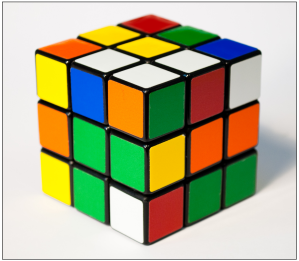
Categorical Quantum Mechanics: Quantum and Classical Information

Chris Heunen

May 20, 2014

Overview



Yesterday:

- Classical communication
  Frobenius algebra
- Observables
  operator algebra
- Nonstandard models
  possibilistic quantum mechanics

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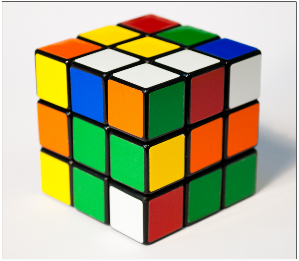

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  Frobenius algebra
- Observables
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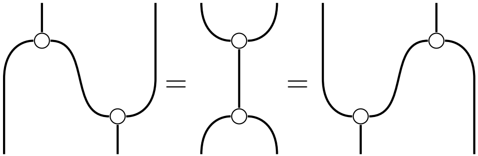
Today:

- Physical processes
  complete positivity
- Classical vs quantum channels
  state spaces
- Drawing theories apart
  nonstandard models

Recall

- Processes $H \to H$ form Frobenius algebra $(A, \quad)$
  (because map-state duality)



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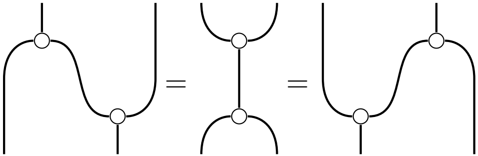

- In category of Hilbert spaces: $A \cong \mathbb{M}_{n_1} \oplus \cdots \oplus \mathbb{M}_{n_k}$
  (commutative $\Rightarrow A \cong \mathbb{C} \oplus \cdots \oplus \mathbb{C}$)

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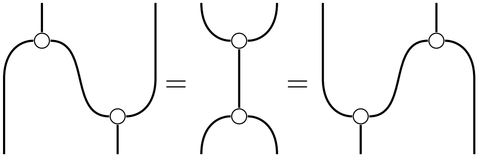

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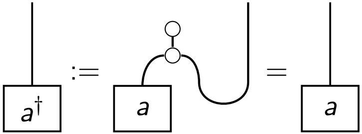
- But can only access observables $a^\dagger = a \in \mathbb{M}_n$



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  $\Leftrightarrow f$ preserves positivity ($a \geq 0 \Rightarrow f(a) \geq 0$)

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- $f\colon A \to B$ completely positive when

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  i.e. account for dynamics
  (some debate about whether other maps are unphysical)

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  (some debate about whether other maps are unphysical)
  e.g. completely positive maps $\mathbb{C}^m \to \mathbb{M}_n$ are POVMs!
- Also interesting mathematically: $A \otimes \mathbb{M}_n(\mathbb{C}) \cong \mathbb{M}_n(A)$
  $a \otimes e_{ij} \mapsto$ block matrix with a in (i, j)-th block

Complete positivity: history

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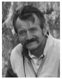
"Positive definite operator functions on a commutative group"
Izvestiya Rossiiskoi Akademii Nauk USSR Matematicheskaya 7:237–244, 1943

"Positive functions on C*-algebras"
Proceedings of the American Mathematical Society 6:211–216, 1955

"Subalgebras of C*-algebras"
Acta Mathematica 123(1):141–224, 1969



"General state changes in quantum theory"
Annals of Physics 64(2):311–335, 1971

"Completely positive linear maps on complex matrices"
Linear Algebra and Its Applications 10(3):285–290, 1975

Injectivity

An object A is injective when "arrows into it can be extended"

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Injective *wrt completely positive maps* for $*$-algebras iff:

- hyperfinite (dense union of finite-dimensional algebras)
- amenable (all derivations are inner)
- nuclear (good notion of tensor product)
- conditional expectation (from $B(H)$ onto A)

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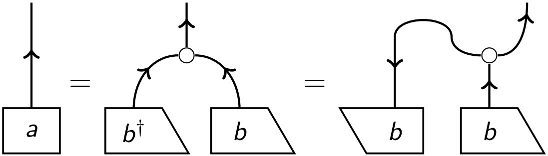
Posivity: abstractly

"Element" $a\colon I \to A$ of Frobenius algebra (A, μ) is positive when



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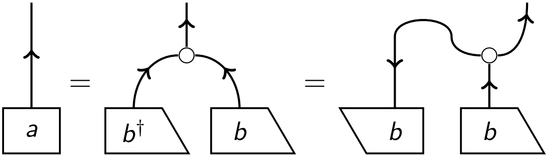
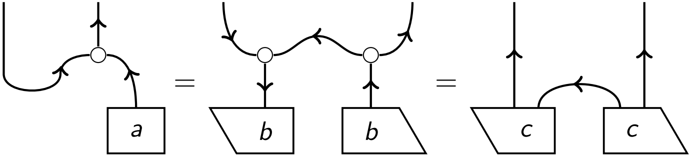


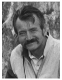
for some $b\colon I \to A$. This implies (in Hilbert spaces is equivalent to)



for some $c\colon I \to X \otimes A$. Take this as definition.

Complete positivity: abstractly

$f : \mathbb{M}_m \to \mathbb{M}_n$ completely positive $\iff$ $f(\rho) = \sum_{i=1}^{k} g_i^\dagger \circ \rho \circ g_i$ for some $g_i : \mathbb{C}^n \to \mathbb{C}^m$



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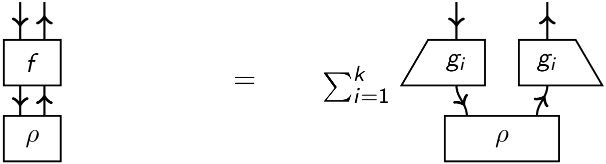

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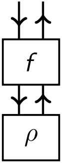
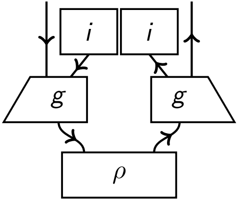


$=$ $\sum_{i=1}^{k}$



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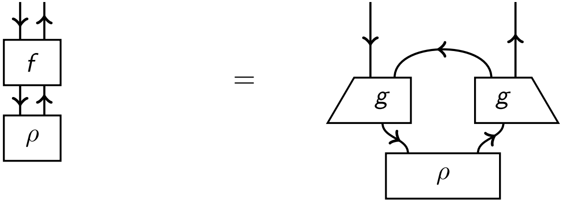

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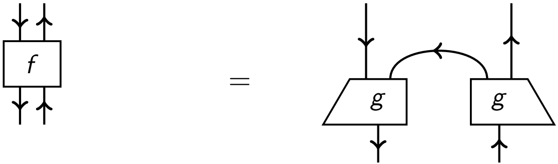
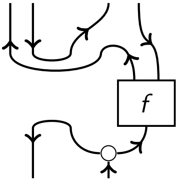


(g is called Kraus map, $\mathbb{C}^k$ the ancilla system, they are not unique)

Complete positivity: abstractly

$f\colon A \to \mathbb{M}_n$ completely positive $\iff$ $f(a) = \sum_{i=1}^{k} g_i^\dagger \circ \pi(a) \circ g_i$
$g\colon \mathbb{C}^n \to \mathbb{C}^k \otimes \mathbb{M}_m$, $\pi\colon A \to \mathbb{M}_m$



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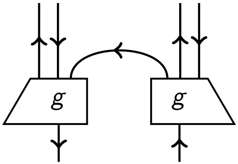

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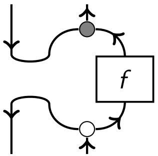

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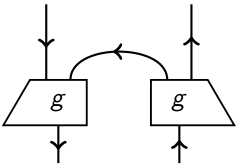
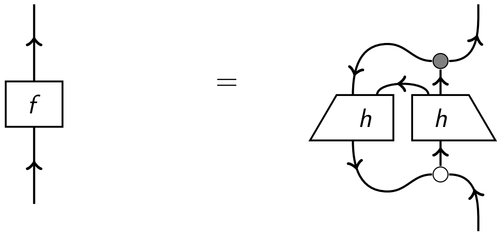


Complete positivity: abstractly

$f: A \to B$ completely positive $\iff$ for some $h: A \to X \otimes B$



Complete positivity: abstractly

The following are equivalent:

- $f\colon (A, \;) \to (B, \;)$ is completely positive
- $f \otimes \mathrm{id}_C \colon A \otimes C \to B \otimes C$ is positive for all $(C, \;)$
- $f \otimes \mathrm{id}_C \colon A \otimes C \to B \otimes C$ is positive for $C = (X^* \otimes X, \;)$

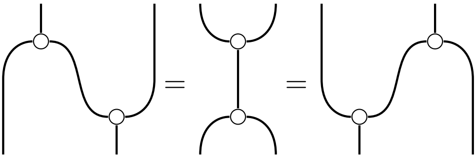


"Categories of quantum and classical channels"
Quantum Information Processing, 2014

Complete positivity: categorically

From a dagger compact category **C** of *pure processes*,
define a new one $\mathrm{CP}^*[\mathbf{C}]$ of *mixed processes*:

- Objects are dagger special Frobenius algebras in **C**



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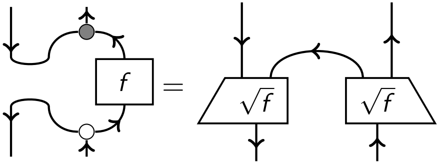

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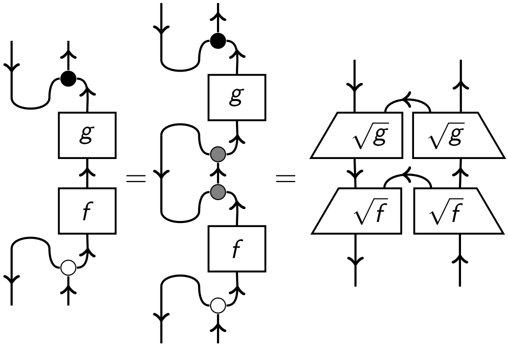

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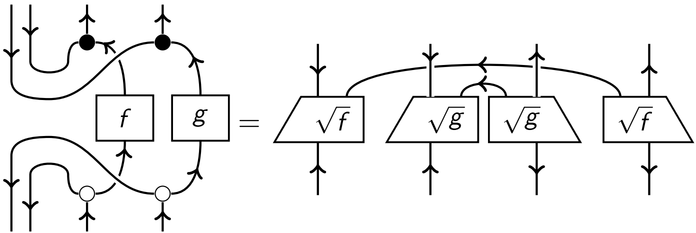

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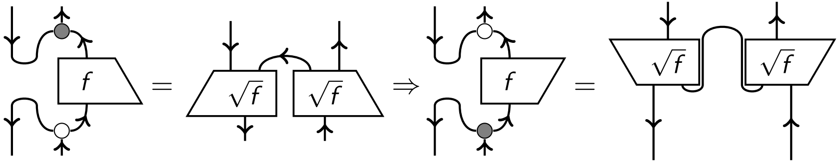

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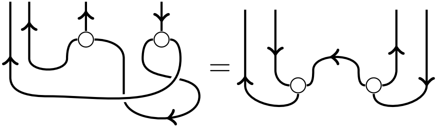
- Dual object of (A, ) is (A^*, )



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Then: CP*[**FHilb**] is *-algebras and completely positive maps

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Quantum Information Processing, 2014

Operator algebra: classical and quantum

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"On normed rings"
Doklady Akademii Nauk SSSR 23:430–432, 1939

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  then $C(X) = \{f : X \to \mathbb{C}\}$ is a commutative operator algebra.
  Any commutative operator algebra is of this form!
- Can recover states (as maps $C(X) \to \mathbb{C}$)
  Constructions on states transfer to observables:

$$X + Y \mapsto C(X) \otimes C(Y)$$
$$X \times Y \mapsto C(X) \oplus C(Y$$

States determine everything

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- Quantum: if H is a Hilbert space,
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  Any operator algebra embeds into one of this form!
- Recover states? Do states determine everything?

"On normed rings"
Doklady Akademii Nauk SSSR 23:430–432, 1939


"Imbedding of normed rings into operators on a Hilbert space"
Matematicheskii Sbornik 12(2):197–217, 1943

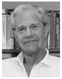
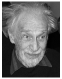
Operator algebra: state spaces

$$\begin{tikzcd}
\text{commutative operator algebras} \arrow[r, leftrightarrow] \arrow[d] & \text{state spaces} \arrow[d, dashed, "G"] \\
\text{operator algebras} \arrow[r, dashed, "F"'] & \text{quantum state spaces}
\end{tikzcd}$$

If G continuous, then F degenerates. ($F(\mathbb{M}_n) = \emptyset$ for $n \geq 3$)

"Obstructing extensions of the functor Spec"
Israel Journal of Mathematics 192(2):667–698, 2012


"Extending obstructions to functorial spectra"
Theory and Applications of Categories, 2014

"The problem if hidden variables in quantum mechanics"
Journal of mathematics and Mechanics 17(1):59–87, 1967

Completely classical systems

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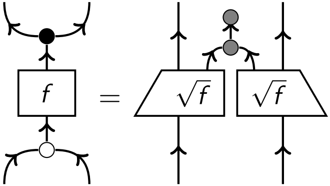

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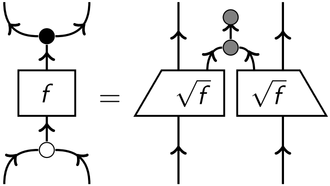

- $\mathrm{CP}^*_c[\mathbf{FHilb}]$ = Hilbert spaces with chosen basis
  and matrices with entries ≥ 0
- Stochastic matrices = transition probabilities of Markov chains

Completely quantum systems

- Recall: $\mathbb{M}_n$ is $(H^* \otimes H, \ )$ for $H = \mathbb{C}^n$

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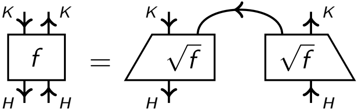

composition, tensor, dagger, etc. as in $\mathbf{C}$

"Dagger compact closed categories and completely positive maps"
Quantum Physics and Logic, ENTCS 170:139–163, 2007

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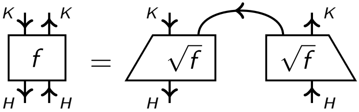
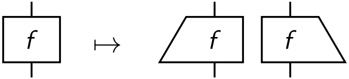


  composition, tensor, dagger, etc. as in $\mathbf{C}$
- "Pure" embedding that preserves tensor, dagger, etc.

$$\begin{array}{ccc} \mathbf{C} & \to & \mathrm{CP}^*_q[\mathbf{C}] \\ H & \mapsto & (H^* \otimes H, \;) \end{array}$$



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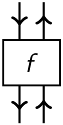
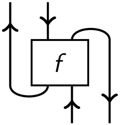
Completely quantum systems: map-state duality



$\longleftrightarrow$

$\widehat{f} :=$ 

$f\colon H^* \otimes H \to K^* \otimes K$

$\mathbb{M}_m \to \mathbb{M}_n$

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$\mathbb{M}_{mn}$

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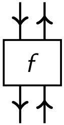

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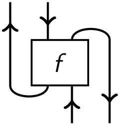
$\widehat{f} :=$ 

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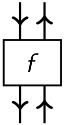
f self-adjoint $\longleftrightarrow$ $\widehat{f}$ preserves adjoints

$(f^\dagger = f)$

$(\widehat{f}(a^\dagger) = \widehat{(f)}(a)^\dagger)$

Completely quantum systems: map-state duality



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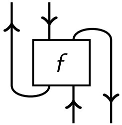
$\widehat{f} :=$ 

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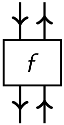

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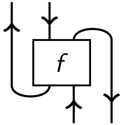
$\widehat{f} :=$ 

$f\colon H^* \otimes H \to K^* \otimes K$		$\widehat{f}\colon H^* \otimes K \to H^* \otimes K$
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f unital	$\longleftrightarrow$	$\widehat{f}$ has trivial left partial trace
$(f(1_m) = 1_n)$		$(\mathrm{tr}_n(\widehat{f}) = 1_m)$
f preserves trace	$\longleftrightarrow$	$\widehat{f}$ has trivial right partial trace
$(\mathrm{tr}(a) = \mathrm{tr}(f(a)))$		$(\mathrm{tr}_m(\widehat{f}) = 1_n)$

Completely quantum systems: environment structures

- Can axiomatise dagger compact categories of the form $\mathrm{CP}^*_q[\mathbf{C}]$
  Idea: $A = (H^* \otimes H, \;)$ always allows map $\frown\colon A \to I$
  (~~deleting map~~ partial trace)

Completely quantum systems: environment structures

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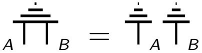
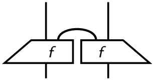
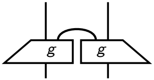
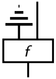
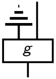
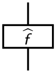
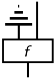
- Environment structure: $\mathbf{C} \hookrightarrow \widehat{\mathbf{C}}$ with maps  in $\widehat{\mathbf{C}}$ satisfying:

coherence: 

doubling:  $=$  in $\mathbf{C}$ $\Leftrightarrow$  $=$  in $\widehat{\mathbf{C}}$

purification: each  in $\widehat{\mathbf{C}}$ is of the form  for some f in $\mathbf{C}$

"Environment and classical channels in CQM"
Computer Science Logic LNCS 6247:230–224, 2010

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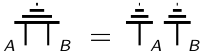
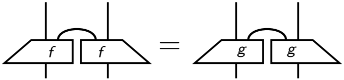
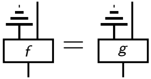
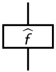
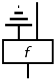
coherence: 

doubling:  in $\mathbf{C}$ $\Leftrightarrow$  in $\widehat{\mathbf{C}}$

purification: each  in $\widehat{\mathbf{C}}$ is of the form  for some f in $\mathbf{C}$

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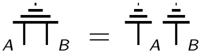
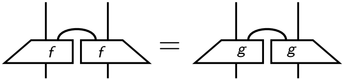
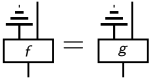
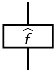
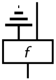
coherence: 

doubling:  in $\mathbf{C}$ $\Leftrightarrow$  in $\widehat{\mathbf{C}}$

purification: each  in $\widehat{\mathbf{C}}$ is of the form  for some f in $\mathbf{C}$

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- Question: axiomatise $\mathrm{CP}^*[\mathbf{C}]$

 

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Operator algebra: infinite dimension

Different generalisations of $\mathbb{C}^n$ and $\mathbb{M}_n$:

- C*-algebras
  $*$-algebra of operators that is closed
- AW*-algebras
  abstract/algebraic version of W*-algebra
- von Neumann algebras / W*-algebras
  $*$-algebra of operators that is weakly closed

In finite dimension coincide

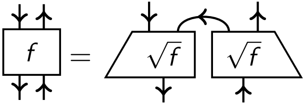
"On normed rings"
Doklady Akademii Nauk SSSR 23:430–432, 1939

"Projections in Banach algebras"
Annals of Mathematics 53(2):235–249, 1951

"On rings of operators"
Annals of Mathematics 37(1):116–229, 1936

Complete positivity: infinite dimension

- Recall $f\colon (A^* \otimes A, \ ) \to (B^* \otimes B, \ )$ completely positive if



"Pictures of completely positivity in arbitrary dimension"
Information and Computation, 2014

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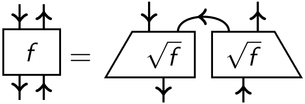

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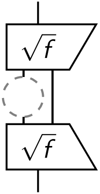


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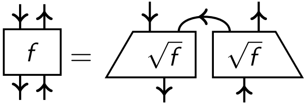

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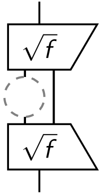

- **C** dagger symmetric monoidal $\Rightarrow \mathrm{CP}^*_\infty[\mathbf{C}]$ symmetric monoidal
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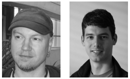

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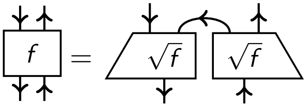
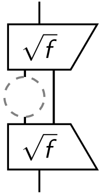


- Can reformulate without cap: 

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- $\mathrm{CP}^*_\infty[\mathbf{Hilb}]$ = type **I** factor W*-algebras and normal c.p. maps

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Information and Computation, 2014

Quantum and classical interaction: infinite dimension

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Quantum and classical interaction: infinite dimension

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- If $C \subseteq \mathbb{M}_n(A)$ commutative,
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"Diagonalizing matrices over AW*-algebras"
Journal of Functional Analysis 264(8):1873–1898, 2013

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- A is injective (and vice versa if A commutative)


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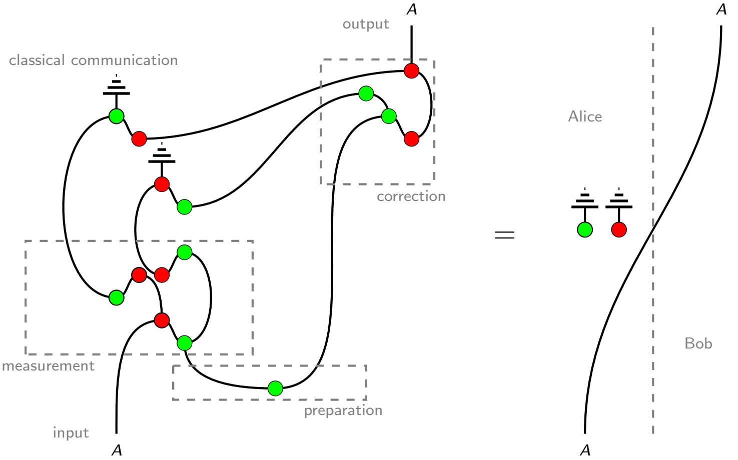
"Projective topological spaces"
Illinois Journal of Mathematics 2(4):482–489, 1958

Quantum teleportation

If $(A, \; , \;)$ and $(A, \; , \;)$ are complementary Frobenius algebras in a dagger compact category **C**, then the following holds in CP*[**C**]:



Nonstandard models: complete positivity

Recall **Rel**:

- Objects are sets
- Arrows are relations
- Tensor product is Cartesian product
- Dagger special Frobenius algebras are groupoids

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When is a map completely positive?

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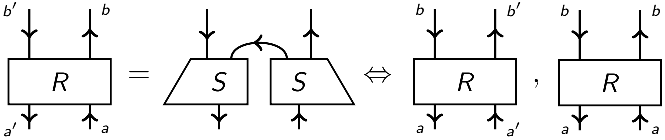

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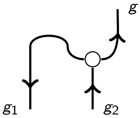

$\Leftrightarrow g_1 \circ g = g_2$

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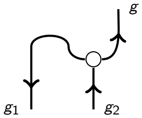

$$\Leftrightarrow g = g_1^{-1} \circ g_2$$

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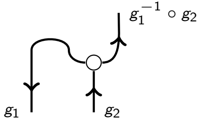

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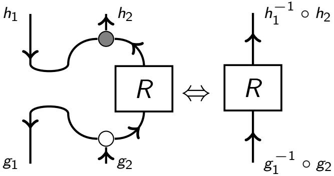

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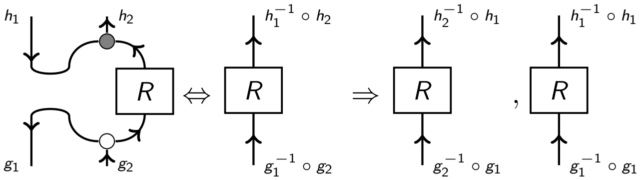

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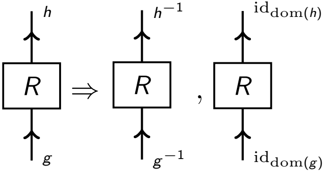

Nonstandard models: complete positivity

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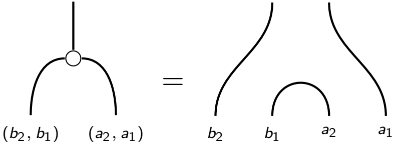
A relation $R \subseteq \mathbf{G} \times \mathbf{H}$ between groupoids respects inverses when $(g, h) \in R$ implies $(g^{-1}, h^{-1}) \in R$ and $(\mathrm{id}_{\mathrm{dom}(g)}, \mathrm{id}_{\mathrm{dom}(h)}) \in R$.

$\mathrm{CP}^*[\mathbf{Rel}]$ = groupoids and relations respecting inverses.

Nonstandard models: completely quantum systems

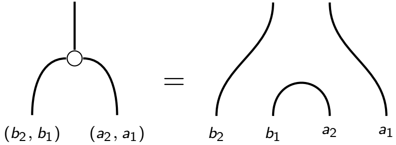
Question: What are the algebras $(A^* \times A, \ )$ in **Rel**?



$$= \begin{cases} (b_2, a_1) & \text{if } b_1 = a_2 \\ \text{undefined} & \text{otherwise} \end{cases}$$

Nonstandard models: completely quantum systems

Question: What are the algebras $(A^* \times A, \ )$ in **Rel**?



$$= \begin{cases} (b_2, a_1) & \text{if } b_1 = a_2 \\ \text{undefined} & \text{otherwise} \end{cases}$$

$\Rightarrow$ identity arrows are (a, a)
$\Rightarrow$ objects correspond to $a \in A$
$\Rightarrow$ $\mathrm{dom}(a_2, a_1) = a_1$, $\mathrm{cod}(a_2, a_1) = a_2$
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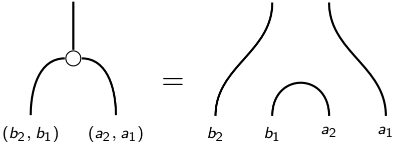

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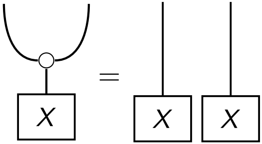
Answer: indiscrete groupoids (exactly one process $a \to b$)

Nonstandard models: subsystems

- Copyable states $X \subseteq \mathbf{G}$





"Relative Frobenius algebras are groupoids"
Journal of Pure and Applied Algebra 217:114–124, 2013



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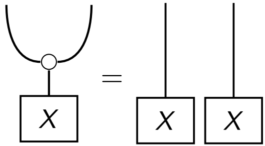

$$\Rightarrow \quad X^2 = \{(g, h) \mid g \circ h \in X\}$$

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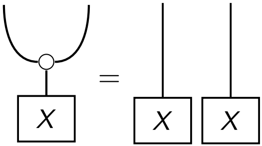

$$\Rightarrow \quad X^2 = \{(g,h) \mid g \circ h \in X\} = \{(f, g \circ f^{-1}) \mid f \in X\}$$

  

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Nonstandard models: subsystems

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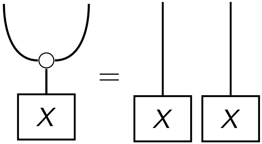

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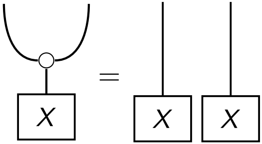
Nonstandard models: subsystems

- Copyable states $X \subseteq \mathbf{G}$ are endohomset connected components



$$\begin{aligned} \Rightarrow \quad & X^2 = \{(g,h) \mid g \circ h \in X\} \\ & = \{(f, g \circ f^{-1}) \mid f \in X\} \\ \Rightarrow \quad & \text{if } f \in X \text{ and } \mathrm{dom}(f) = \mathrm{dom}(g) \\ & \text{then } g \in X \\ \Rightarrow \quad & \text{if } f \in X \text{ then } f \circ f \text{ defined} \end{aligned}$$



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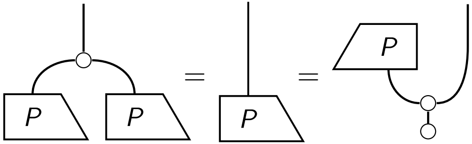

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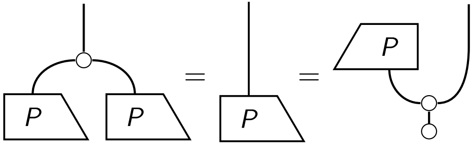

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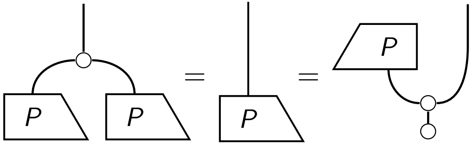

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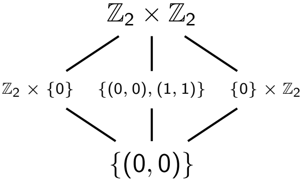
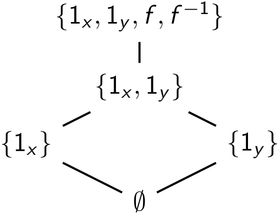
  $(a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c))$





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"Completely positive projections and biproducts"
Quantum Physics and Logic, EPTCS 2013

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- Instead of direct sums, use 2-categories (caveats)




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"Mixed quantum states in higher categories"
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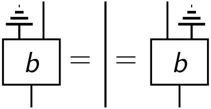
- $(A, ⅄)$ broadcastable when  for some c.p. b

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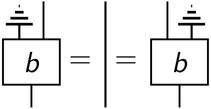
- $(A,$ $)$ broadcastable when  for some c.p. b
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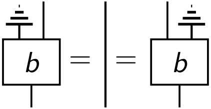
- $(A, \,)$ broadcastable when  for some c.p. b
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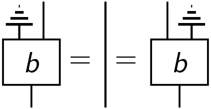
- $(A, \text{⅄})$ broadcastable when  for some c.p. b
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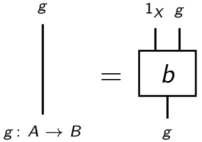

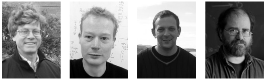

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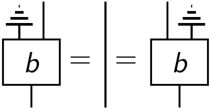
- $(A, \curlywedge)$ broadcastable when  for some c.p. b
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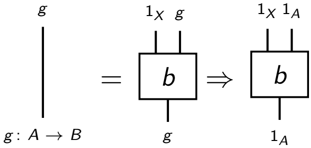

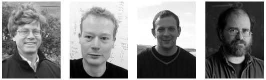

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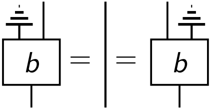
- $(A, \ )$ broadcastable when  for some c.p. b
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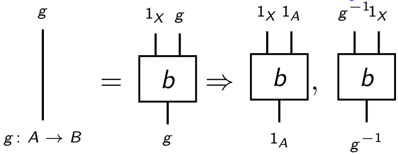

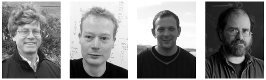

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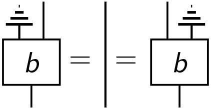
- $(A, \;)$ broadcastable when  for some c.p. b
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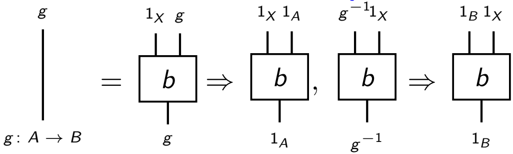

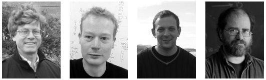

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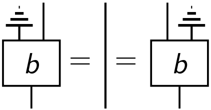
- (A,) broadcastable when  for some c.p. b

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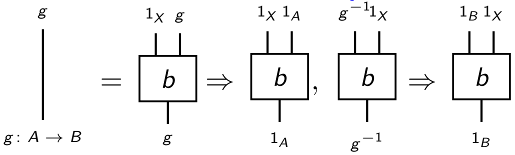
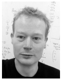


$\Rightarrow A = B$

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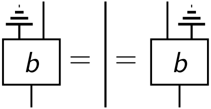
- $(A, \ )$ broadcastable when  for some c.p. b
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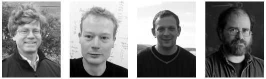
- $(A, \ )$ commutative $\leftrightarrows$ $(A, \ )$ broadcastable



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Conclusion

What have we learnt?

- Frobenius algebras model classical and quantum information
- Physical processes are completely positive channels
- Teleportation uses both classical and quantum information
- In Hilbert spaces: operator algebra
- In possibilistic mechanics: groupoids
- Nonstandard models break distributivity, no-broadcasting

Open questions: interaction between classical and quantum