Contextual Semantics: From Quantum Mechanics to Logic, Databases, Constraints, and Complexity Lecture 1

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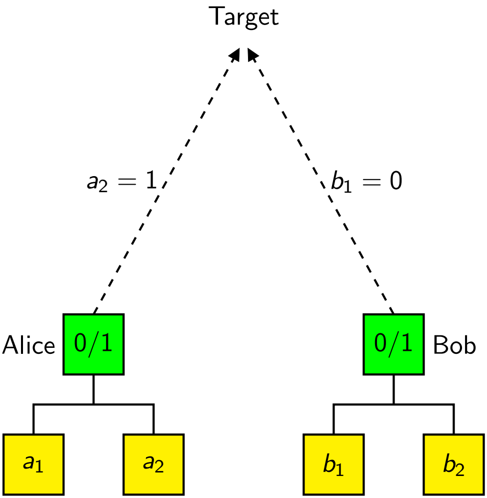
This is the idea of contextual semantics.

Alice and Bob look at bits



A Probabilistic Model Of An Experiment

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Example: The Bell Model

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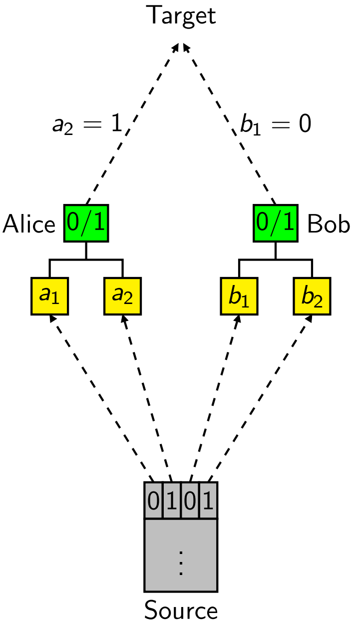
How can we explain this behaviour?

Classical Correlations



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Using elementary probability theory, we can calculate:

$$p_N \;\leq\; \mathrm{Prob}(\bigvee_{i=1}^{N-1} \neg\phi_i) \;\leq\; \sum_{i=1}^{N-1} \mathrm{Prob}(\neg\phi_i) \;=\; \sum_{i=1}^{N-1}(1 - p_i) \;=\; (N-1) - \sum_{i=1}^{N-1} p_i.$$

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Hence we obtain the inequality

$$\sum_{i=1}^{N} p_i \;\leq\; N - 1.$$

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The violation of the logical Bell inequality is $1/4$.

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The support of the Hardy model:

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Alice and Bob's choices are now of **measurement setting** (e.g. which direction to measure spin) rather than "which register to load".

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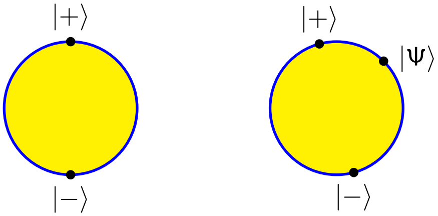
States of the system can be described by complex unit vectors in $\mathbb{C}^2$. These can be visualized as points on the unit 2-sphere:



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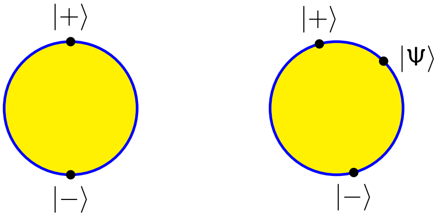

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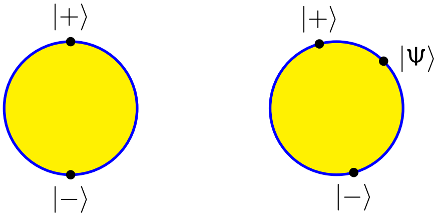

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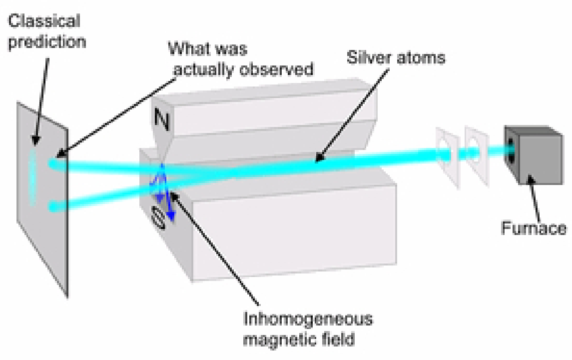
Note the appearance of **quantization** here: there are not a continuum of possible outcomes for each measurement, but only two!

The Stern-Gerlach Experiment



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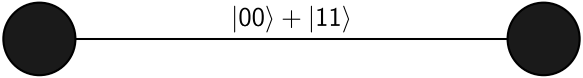

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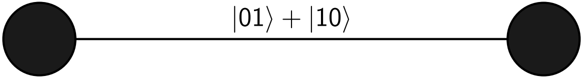

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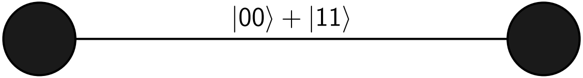

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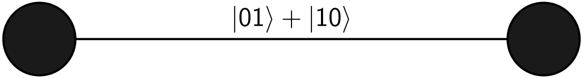

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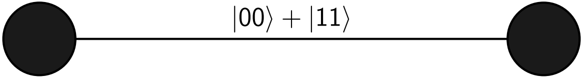

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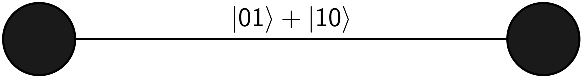

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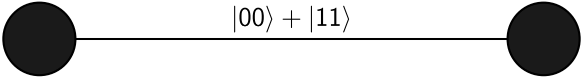

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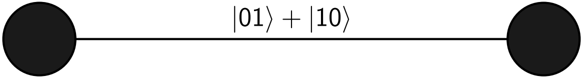

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Bell's theorem: QM is **essentially non-local**.

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Important note: this is **physically realizable**!

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Extensively tested experimentally.

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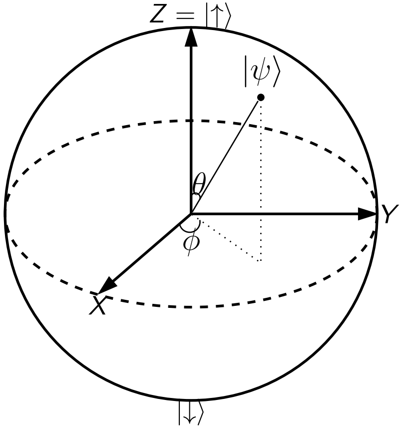

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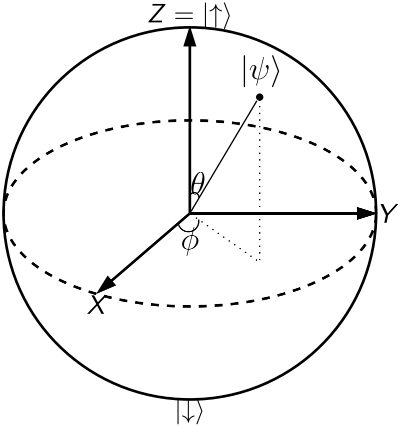

Spin measurements lying in the equatorial plane of the Bloch sphere
Spin Up: $(|\uparrow\rangle + e^{i\phi}|\downarrow\rangle)/\sqrt{2}$, Spin Down: $(|\uparrow\rangle + e^{i(\phi+\pi)}|\downarrow\rangle)/\sqrt{2}$

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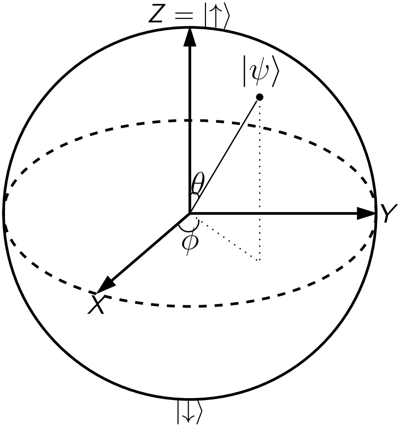

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X itself, $\phi = 0$:
Spin Up $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ and Spin Down $(|\uparrow\rangle - |\downarrow\rangle)/\sqrt{2}$.

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The event in yellow is represented by

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Probability of this event M when measuring (a, b') on $B = (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)/\sqrt{2}$ is given by Born rule:

$$|\langle B|M\rangle|^2.$$

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The other entries can be computed similarly.

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Each row of the table specifies a **probability distribution** on events O^C for a given choice of measurements C.

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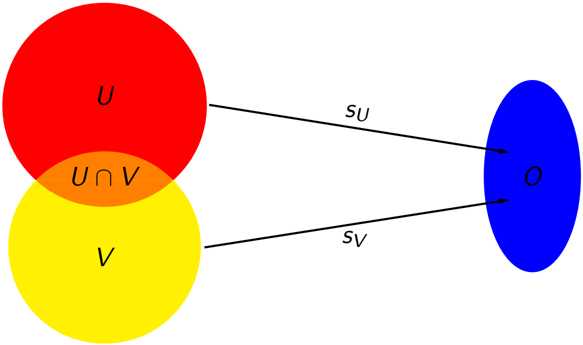
The fact that the behaviour of these observable outcomes cannot be accounted for by some context-independent global description of reality corresponds to the geometric fact that these local sections cannot be glued together into a **global section**.

Gluing functional sections



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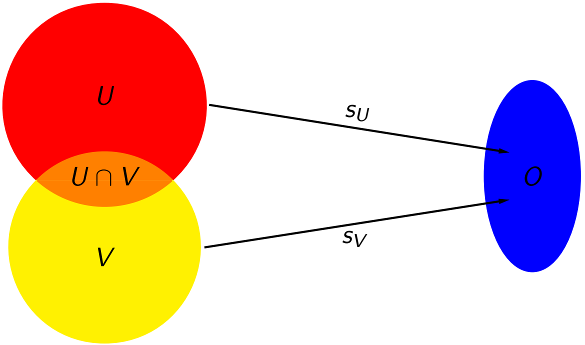

If $s_U|_{U \cap V} = s_V|_{U \cap V}$, they can be glued to form

$$s : U \cup V \longrightarrow O$$

such that $s|_U = s_U$ and $s|_V = s_V$.

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In fact, it is the composition of $\mathcal{E} : U \mapsto O^U$ and the covariant distribution functor $\mathcal{D}_R$.

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N.B. Vorob'ev theorem.

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This uses the following measurement cover $\mathcal{U} = \{U_1, \ldots, U_9\}$:

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Yields a proof of the **Kochen-Specker theorem**.

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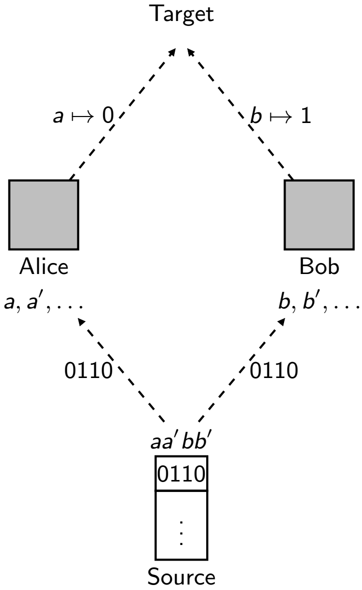
In other words, Bob's choice of measurement cannot influence Alice's outcome.

Hidden Variables: The Mermin instruction set picture



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If d is a global section for the model $\{e_C\}$, we recover the predictions of the model by **averaging over the values of these hidden variables**:

$$e_C(s) \; = \; d|C(s) \; = \sum_{s' \in \mathcal{E}(X), s'|C = s} d(s') \; = \sum_{s' \in \mathcal{E}(X)} \delta_{s'|C}(s) \cdot d(s').$$

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So:

existence of a local hidden-variable model for a given empirical model
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Hence:

No such h.v. model exists (the empirical model is **non-local**/**contextual**)
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there is an **obstruction to the existence of a global section**

Methods for showing obstructions to global sections

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Thus in terms of well-known examples, we have

$$\text{Bell} < \text{Hardy} < \text{GHZ}$$

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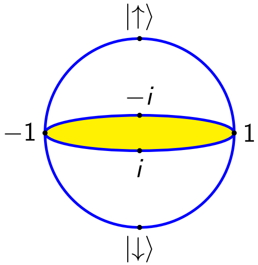
This computation is controlled by the product of the $|\downarrow\rangle$-coefficients of the basis vectors: cyclic group generated by $i \;\cong\; \mathbb{Z}_4$.



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NB: a model with these properties can be realized in quantum mechanics.

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The remaining cases are where all Y's receive the same value. Then any pair of X's must receive different values. But taking any 3 X's, this yields a contradiction, since there are only two values, so some pair must receive the same value.

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- We say that a state is strongly non-local if for **some** choice of local observables for each party, the resulting empirical model is strongly non-local.
- We can similarly define logical non-locality for states; we say that a state is logically non-local if for some choice of local observables, the resulting empirical model is logically non-local; while the state is **not** strongly non-local.

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- Finally, a state is weakly non-local if it is non-local, but neither of the previous two cases apply.

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We believe that an answer to this problem will shed considerable light on the structure of multipartite states, not least because it will necessitate solving the following task:

> *Given a multipartite state, find local observables which witness its highest degree of non-locality.*

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For example, the only strongly contextual bipartite models are the PR-boxes, which are of course not quantum realizable. By contrast, for all $n > 2$, the n-partite GHZ states are strongly contextual.

Proposition

All bipartite entangled states **except** *the maximally entangled ones are logically non-local.*

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However, as we shall see, for $n > 2$ a different picture emerges.

A Striking Result

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Let $P(n)$ be the class of n-qubit pure states which, up to permutation, can be written as tensor products of 1-qubit and 2-qubit maximally entangled states. Let $L(n)$ be the set of logically non-local n-qubit states.

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The general result is proved non-constructively.

Permutation Symmetric States

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For each $n \geq 2$, $0 < k < n$ the Dicke state $S(n,k)$ is defined as:

$$S(n,k) \; := \; K \sum_{\text{perm}} |0^k 1^{n-k}\rangle.$$

where $K = \binom{n}{k}^{-1/2}$ is a normalization constant, and we sum over all products of k 0-kets and $n-k$ 1-kets.

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All permutation symmetric states are logically non-local.

Functionally dependent balanced states

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A balanced $n+1$-qubit quantum state with a functional dependency given by an n-ary Boolean function $F : \{0,1\}^n \rightarrow \{0,1\}$has the form

$$\Psi_F = \frac{1}{\sqrt{2^n}} \sum_{q_1 q_2 \ldots q_n = 00\ldots0}^{11\ldots1} |q_1 q_2 \ldots q_n F(q1, q2, \ldots, q_n)\rangle$$

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Proposition

All balanced functionally dependent states are in $P(n)$ or $L(n)$.