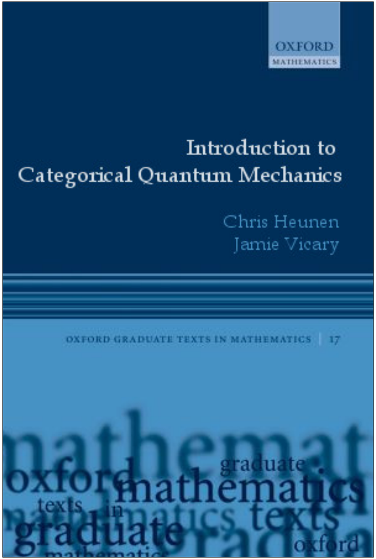
Categorical Quantum Mechanics: Frobenius algebra

Chris Heunen
University of Oxford

May 19, 2014

Overview



Today:

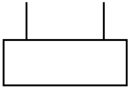
- Classical communication
  Frobenius algebra, “spiders”
- Observables
  operator algebra
- Nonstandard models
  possibilistic quantum mechanics

Tomorrow:

- Physical processes
  complete positivity
- Classical vs quantum channels
  state spaces
- Drawing theories apart
  nonstandard models

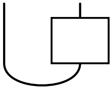
Introduction

state

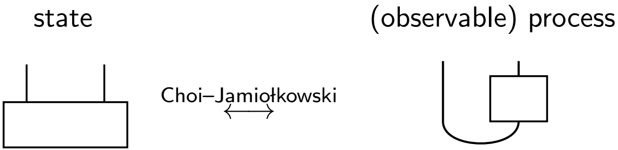


Choi–Jamiołkowski $\longleftrightarrow$

(observable) process

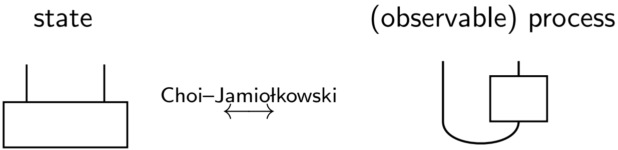


Introduction



Closure: encode composition of (observable) processes in states

Introduction

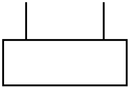


Closure: encode composition of (observable) processes in states

Processes compose → states compose → algebra

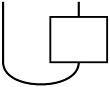
Introduction

state



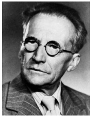
$\xleftrightarrow{\text{Choi–Jamiołkowski}}$

(observable) process



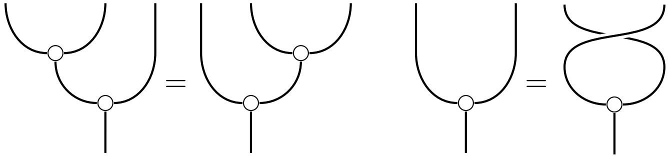
Closure: encode composition of (observable) processes in states

Processes compose $\rightarrow$ states compose $\rightarrow$ algebra

-  states evolve, observables fixed
-  states fixed, observables evolve

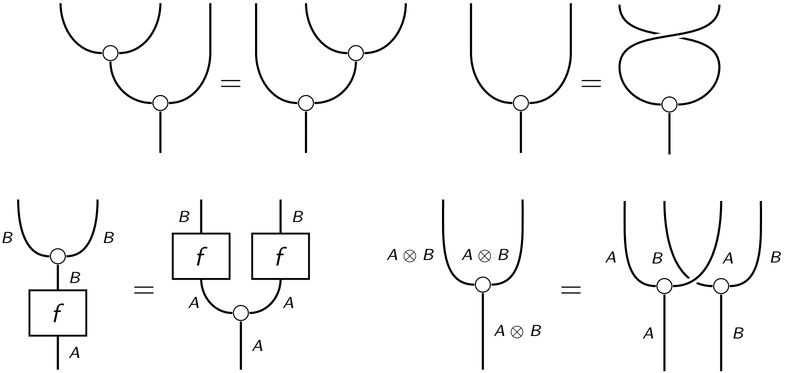
No cloning

Suppose we could copy systems



No cloning

Suppose we could copy systems uniformly



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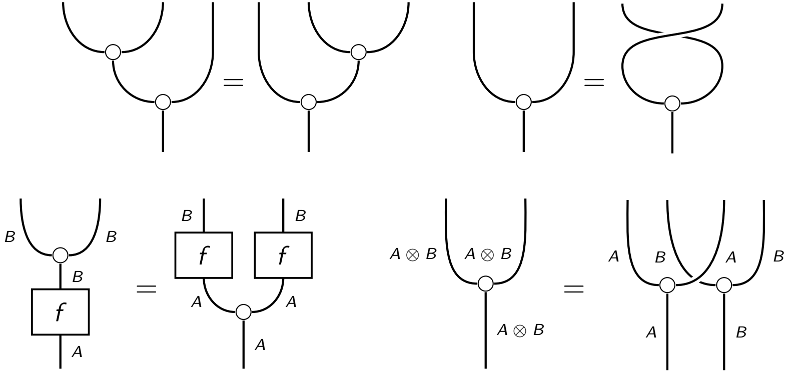


Then our dagger compact category must degenerate

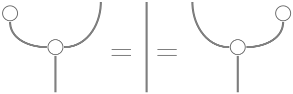
(the only processes $A \to A$ are multiplication by a scalar)

“No-cloning in categorical quantum mechanics”

Semantic techniques in quantum computation (eds. Gay, Mackie) 1–28, Cambridge Univ. Press, 2010

No deleting

Suppose we could delete systems



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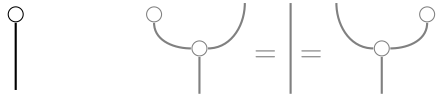

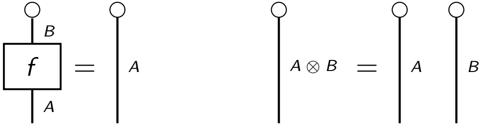

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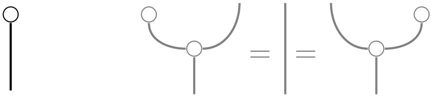

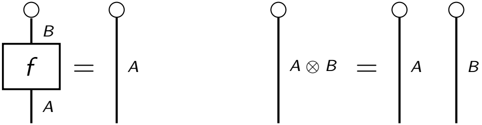

Then our dagger compact category must degenerate

(there is at most one process $A \to B$)

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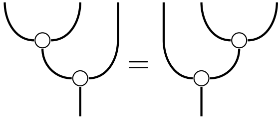
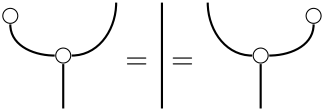
Semantic techniques in quantum computation (eds. Gay, Mackie) 1–28, Cambridge Univ. Press, 2010

Frobenius algebra

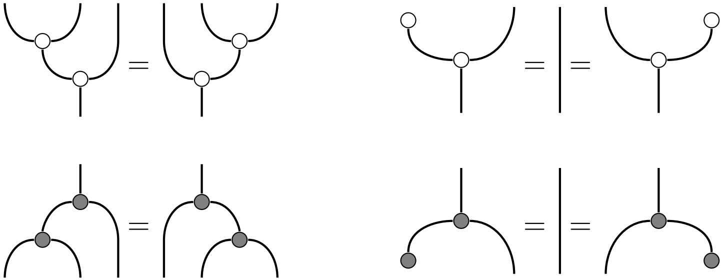
A coalgebra is:





Frobenius algebra

An algebra-coalgebra-pair is:



Frobenius algebra

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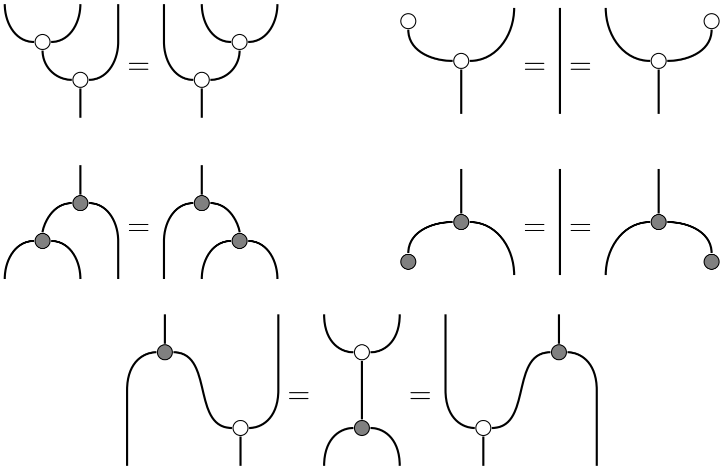

Frobenius algebra

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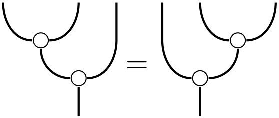

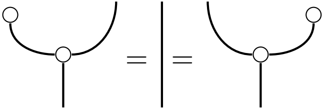

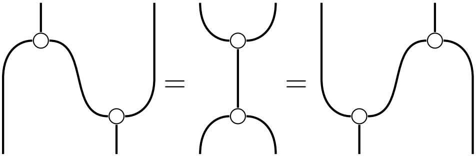

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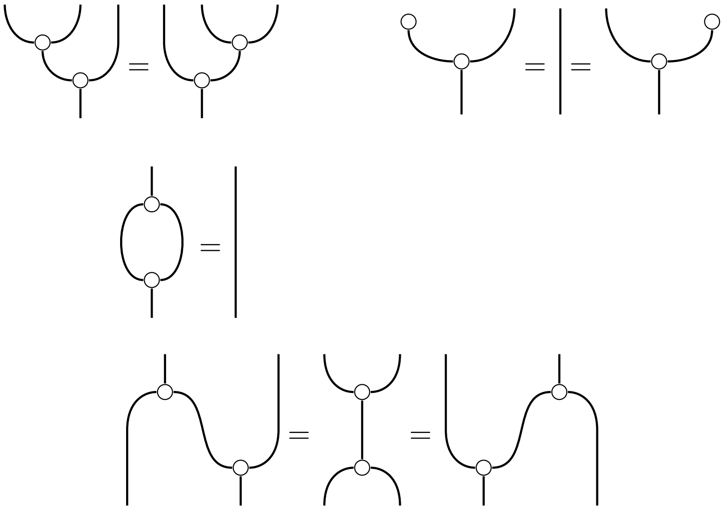
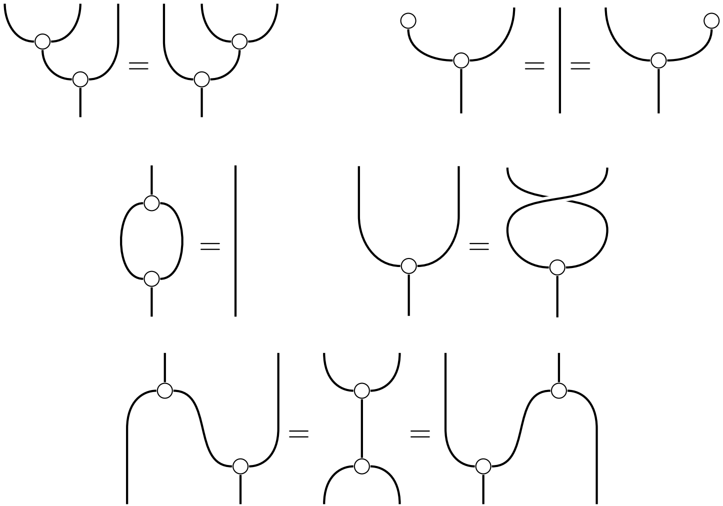


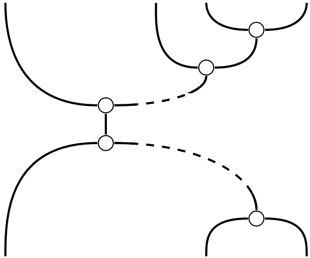
Frobenius algebra

A classical structure is:



Frobenius algebra: generators and relations

Any connected diagram built from the components of a special Frobenius algebra equals the following normal form:



"2D topological quantum field theories and Frobenius algebras"
Journal of Knot Theory and its Ramifications 5:569–587, 1996

"Frobenius algebras and 2D topological quantum field theories"
Cambridge University Press, 2003

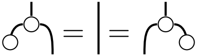
Proof of noncommutative spider theorem

Pick a . Push down past all . What can we meet on the way?

Proof of noncommutative spider theorem

Pick a . Push down past all . What can we meet on the way?

- Meeting  makes  vanish.



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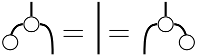

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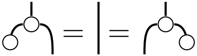
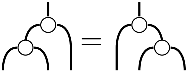


- Can push chosen past another by associativity.



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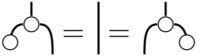

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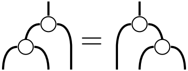
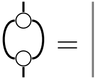
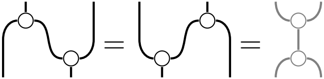


- Can meet a  in three ways:





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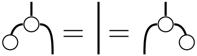

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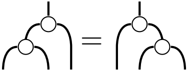

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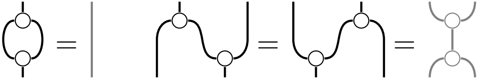

Push all above all , killing all .

Proof of noncommutative spider theorem

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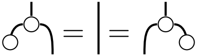

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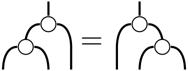

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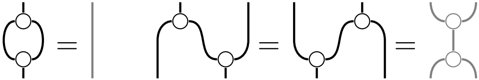
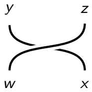


Push all above all , killing all .
Use (co)associativity to end up with desired normal form.

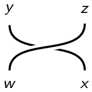
Proof of commutative spider theorem

Isolate a swap.



Proof of commutative spider theorem

Isolate a swap. By assumption region w connected to x or y.



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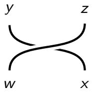

- Case x: connecting subdiagram has strictly less , so may assume normal form. Use coassociativity to get directly above a . Eliminate using commutativity.

Proof of commutative spider theorem

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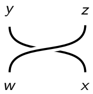
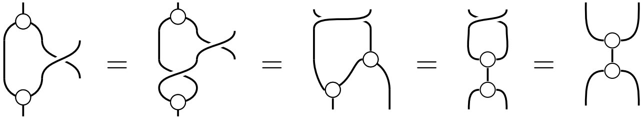


- Case x: connecting subdiagram has strictly less , so may assume normal form. Use coassociativity to get  directly above a . Eliminate  using commutativity.
- Case y: both subdiagrams w and y strictly less , so may assume normal form. Local situation:



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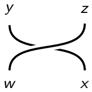

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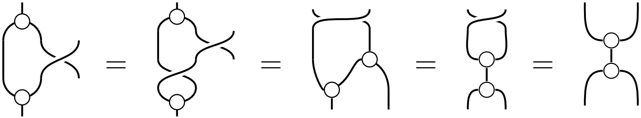
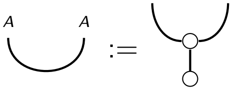


Eliminate all one by one; use noncommutative spider theorem.

Frobenius algebras have duals

In any monoidal category, Frobenius algebras are self-dual



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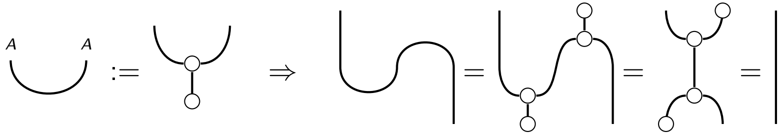

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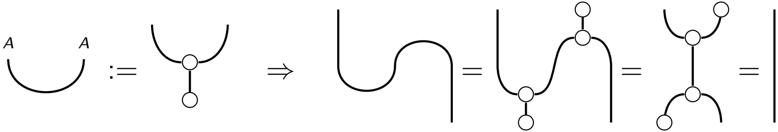
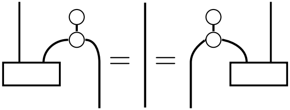


So Frobenius algebras give more information than compactness.

(And: Frobenius algebras must be finite-dimensional.)

Frobenius algebra: alternative definitions

Alternative definition 2: a Frobenius algebra is an algebra (,)
with a nondegenerate form , *i.e.* is part of a self-duality.



"Theorie der hyperkomplexen Größen I"
Sitzungsberichte der Preussischen Akademie der Wissenschaften 504–537, 1903

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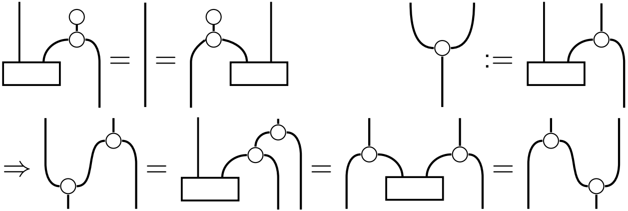

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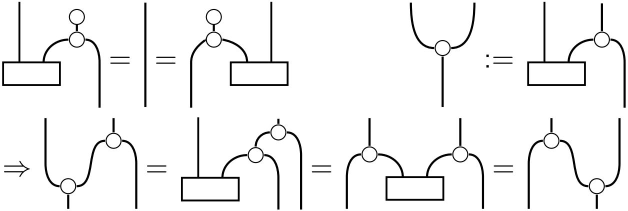

Alternative definition 3: a Frobenius algebra is an algebra-coalgebra pair such that is a homomorphism of -modules.

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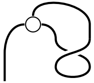
Frobenius algebra: minimal definition

- Compactness implies unitality:  $:=$ 

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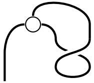
- Compactness implies unitality:  := 
- Speciality and Frobenius law imply associativity

Frobenius algebra: minimal definition

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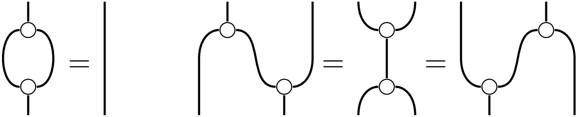
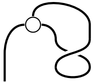
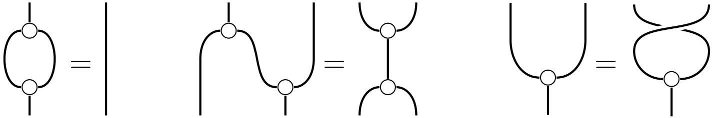
For a special dagger Frobenius algebra  it suffices to check:



Frobenius algebra: minimal definition

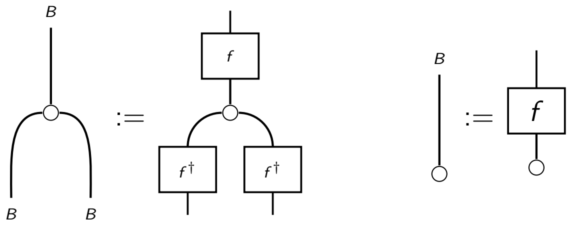
- Compactness implies unitality:  := 
- Speciality and Frobenius law imply associativity
- Commutativity implies one equation suffices for Frobenius law

For a classical structure  it suffices to check:



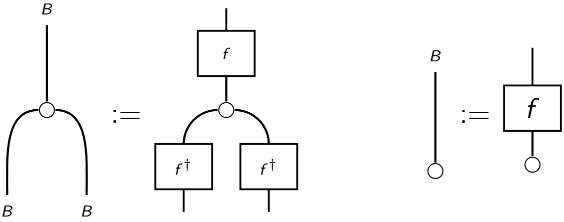
Frobenius algebra: abstract examples

- If $(A, \;)$ is a Frobenius algebra and $A \xrightarrow{f} B$ is unitary, then B is a Frobenius algebra with

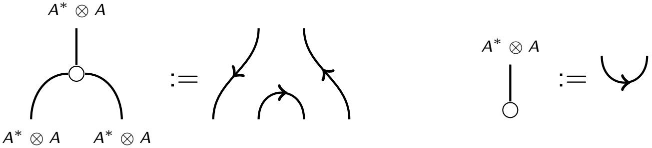


Frobenius algebra: abstract examples

- If $(A,$ $)$ is a Frobenius algebra and $A \xrightarrow{f} B$ is unitary, then B is a Frobenius algebra with

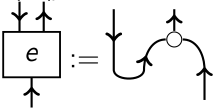


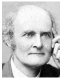
- If A is any object in a dagger compact category, then $A^* \otimes A$ is a Frobenius algebra with



Operator algebra: embedding

Any Frobenius algebra $(A, \ )$ embeds into $(A^* \otimes A, \ )$:





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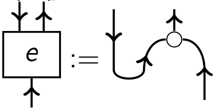
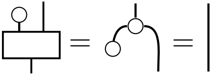
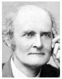


- is embedding: 



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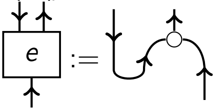
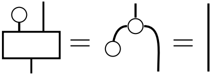
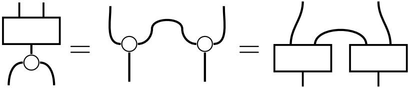
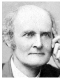


- is embedding: 
- preserves multiplication: 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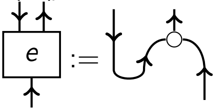
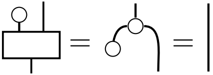
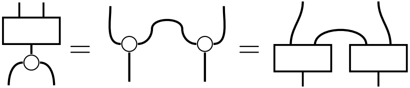
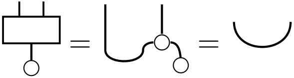
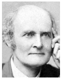


- is embedding: 
- preserves multiplication: 
- preserves unit: 



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Operator algebra: concretely

A $*$-algebra is a subset $A \subseteq \mathbb{M}_n(\mathbb{C})$ closed under multiplication, addition, adjoints, scalar multiplication, and contains the identity.
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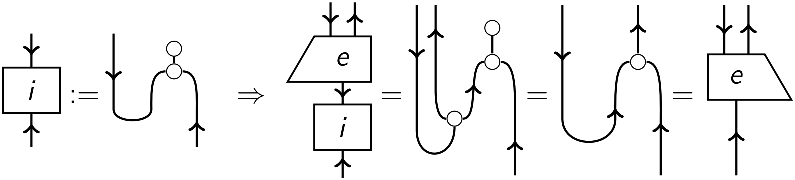
Special dagger Frobenius algebras (A, ) in the category of Hilbert spaces are precisely $*$-algebras:



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- Can axiomatize $*$-algebras without acting on a Hilbert space.

 

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$$\Yup : |\, i \,\rangle \mapsto |\, i \,\rangle \otimes |\, i \,\rangle \qquad \multimapdotbothvert : |\, i \,\rangle \mapsto 1$$

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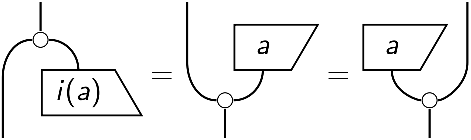

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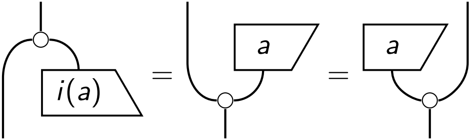

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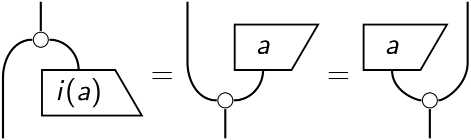

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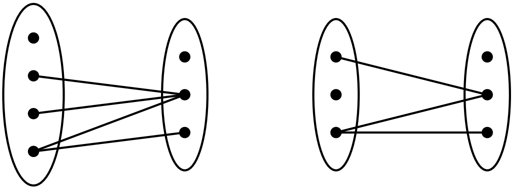
Drawing theories apart

Possibilistic quantum theory:

- systems: *sets* A
- processes $A \to B$: *relations* $R \subseteq A \times B$
- composition: $S \circ R = \{(a, c) \mid \exists b \colon (a, b) \in R, (b, c \in S)\}$

$A \xrightarrow{R} B$ $\qquad$ $B \xrightarrow{S} C$



- parallel composition: Cartesian *product*

Nonstandard models

Category of sets and relations:

- scalars: Boolean values $(\{\mathrm{true}, \mathrm{false}\}, \wedge, \vee)$

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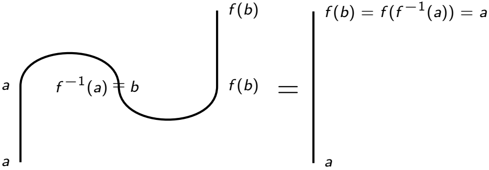
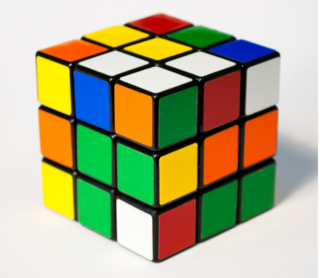


- trace: detects whether relation has fixed point
- dimension: detects whether set is empty

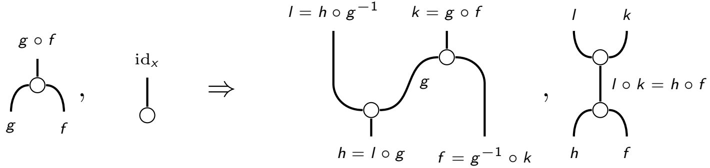
Groupoids

A groupoid is a category whose processes are invertible



Frobenius algebras: possibilistic

The set of arrows of a groupoid is a Frobenius algebra in **Rel**:



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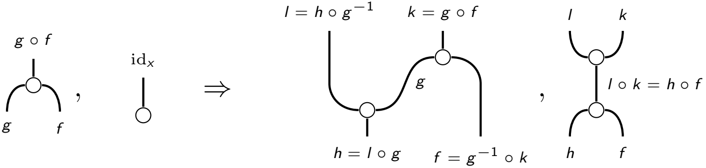
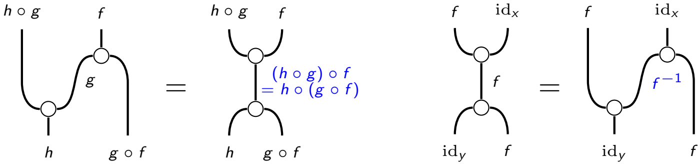


A Frobenius algebra in **Rel** is the set of arrows of a groupoid:
 is a partial function because of speciality, and for $f \colon x \to y$



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- In Hilbert spaces: operator algebra
- In possibilistic mechanics: groupoids
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Tomorrow:

- Physical processes: maps between state spaces
- Complete positivity: channels between operator algebras
- Nonstandard models: different quantum information theory