etat, Université Paris VII, 1972.

[Hug99] D.J.D.Hughes. *Hypergame Semantics: Full Completeness for System F.* D.Phil. thesis, University of Oxford, submitted 1999.

[HRR90] J.Hyland, E.Robinson, G.Rosolini. Algebraic types in PER models, *MFPS* Conf. Proc., M.Main *et al.* eds, LNCS **442**, 1990, 333–350.

[HO96] M.Hyland, L.Ong. On full abstraction for PCF, *Inf. and Comp.*, 1996, to appear.

[Lai97] J.Laird. Full abstraction for functional languages with control, *LICS'97.*

[Law70] F.Lawvere. Equality in hyperdoctrines and the comprehension schema as an adjoint functor, Proc. Symp. on Applications of Categorical Logic, 1970.

[McC96] G.McCusker. Games and full abstraction for FPC, *LICS'96.*

[Nic94] H.Nickau. Hereditarily sequential functionals, Proc. of the Symposium *Logical Foundations for Computer Science*, LNCS **813**, 1994.

[Pit88] A.Pitts. Polymorphism is set-theoretic constructively, *CTCS'88* Conf. Proc., D.Pitt ed., LNCS **283**, 1988.

[See87] R.Seely. Linear logic, $*$-autonomous categories and cofree coalgebras, in *Category theory, computer science and logic*, American Math. Society, 1987.

[See90] R.Seely. Polymorphic linear logic and topos models, Math. Reports, Academy of Science (Canada) XII, 1990.

[Sta88] Statman. λ-definable functionals and $\beta\eta$-conversion, *Arch. Math. Logik* **23**, 1983, 21–26.